Knots in Three and Four Dimensions

Dror Bar-Natan, http://drorbn.net/syr21 (see “handout”!)

Syracuse by Web, April 2021

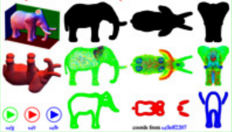
Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

This talk is best viewed with your own webcam on!

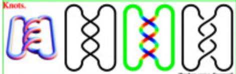
Knots in Three and Four Dimensions

Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

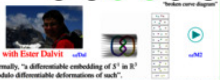
Warmup: Flatlanders View an Elephant.



Knots.



"broken curve diagram"

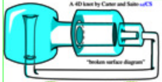


with Ester Dalvit ω/Dal ω/M2

Formally, "a differentiable embedding of S^1 in $\mathbb{R}^3$ modulo differentiable deformations of such".

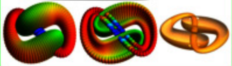
2-Knots / 4D Knots. Formally, "a differentiable embedding of S^2 in $\mathbb{R}^4$ modulo differentiable deformations of such".

A 4D knot by Carter and Saito ω/CS

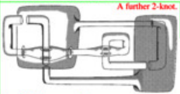


"broken surface diagram"

Carter, Banach, Saito

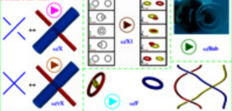


A further 2-knot.

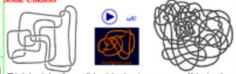


ω/CS

Some Movies



Some Unknots



Thistlethwaite's unknot Scharein's relaxation Haken's unknot

Reidemeister's Theorem. (a) Every knot has a "broken curve diagram", made only of curves and "crossings" like ⤫. (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of "Reidemester moves":

Kurt Reidemeister

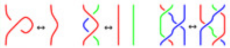
Topology is locally analysis and globally algebra

3-Colourings. Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.

Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{trefoil}) = 9$; so $\bigcirc \neq$ trefoil.

Riddle. Is $\lambda(K)$ always a power of 3?

Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:



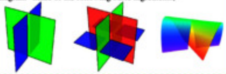
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

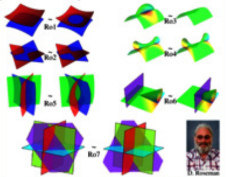
www.katlas.org

Knots in Three and Four Dimensions, 2

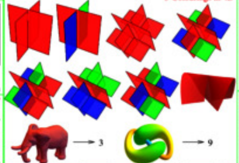
Theorem. Every 2-knot can be represented by a "broken surface diagram" made of the following basic ingredients,



…and any two representations of the same knot differ by a sequence of the following "Roseman moves":

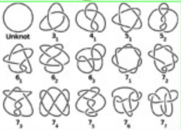


3-Colourings in 4D



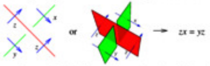
A Knot Table

There are many more!



ω/KT

A Stronger Invariant. There is an assigment of groups to knots / 2-knots as follows. Put an arrow "under" every un-broken curve / surface in a broken curve / surface diagram and label it with the name of a group generator. Then mod out by relations as below.

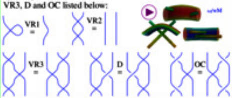


Facts. The resulting "Fundamental group" $\pi_1(K)$ of a knot / 2-knot K is a very strong but not very computable invariant of K. Though it has computable projections; e.g., for any finite G, count the homomorphisms from $\pi_1(K)$ to G.

Exercise. Show that $|\text{Hom}(\pi_1(K) \to S_3)| = \lambda(K) + 3$.

Satoh's Conjecture. (Satoh, *Virtual Knot Presentations of Ribbon Torus-Knots*, J. Knot Theory and its Ramifications **9** (2000) 531–542). Two long w-knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D iff they differ by a sequence of R-moves as above and the "w-moves" VR1–VR3, D and OC listed below:

Shin Satoh



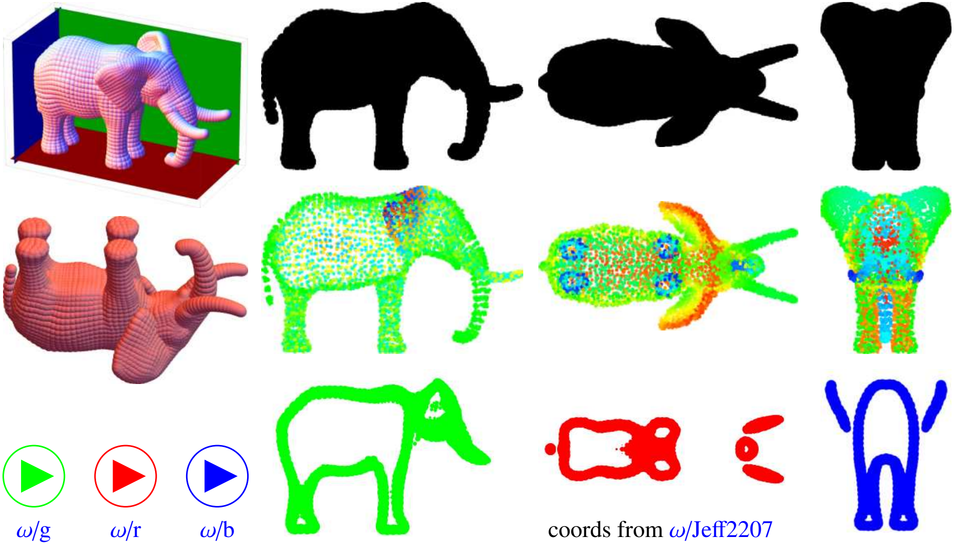
Some knot theory books.

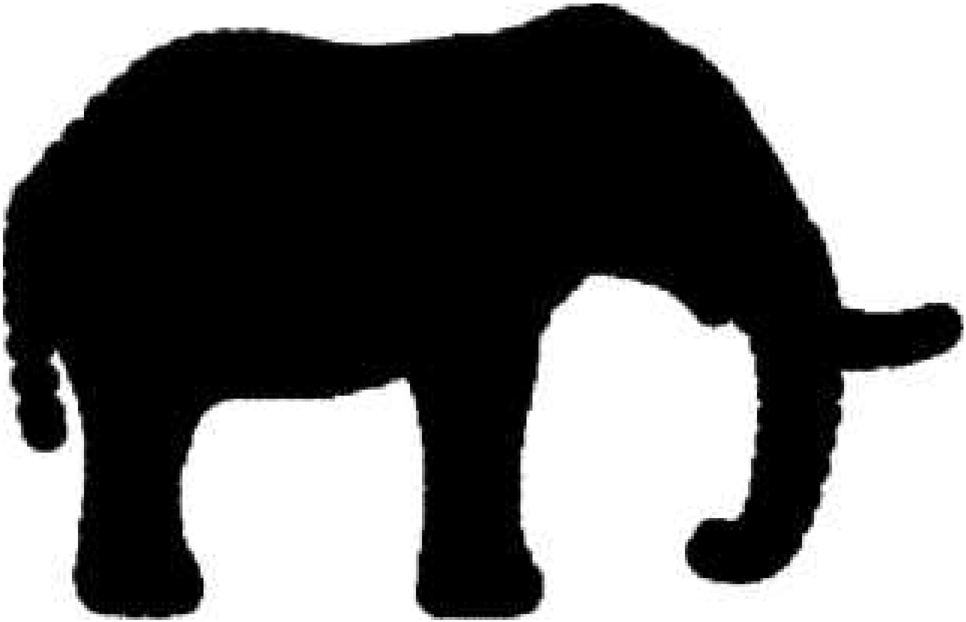
- Colin C. Adams, *The Knot Book, an Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, 2004.
- Meike Akveld and Andrew Jobbings, *Knots Unravelled, from Strings to Mathematics*, Arbelos 2011.
- J. Scott Carter and Masahico Saito, *Knotted Surfaces and Their Diagrams*, American Mathematical Society, 1997.
- Peter Cromwell, *Knots and Links*, Cambridge University Press, 2004.
- W.B. Raymond Lickorish, *An Introduction to Knot Theory*, Springer 1997.

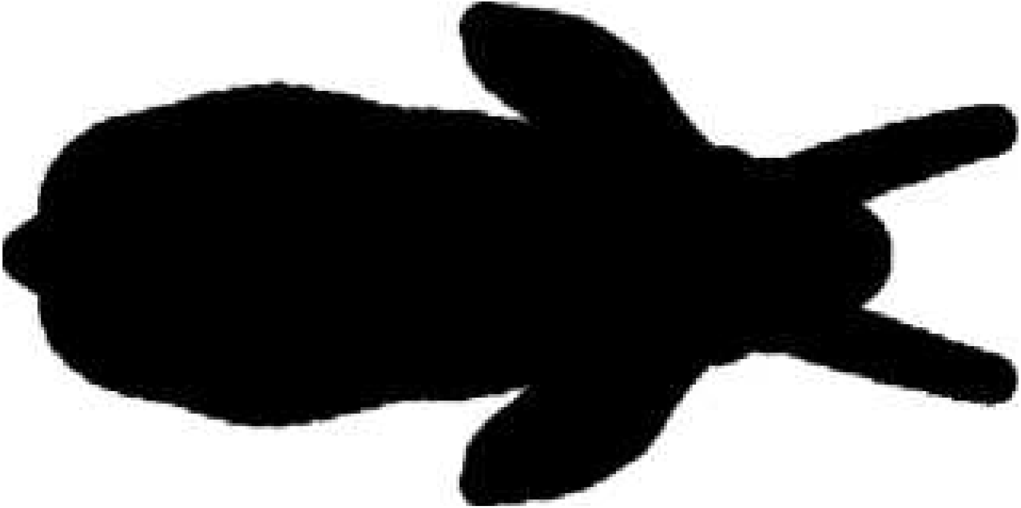
Thanks for inviting me to Syracuse! As most of you have never seen it, here's a picture of the lecture room:

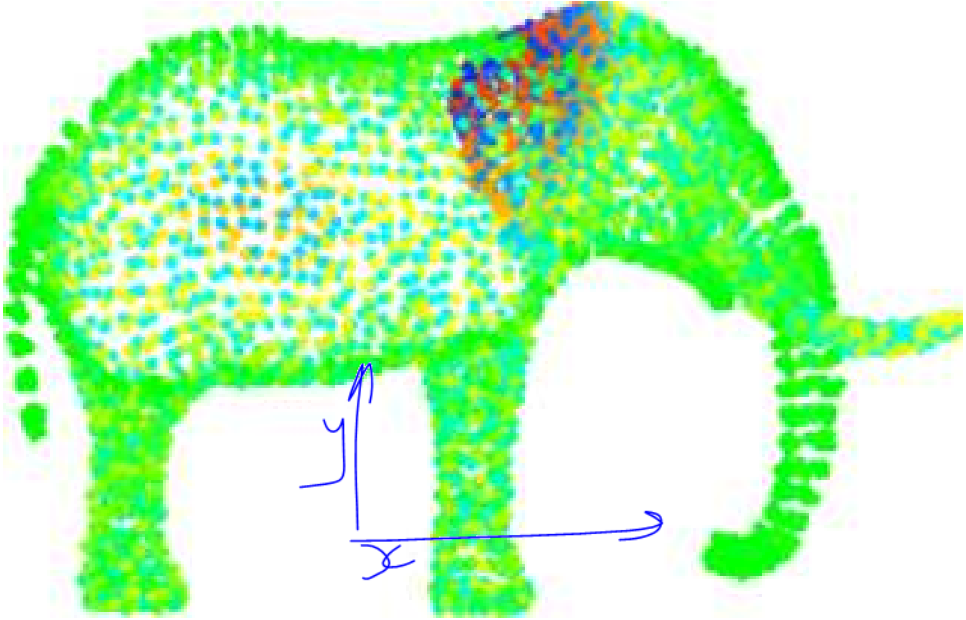
If you can, please turn your video on! (And mic, whenever needed).

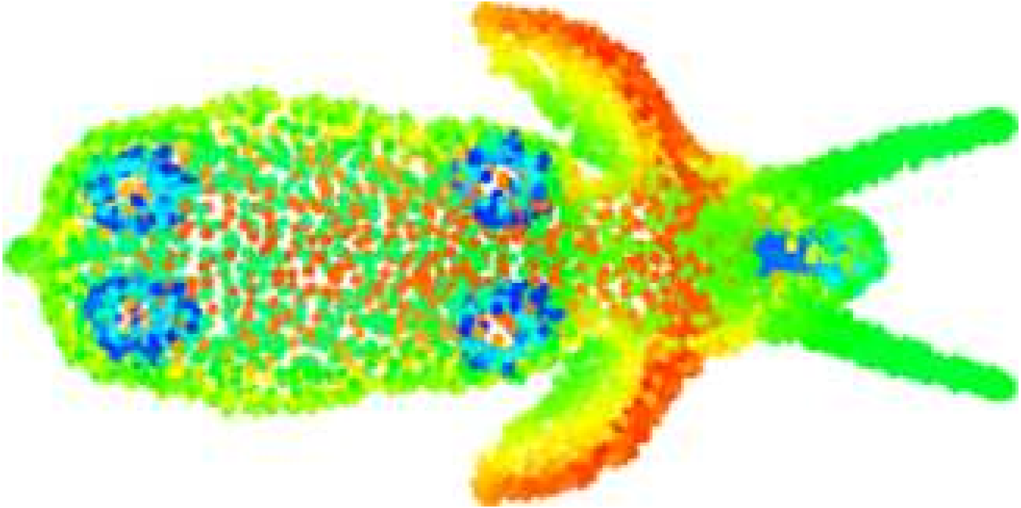
Warmup: Flatlanders View an Elephant.

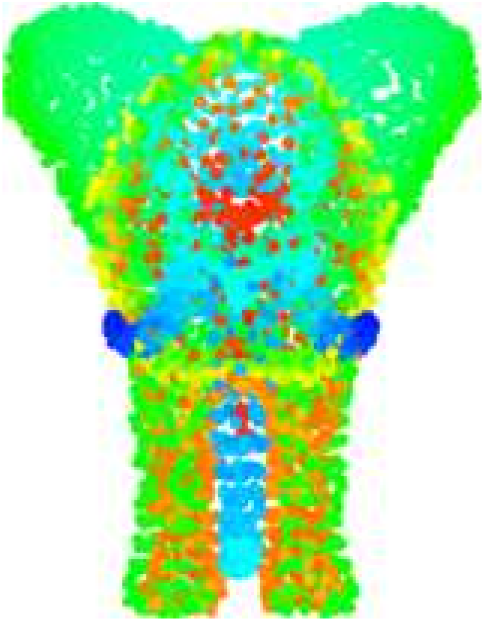


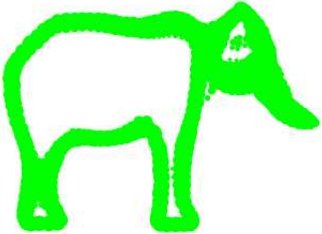


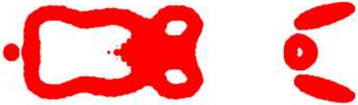



y
x



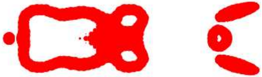






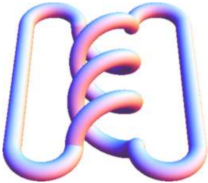
coords from ω/Jeff2207

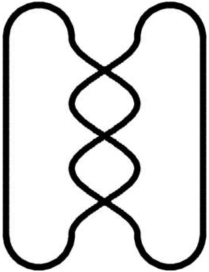
ω/g ω/r ω/b

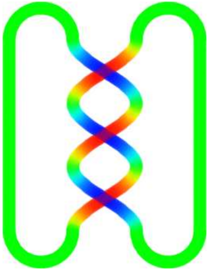


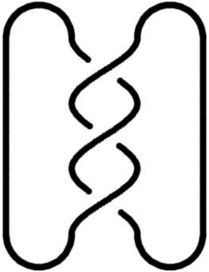
coords from ω/Jeff2207

Knots.



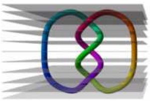


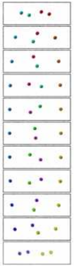




"broken curve diagram"

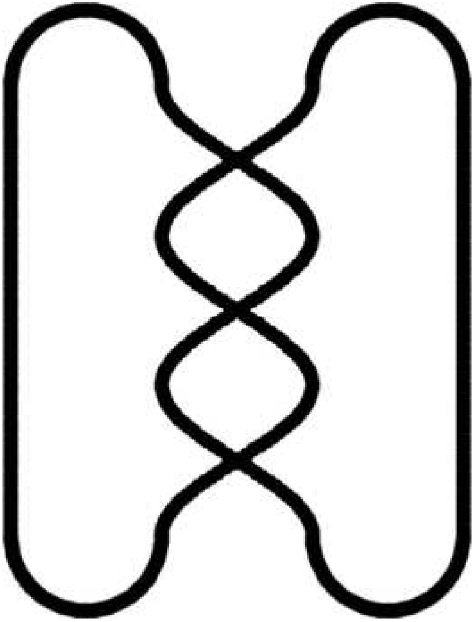
with Ester Dalvit ω/Dal

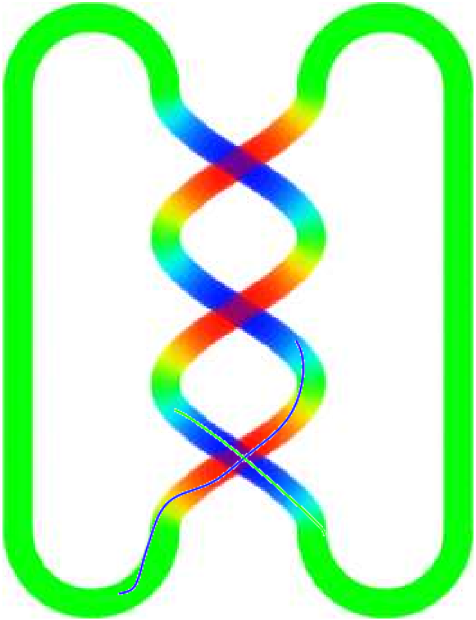


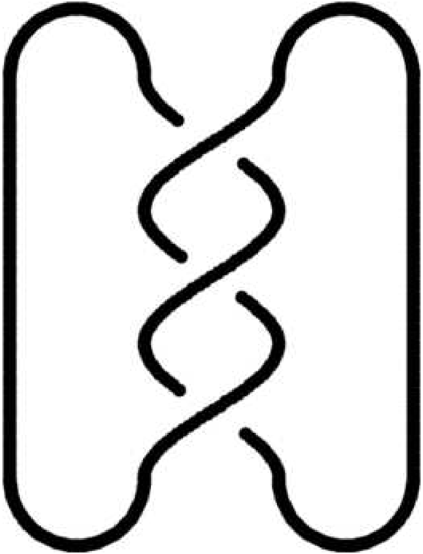


ω/M2

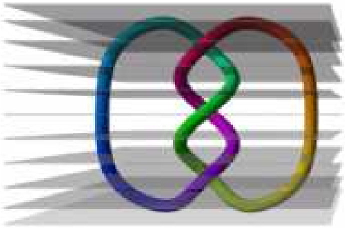
Formally, "a differentiable embedding of S^1 in $\mathbb{R}^3$ modulo differentiable deformations of such".

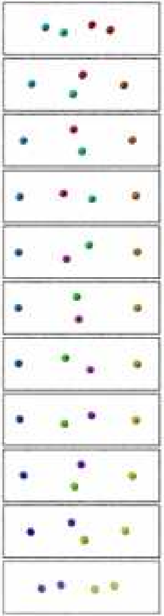




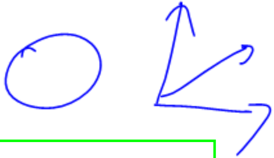


“broken curve diagram”





ω/M2

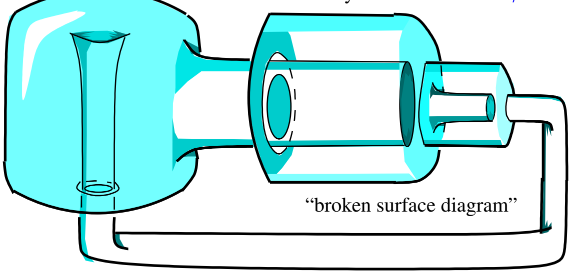


Formally, “a differentiable embedding of S^1 in $\mathbb{R}^3$ modulo differentiable deformations of such”.

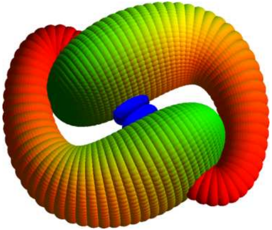
2-Knots / 4D Knots.

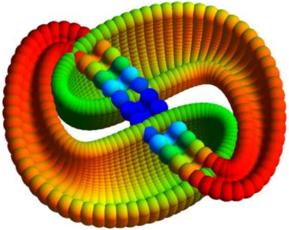
Formally, "a differentiable embedding of S^2 in $\mathbb{R}^4$ modulo differentiable deformations of such".

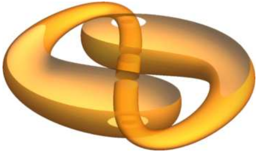
A 4D knot by Carter and Saito ω/CS

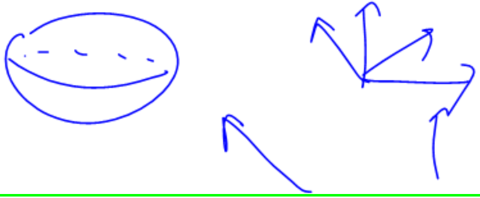


Carter, Banach, Saito





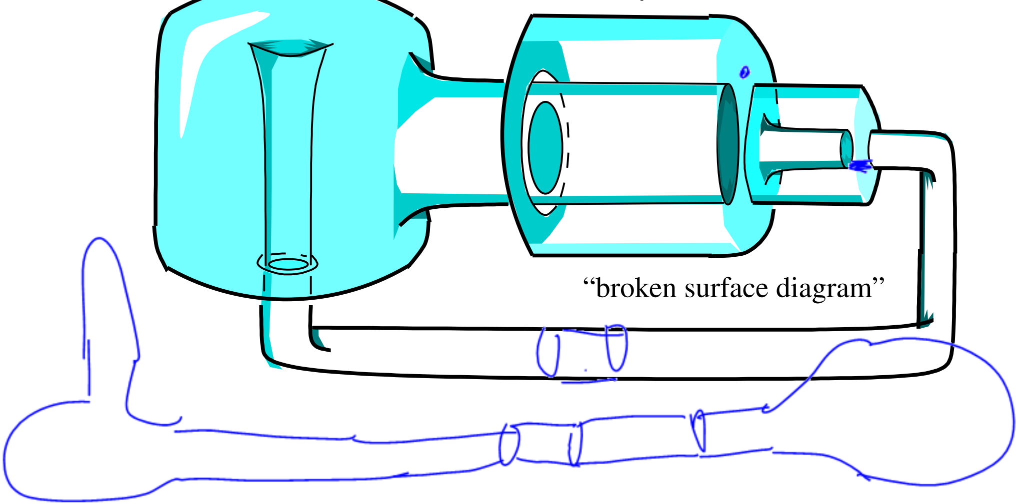




Formally, "a differentiable embedding of S^2 in $\mathbb{R}^4$ modulo differentiable deformations of such".

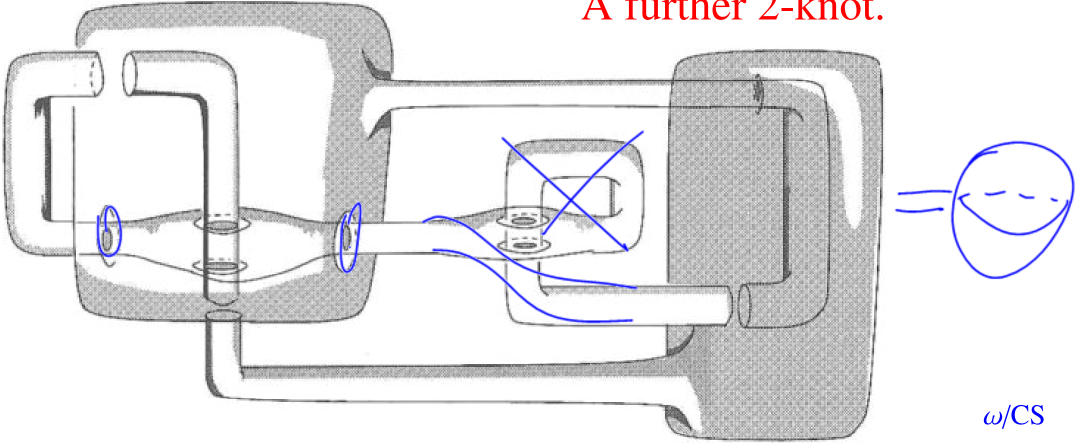
$S^2 \longrightarrow \mathbb{R}^4$

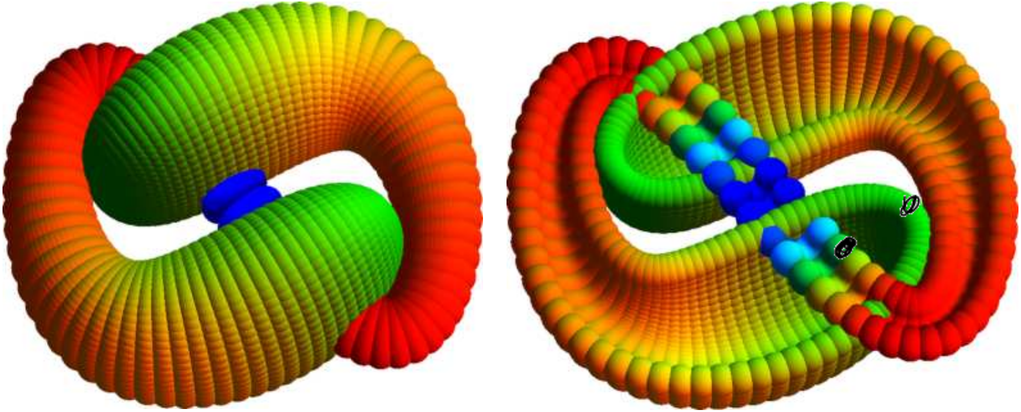
A 4D knot by Carter and Saito ω/CS

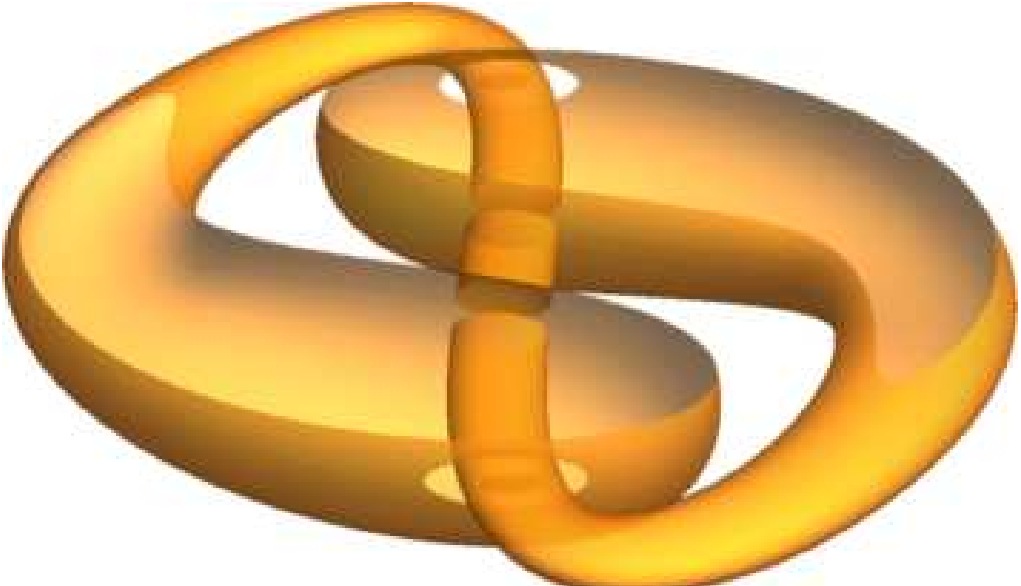


$S^2 \to \mathbb{R}^4$

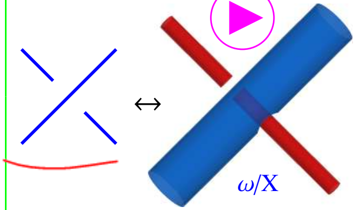
A further 2-knot.



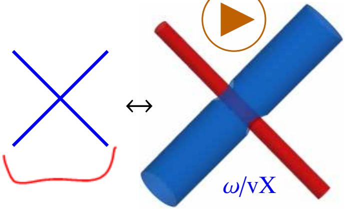




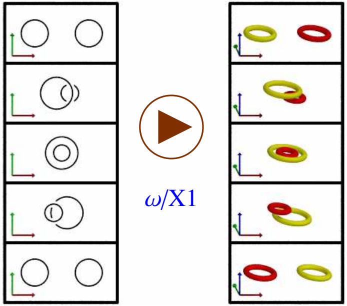
Some Movies



ω/X



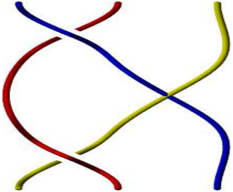
ω/vX

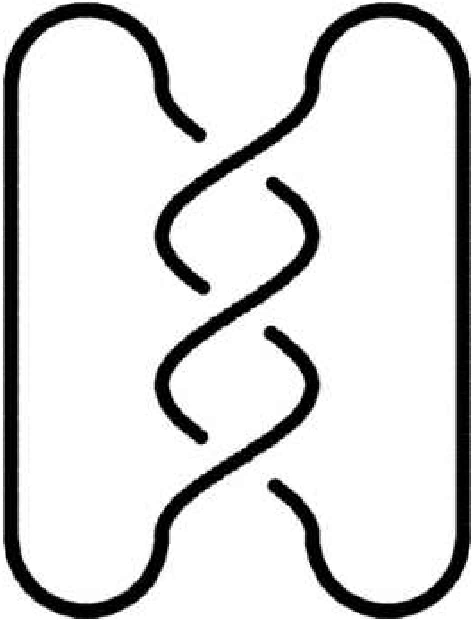


ω/X1

ω/Bub

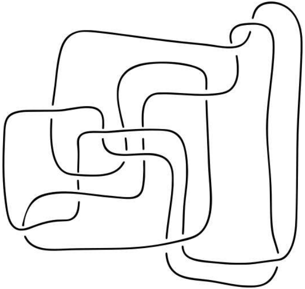
ω/F





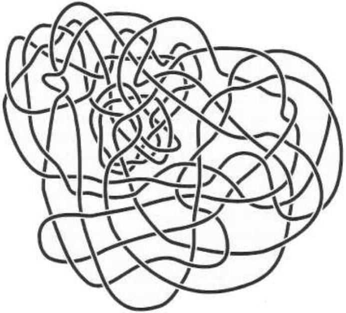
Some Unknots

ω/U

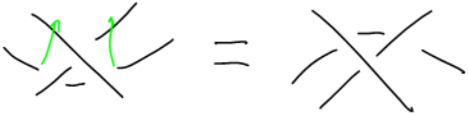


Thistlethwaite's unknot

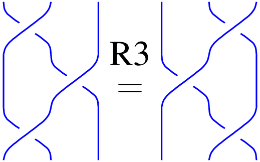
Scharein's relaxation

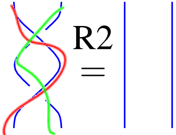


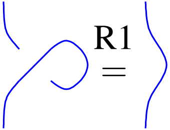
Haken's unknot



Reidemeister's Theorem. (a) Every knot has a "broken curve diagram", made only of curves and "crossings" like ⤫. (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of "Reidemester moves":

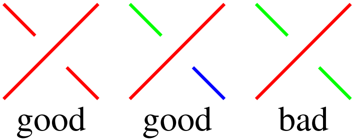




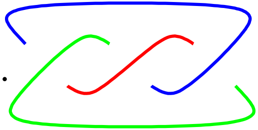


Kurt Reidemeister

Topology is locally analysis and globally algebra



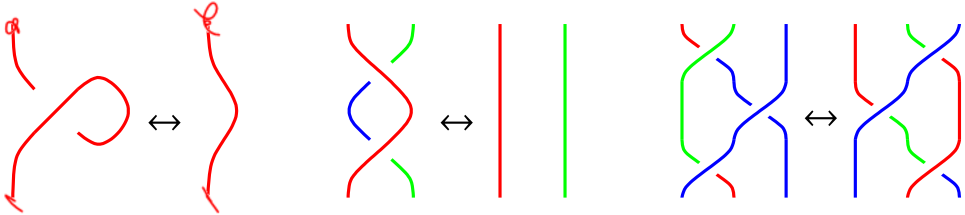
3-Colourings. Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.



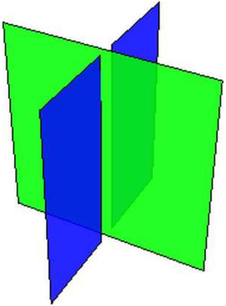
Example. $\lambda(\bigcirc) = 3$ while $\lambda($$) = 9$; so $\bigcirc \neq$ .

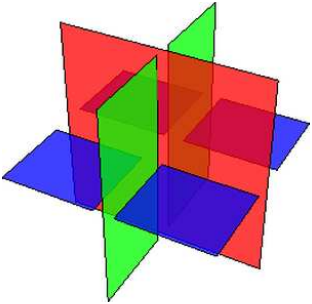
Riddle. Is $\lambda(K)$ always a power of 3?

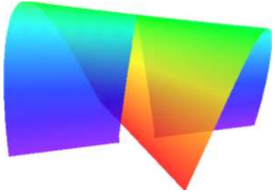
Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:



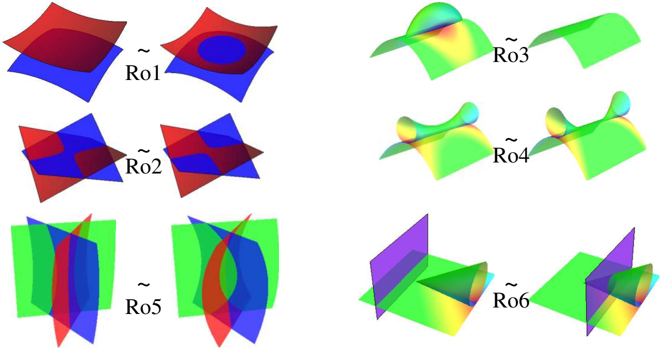
Theorem. Every 2-knot can be represented by a "broken surface diagram" made of the following basic ingredients,

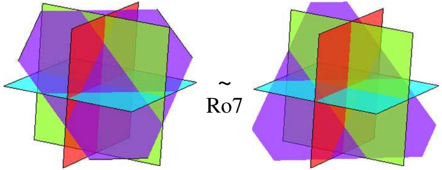






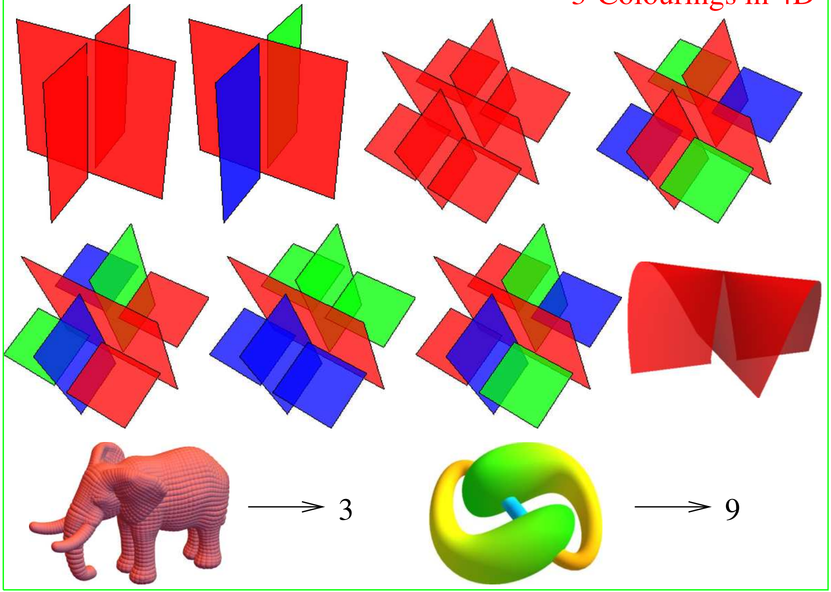
…and any two representations of the same knot differ by a sequence of the following “Roseman moves”:

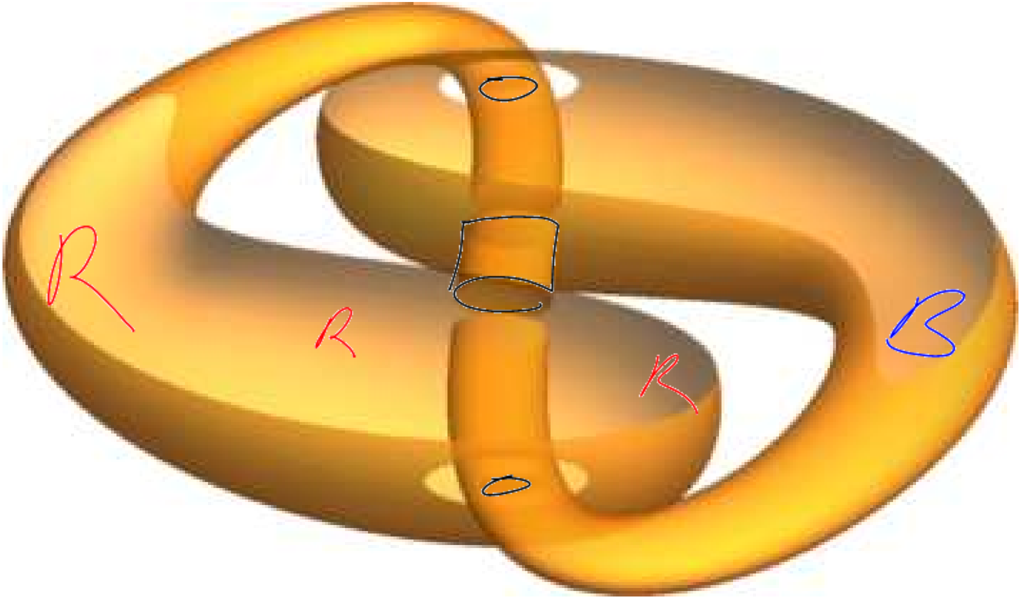





D. Roseman

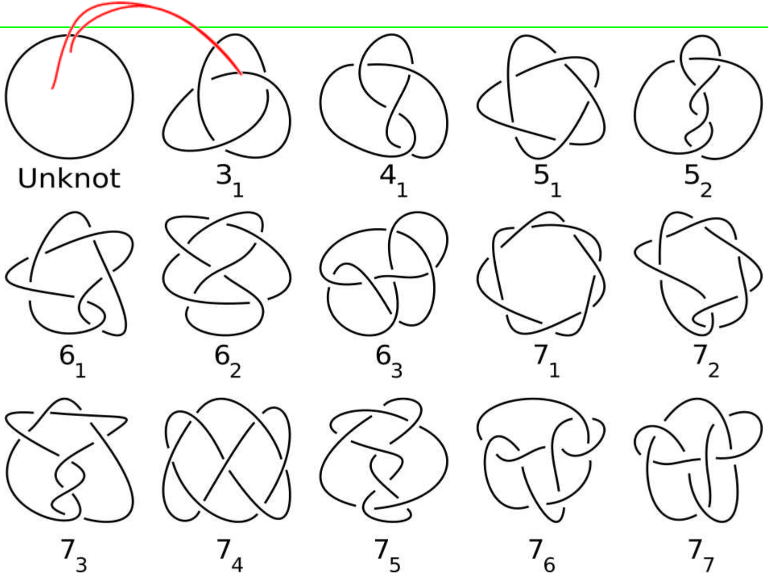
3-Colourings in 4D




R
R
R
B

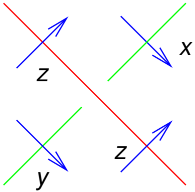
A Knot Table

There are many more!

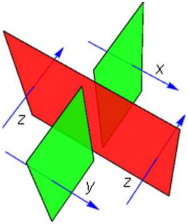


ω/KT

A Stronger Invariant. There is an assigment of groups to knots / 2-knots as follows. Put an arrow "under" every un-broken curve / surface in a broken curve / surface diagram and label it with the name of a group generator. Then mod out by relations as below.



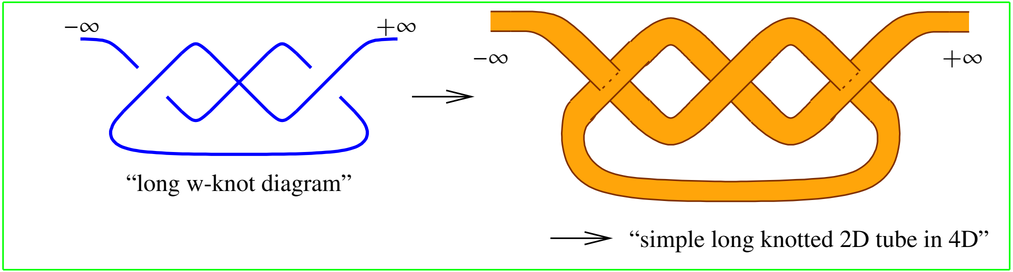
or



$$\longrightarrow \qquad zx = yz$$

Facts. The resulting "Fundamental group" $\pi_1(K)$ of a knot / 2-knot K is a very strong but not very computable invariant of K. Though it has computable projections; e.g., for any finite G, count the homomorphisms from $\pi_1(K)$ to G.

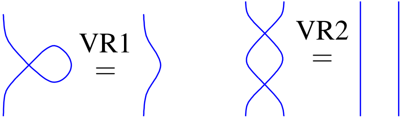
Exercise. Show that $|\mathrm{Hom}(\pi_1(K) \to S_3)| = \lambda(K) + 3$.

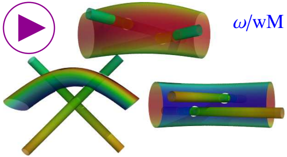


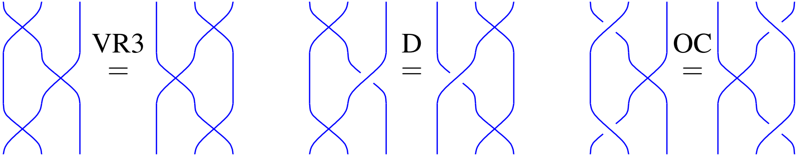
−∞
+∞
“long w-knot diagram”
−∞
+∞
“simple long knotted 2D tube in 4D”

Satoh's Conjecture. (Satoh, *Virtual Knot Presentations of Ribbon Torus-Knots,* J. Knot Theory and its Ramifications **9** (2000) 531–542). Two long w-knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D iff they differ by a sequence of R-moves as above and the "w-moves" VR1–VR3, D and OC listed below:

Shin Satoh







Some knot theory books.

- Colin C. Adams, *The Knot Book, an Elementary Introduction to the Mathematical Theory of Knots,* American Mathematical Society, 2004.
- Meike Akveld and Andrew Jobbings, *Knots Unravelled, from Strings to Mathematics,* Arbelos 2011.
- J. Scott Carter and Masahico Saito, *Knotted Surfaces and Their Diagrams,* American Mathematical Society, 1997.
- Peter Cromwell, *Knots and Links,* Cambridge University Press, 2004.
- W.B. Raymond Lickorish, *An Introduction to Knot Theory,* Springer 1997.