

Pensieve header: Computations in the ybax algebra using the Drinfel'd double (at $\epsilon^2=0$).
Continues ExpDoubleEpsilonSquare5@@.nb.

The double at $\epsilon^2 = 0$

Utilities

Canonical Form:

ybax

```
In[ ]:= CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := ExpandDenominator@
  ExpandNumerator@Together[Expand[ε] /.  $e^x e^y \rightarrow e^{x+y}$  /.  $e^x \rightarrow e^{CF[x]}$ ];
```

The Kronecker δ :

ybax

```
In[ ]:= Kδ /: Kδ[i_, j_] := If[i === j, 1, 0];
```

Equality and multiplication of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

ybax

```
In[ ]:=  $\mathbb{E}$  /:  $\mathbb{E}[L1_, Q1_, P1_] \equiv \mathbb{E}[L2_, Q2_, P2_] :=$ 
  CF[L1 == L2] ^ CF[Q1 == Q2] ^ CF[Normal[P1 - P2] == 0];
 $\mathbb{E}$  /:  $\mathbb{E}[L1_, Q1_, P1_] \mathbb{E}[L2_, Q2_, P2_] := \mathbb{E}[L1 + L2, Q1 + Q2, P1 * P2];$ 
```

Zip and Bind

ybax

```
In[ ]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)i;
```

ybax

```
In[ ]:= expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ε_] := Expand[ε];
Zip[P_] := P;
Zip[ε_, εs_][P_] := (expand[P // Zip[εs]] /.  $f_ \cdot \zeta^{d \cdot} \rightarrow \partial_{\{\zeta^*, d\}} f$ ) /.  $\zeta^* \rightarrow 0$ 
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

ybax

```

In[ ]:= QZip $_{\zeta\mathcal{S}\text{-List}, \text{simp}}$ @ $\mathbb{E}[L\_ , Q\_ , P\_ ] := \text{Module}[\{\zeta, z, z\mathcal{S}, c, y\mathcal{S}, \eta\mathcal{S}, qt, zrule, Q1, Q2\},
  z\mathcal{S} = \text{Table}[\zeta^*, \{\zeta, \zeta\mathcal{S}\}];
  c = Q /. \text{Alternatives}@@(\zeta\mathcal{S} \cup z\mathcal{S}) \rightarrow \emptyset;
  y\mathcal{S} = \text{Table}[\partial_\zeta(Q /. \text{Alternatives}@@z\mathcal{S} \rightarrow \emptyset), \{\zeta, \zeta\mathcal{S}\}];
  \eta\mathcal{S} = \text{Table}[\partial_z(Q /. \text{Alternatives}@@\zeta\mathcal{S} \rightarrow \emptyset), \{z, z\mathcal{S}\}];
  qt = \text{Inverse}@\text{Table}[K\delta_{z, \zeta^*} - \partial_{z, \zeta}Q, \{\zeta, \zeta\mathcal{S}\}, \{z, z\mathcal{S}\}];
  zrule = \text{Thread}[z\mathcal{S} \rightarrow qt.(z\mathcal{S} + y\mathcal{S})];
  Q2 = (Q1 = c + \eta\mathcal{S}.z\mathcal{S} /. zrule) /. \text{Alternatives}@@z\mathcal{S} \rightarrow \emptyset;
  simp /@  $\mathbb{E}[L, Q2, \text{Det}[qt] e^{-Q2} \text{Zip}_{\zeta\mathcal{S}}[e^{Q1}(P /. zrule)]]];
  QZip $_{\zeta\mathcal{S}\text{-List}} := \text{QZip}_{\zeta\mathcal{S}, \text{CF}}$ ;$$ 
```

Upper to lower and lower to Upper:

ybax

```

In[ ]:= U21 = {B $_{i-}$ p \rightarrow e^{-p} b_i, B $_{i-}$ p \rightarrow e^{-p} b, T $_{i-}$ p \rightarrow e^p t_i, T $_{i-}$ p \rightarrow e^p t, \mathcal{A}_{i-}^p \rightarrow e^p \alpha_i, \mathcal{A}_{i-}^p \rightarrow e^p \alpha};
  L2U = {e $^{c-}$  b $_{i+d-}$  \rightarrow B $_{i-}^c e^d, e $^{c-}$  b+d $_{i-}$  \rightarrow B $^{-c} e^d,$ 
  e $^{c-}$  t $_{i+d-}$  \rightarrow T $_{i-}^c e^d, e $^{c-}$  t+d $_{i-}$  \rightarrow T $^c e^d,$ 
  e $^{c-}$  \alpha $_{i+d-}$  \rightarrow \mathcal{A}_{i-}^c e^d, e $^{c-}$  \alpha+d $_{i-}$  \rightarrow \mathcal{A}^c e^d,
  e $^{\zeta-}$  \rightarrow e^{\text{Expand}@\zeta}}$ };$ 
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single “P”. Here the z’s are b and α and the ζ ’s are β and a .

ybax

```

In[ ]:= LZip $_{\zeta\mathcal{S}\text{-List}, \text{simp}}$ @ $\mathbb{E}[L\_ , Q\_ , P\_ ] := \text{Module}[\{\zeta, z, z\mathcal{S}, c, y\mathcal{S}, \eta\mathcal{S}, lt, zrule, L1, L2, Q1, Q2\},
  z\mathcal{S} = \text{Table}[\zeta^*, \{\zeta, \zeta\mathcal{S}\}];
  c = L /. \text{Alternatives}@@(\zeta\mathcal{S} \cup z\mathcal{S}) \rightarrow \emptyset;
  y\mathcal{S} = \text{Table}[\partial_\zeta(L /. \text{Alternatives}@@z\mathcal{S} \rightarrow \emptyset), \{\zeta, \zeta\mathcal{S}\}];
  \eta\mathcal{S} = \text{Table}[\partial_z(L /. \text{Alternatives}@@\zeta\mathcal{S} \rightarrow \emptyset), \{z, z\mathcal{S}\}];
  lt = \text{Inverse}@\text{Table}[K\delta_{z, \zeta^*} - \partial_{z, \zeta}L, \{\zeta, \zeta\mathcal{S}\}, \{z, z\mathcal{S}\}];
  zrule = \text{Thread}[z\mathcal{S} \rightarrow lt.(z\mathcal{S} + y\mathcal{S})];
  L2 = (L1 = c + \eta\mathcal{S}.z\mathcal{S} /. zrule) /. \text{Alternatives}@@z\mathcal{S} \rightarrow \emptyset;
  Q2 = (Q1 = Q /. U21 /. zrule) /. \text{Alternatives}@@z\mathcal{S} \rightarrow \emptyset;
  simp /@  $\mathbb{E}[L2, Q2, \text{Det}[lt] e^{-L2-Q2} \text{Zip}_{\zeta\mathcal{S}}[e^{L1+Q1}(P /. U21 /. zrule)]] // L2U$ ];
  LZip $_{\zeta\mathcal{S}\text{-List}} := \text{LZip}_{\zeta\mathcal{S}, \text{CF}}$ ;$ 
```

ybax

```

In[ ]:= Bind $_{\{i\}}$ [L_, R_] := L R;
  Bind $_{\{i\mathcal{S}\_}}$ [L $_{\mathcal{E}}$ , R $_{\mathcal{E}}$ ] := \text{Module}[\{n\},
    Times[
      L /. \text{Table}[(v : b | B | t | T | a | x | y) $_i \rightarrow v_{n\mathcal{E}i}, \{i, \{i\mathcal{S}\}\}],$ 
      R /. \text{Table}[(v : \beta | \tau | \alpha | \mathcal{A} | \xi | \eta) $_i \rightarrow v_{n\mathcal{E}i}, \{i, \{i\mathcal{S}\}\}],$ 
    ] // LZipFlatten@Table[\{\beta $_{n\mathcal{E}i}, \tau_{n\mathcal{E}i}, \alpha_{n\mathcal{E}i}\}, \{i, \{i\mathcal{S}\}\}] // QZipFlatten@Table[\{\xi $_{n\mathcal{E}i}, \eta_{n\mathcal{E}i}\}, \{i, \{i\mathcal{S}\}\}];
  B $_{L\text{-List}}$ [L_, R_] := Bind $_L[L, R]$ ; B $_{i\mathcal{S}\_}$ [L_, R_] := Bind $_{\{i\mathcal{S}\}}[L, R]$ ;$$ 
```

The two halves

ybax

$$\begin{aligned}
 \text{In}[*]:= & \mathbf{am}_{i_{-},j_{-}\rightarrow k_{-}} := \mathbb{E} \left[(\alpha_i + \alpha_j) \mathbf{a}_k, (e^{-\alpha_j} \xi_i + \xi_j) \mathbf{x}_k, 1 + \mathbf{0}[\epsilon]^2 \right] \\
 & \mathbf{a}\Delta_{i_{-}\rightarrow j_{-},k_{-}} := \mathbb{E} \left[\alpha_i (\mathbf{a}_j + \mathbf{a}_k), \xi_i (\mathbf{x}_j + \mathbf{x}_k), 1 + \epsilon \xi_i \mathbf{x}_k (-\mathbf{a}_j + \xi_i \mathbf{x}_j / 2) + \mathbf{0}[\epsilon]^2 \right] \\
 & \mathbf{aS}_{i_{-}} := \mathbb{E} \left[-\alpha_i \mathbf{a}_i, -e^{\alpha_i} \xi_i \mathbf{x}_i, 1 - \epsilon e^{\alpha_i} \xi_i \mathbf{x}_i (\mathbf{a}_i + e^{\alpha_i} \xi_i \mathbf{x}_i / 2) + \mathbf{0}[\epsilon]^2 \right] \\
 & \mathbf{aSi}_{i_{-}} := \mathbb{E} \left[-\alpha_i \mathbf{a}_i, -e^{\alpha_i} \xi_i \mathbf{x}_i, 1 - \epsilon e^{\alpha_i} \xi_i \mathbf{x}_i (\mathbf{a}_i - 1 + e^{\alpha_i} \xi_i \mathbf{x}_i / 2) + \mathbf{0}[\epsilon]^2 \right]
 \end{aligned}$$

ybax

$$\begin{aligned}
 \text{In}[*]:= & \mathbf{bm}_{i_{-},j_{-}\rightarrow k_{-}} := \mathbb{E} \left[(\beta_i + \beta_j) \mathbf{b}_k, (\eta_i + \eta_j) \mathbf{y}_k, 1 - \epsilon \eta_j \mathbf{y}_k \beta_i + \mathbf{0}[\epsilon]^2 \right] \\
 & \mathbf{b}\Delta_{i_{-}\rightarrow j_{-},k_{-}} := \mathbb{E} \left[\beta_i (\mathbf{b}_j + \mathbf{b}_k), \eta_i (e^{-\beta_k} \mathbf{y}_j + \mathbf{y}_k), 1 + \epsilon \eta_i^2 \mathbf{y}_j \mathbf{y}_k e^{-\beta_k} / 2 + \mathbf{0}[\epsilon]^2 \right] \\
 & \mathbf{bS}_{i_{-}} := \mathbb{E} \left[-\beta_i \mathbf{b}_i, -e^{\beta_i} \eta_i \mathbf{y}_i, 1 - \epsilon e^{\beta_i} \eta_i \mathbf{y}_i (\beta_i + e^{\beta_i} \eta_i \mathbf{y}_i / 2) + \mathbf{0}[\epsilon]^2 \right] \\
 & \mathbf{bSi}_{i_{-}} := \mathbb{E} \left[-\beta_i \mathbf{b}_i, -e^{\beta_i} \eta_i \mathbf{y}_i, 1 - \epsilon e^{\beta_i} \eta_i \mathbf{y}_i (\beta_i - 1 + e^{\beta_i} \eta_i \mathbf{y}_i / 2) + \mathbf{0}[\epsilon]^2 \right]
 \end{aligned}$$

First check that on the generators this agrees with our conventions in the handout:

$$\begin{aligned}
 \text{In}[*]:= & \left\{ \left\{ \begin{aligned} & "[\mathbf{a}, \mathbf{x}] \rightarrow \left((\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{a}_2 \mathbf{x}_1] \sim \mathbf{B}_{1,2} \sim \mathbf{am}_{1,2\rightarrow 1}) \llbracket 3 \rrbracket - (\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{a}_1 \mathbf{x}_2] \sim \mathbf{B}_{1,2} \sim \mathbf{am}_{1,2\rightarrow 1}) \llbracket 3 \rrbracket \right), \\ & "[\mathbf{b}, \mathbf{y}] \rightarrow \left((\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{y}_2 \mathbf{b}_1] \sim \mathbf{B}_{1,2} \sim \mathbf{bm}_{1,2\rightarrow 1}) \llbracket 3 \rrbracket - (\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{y}_1 \mathbf{b}_2] \sim \mathbf{B}_{1,2} \sim \mathbf{bm}_{1,2\rightarrow 1}) \llbracket 3 \rrbracket \right) \} / \cdot \mathbf{z}_{-1} \rightarrow \mathbf{z}, \\ & "\Delta[\mathbf{y}] \rightarrow \text{Last}[\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{y}_1] \sim \mathbf{B}_1 \sim \mathbf{b}\Delta_{1\rightarrow 1,2}], \\ & "\Delta[\mathbf{b}] \rightarrow \text{Last}[\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{b}_1] \sim \mathbf{B}_1 \sim \mathbf{b}\Delta_{1\rightarrow 1,2}], \\ & "\Delta[\mathbf{a}] \rightarrow \text{Last}[\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{a}_1] \sim \mathbf{B}_1 \sim \mathbf{a}\Delta_{1\rightarrow 1,2}], \\ & "\Delta[\mathbf{x}] \rightarrow \text{Last}[\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{x}_1] \sim \mathbf{B}_1 \sim \mathbf{a}\Delta_{1\rightarrow 1,2}], \\ & \left\{ \begin{aligned} & "S(\mathbf{a}) \rightarrow \left((\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{a}_1] \sim \mathbf{B}_1 \sim \mathbf{aS}_1) \llbracket 3 \rrbracket \right), \\ & "S(\mathbf{x}) \rightarrow \left((\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{x}_1] \sim \mathbf{B}_1 \sim \mathbf{aS}_1) \llbracket 3 \rrbracket \right), \\ & "S(\mathbf{b}) \rightarrow \left((\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{b}_1] \sim \mathbf{B}_1 \sim \mathbf{bS}_1) \llbracket 3 \rrbracket \right), \\ & "S(\mathbf{y}) \rightarrow \left((\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{y}_1] \sim \mathbf{B}_1 \sim \mathbf{bS}_1) \llbracket 3 \rrbracket \right) \end{aligned} \right\} \} / \cdot \mathbf{z}_{-1} \rightarrow \mathbf{z} \end{aligned} \right. \\
 \text{Out}[*]:= & \left\{ \left\{ \begin{aligned} & [\mathbf{a}, \mathbf{x}] \rightarrow -\mathbf{x} + \mathbf{0}[\epsilon]^2, [\mathbf{b}, \mathbf{y}] \rightarrow -\mathbf{y} \epsilon + \mathbf{0}[\epsilon]^2, \{\Delta[\mathbf{y}] \rightarrow (\mathbf{B}_2 \mathbf{y}_1 + \mathbf{y}_2) + \mathbf{0}[\epsilon]^2, \\ & \Delta[\mathbf{b}] \rightarrow (\mathbf{b}_1 + \mathbf{b}_2) + \mathbf{0}[\epsilon]^2, \Delta[\mathbf{a}] \rightarrow (\mathbf{a}_1 + \mathbf{a}_2) + \mathbf{0}[\epsilon]^2, \Delta[\mathbf{x}] \rightarrow (\mathbf{x}_1 + \mathbf{x}_2) - \mathbf{a}_1 \mathbf{x}_2 \epsilon + \mathbf{0}[\epsilon]^2 \}, \\ & \{S(\mathbf{a}) \rightarrow -\mathbf{a} + \mathbf{0}[\epsilon]^2, S(\mathbf{x}) \rightarrow -\mathbf{x} - \mathbf{a} \mathbf{x} \epsilon + \mathbf{0}[\epsilon]^2, S(\mathbf{b}) \rightarrow -\mathbf{b} + \mathbf{0}[\epsilon]^2, S(\mathbf{y}) \rightarrow -\frac{\mathbf{y}}{\mathbf{B}} + \mathbf{0}[\epsilon]^2 \} \end{aligned} \right\} \right.
 \end{aligned}$$

Hopf algebra axioms on both sides separately

(co)-associativity on both sides

$$\begin{aligned}
 \text{In}[*]:= & \left\{ \begin{aligned} & (\mathbf{a}\Delta_{1\rightarrow 1,2} \sim \mathbf{B}_2 \sim \mathbf{a}\Delta_{2\rightarrow 2,3}) \equiv (\mathbf{a}\Delta_{1\rightarrow 1,3} \sim \mathbf{B}_1 \sim \mathbf{a}\Delta_{1\rightarrow 1,2}), (\mathbf{b}\Delta_{1\rightarrow 1,2} \sim \mathbf{B}_2 \sim \mathbf{b}\Delta_{2\rightarrow 2,3}) \equiv (\mathbf{b}\Delta_{1\rightarrow 1,3} \sim \mathbf{B}_1 \sim \mathbf{b}\Delta_{1\rightarrow 1,2}), \\ & (\mathbf{am}_{1,2\rightarrow 1} \sim \mathbf{B}_1 \sim \mathbf{am}_{1,3\rightarrow 1}) \equiv (\mathbf{am}_{2,3\rightarrow 2} \sim \mathbf{B}_2 \sim \mathbf{am}_{1,2\rightarrow 1}), (\mathbf{bm}_{1,2\rightarrow 1} \sim \mathbf{B}_1 \sim \mathbf{bm}_{1,3\rightarrow 1}) \equiv (\mathbf{bm}_{2,3\rightarrow 2} \sim \mathbf{B}_2 \sim \mathbf{bm}_{1,2\rightarrow 1}) \end{aligned} \right\} \\
 \text{Out}[*]:= & \{\text{True}, \text{True}, \text{True}, \text{True}\}
 \end{aligned}$$

Δ is an algebra morphism

$$\begin{aligned}
 \text{In}[*]:= & \left\{ \begin{aligned} & \mathbf{am}_{1,2\rightarrow 1} \sim \mathbf{B}_1 \sim \mathbf{a}\Delta_{1\rightarrow 1,2} \equiv (\mathbf{a}\Delta_{1\rightarrow 1,3} \mathbf{a}\Delta_{2\rightarrow 2,4}) \sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{am}_{3,4\rightarrow 2} \mathbf{am}_{1,2\rightarrow 1}), \\ & \mathbf{bm}_{1,2\rightarrow 1} \sim \mathbf{B}_1 \sim \mathbf{b}\Delta_{1\rightarrow 1,2} \equiv (\mathbf{b}\Delta_{1\rightarrow 1,3} \mathbf{b}\Delta_{2\rightarrow 2,4}) \sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{bm}_{3,4\rightarrow 2} \mathbf{bm}_{1,2\rightarrow 1}) \end{aligned} \right\} \\
 \text{Out}[*]:= & \{\text{True}, \text{True}\}
 \end{aligned}$$

S is convolution inverse of id

$$\text{In[*]} := \left\{ \left(\mathbf{a}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_1 \sim \mathbf{a}\mathbf{S}_1 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{a}\mathbf{m}_{1, 2 \rightarrow 1}, \left(\mathbf{a}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_2 \sim \mathbf{a}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{a}\mathbf{m}_{1, 2 \rightarrow 1} \right\}$$

$$\left\{ \left(\mathbf{b}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_1 \sim \mathbf{b}\mathbf{S}_1 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{b}\mathbf{m}_{1, 2 \rightarrow 1}, \left(\mathbf{b}\Delta_{1 \rightarrow 1, 2} \sim \mathbf{B}_2 \sim \mathbf{b}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{b}\mathbf{m}_{1, 2 \rightarrow 1} \right\}$$

$$\text{Out[*]} := \left\{ \mathbb{E} \left[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathbf{O}[\epsilon]^2 \right], \mathbb{E} \left[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathbf{O}[\epsilon]^2 \right] \right\}$$

$$\text{Out[*]} := \left\{ \mathbb{E} \left[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathbf{O}[\epsilon]^2 \right], \mathbb{E} \left[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathbf{O}[\epsilon]^2 \right] \right\}$$

Si is the inverse of S

$$\text{In[*]} := \left\{ \mathbf{a}\mathbf{S}\mathbf{i}_1 \sim \mathbf{B}_1 \sim \mathbf{a}\mathbf{S}_1 \equiv \mathbb{E} \left[\mathbf{a}_1 \alpha_1, \mathbf{x}_1 \xi_1, \mathbf{1} \right], \mathbf{a}\mathbf{S}_1 \sim \mathbf{B}_1 \sim \mathbf{a}\mathbf{S}\mathbf{i}_1 \equiv \mathbb{E} \left[\mathbf{a}_1 \alpha_1, \mathbf{x}_1 \xi_1, \mathbf{1} \right] \right\}$$

$$\left\{ \mathbf{b}\mathbf{S}\mathbf{i}_1 \sim \mathbf{B}_1 \sim \mathbf{b}\mathbf{S}_1 \equiv \mathbb{E} \left[\mathbf{b}_1 \beta_1, \mathbf{y}_1 \eta_1, \mathbf{1} \right], \mathbf{b}\mathbf{S}_1 \sim \mathbf{B}_1 \sim \mathbf{b}\mathbf{S}\mathbf{i}_1 \equiv \mathbb{E} \left[\mathbf{b}_1 \beta_1, \mathbf{y}_1 \eta_1, \mathbf{1} \right] \right\}$$

$$\text{Out[*]} := \{ \text{True}, \text{True} \}$$

$$\text{Out[*]} := \{ \text{True}, \text{True} \}$$

S is an algebra anti-(co)morphism

$$\text{In[*]} := \left\{ \mathbf{a}\mathbf{m}_{1, 2 \rightarrow 1} \sim \mathbf{B}_1 \sim \mathbf{a}\mathbf{S}_1 \equiv \left(\mathbf{a}\mathbf{S}_1 \mathbf{a}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{a}\mathbf{m}_{2, 1 \rightarrow 1}, \mathbf{b}\mathbf{m}_{1, 2 \rightarrow 1} \sim \mathbf{B}_1 \sim \mathbf{b}\mathbf{S}_1 \equiv \left(\mathbf{b}\mathbf{S}_1 \mathbf{b}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{b}\mathbf{m}_{2, 1 \rightarrow 1} \right\}$$

$$\left\{ \mathbf{a}\mathbf{S}_1 \sim \mathbf{B}_1 \sim \mathbf{a}\Delta_{1 \rightarrow 1, 2} \equiv \mathbf{a}\Delta_{1 \rightarrow 2, 1} \sim \mathbf{B}_{1, 2} \sim \left(\mathbf{a}\mathbf{S}_1 \mathbf{a}\mathbf{S}_2 \right), \mathbf{b}\mathbf{S}_1 \sim \mathbf{B}_1 \sim \mathbf{b}\Delta_{1 \rightarrow 1, 2} \equiv \mathbf{b}\Delta_{1 \rightarrow 2, 1} \sim \mathbf{B}_{1, 2} \sim \left(\mathbf{b}\mathbf{S}_1 \mathbf{b}\mathbf{S}_2 \right) \right\}$$

$$\text{Out[*]} := \{ \text{True}, \text{True} \}$$

$$\text{Out[*]} := \{ \text{True}, \text{True} \}$$

Pairing

ybax

$$\text{In[*]} := \mathbf{tP}_{i, j} := \mathbb{E} \left[\beta_i \alpha_j, \eta_i \xi_j, \mathbf{1} + \epsilon \eta_i^2 \xi_j^2 / 4 \right]$$

$$\text{In[*]} := \mathbf{qfac}[\mathbf{k}_-, \mathbf{q}_-] := (1 - \mathbf{q})^{-\mathbf{k}} \mathbf{QPochhammer}[\mathbf{q}, \mathbf{q}, \mathbf{k}] // \text{FunctionExpand}$$

$$\mathbf{qfe}[\mathbf{k}_-] := \text{Normal}[\text{Series}[\mathbf{qfac}[\mathbf{k}, \mathbf{E}^\rho], \{\rho, \mathbf{0}, \mathbf{1}\}], \{\rho \rightarrow \epsilon\}]$$

$$\text{Table}[\mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{y}_1^r \mathbf{b}_1^s \mathbf{a}_2^t \mathbf{x}_2^u] \sim \mathbf{B}_{1, 2} \sim \mathbf{tP}_{1, 2} \equiv \mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{K}\delta_{r, u} \mathbf{K}\delta_{s, t} \mathbf{qfe}[\mathbf{r} \mathbf{s}!],$$

$$\{\mathbf{r}, \mathbf{0}, \mathbf{4}\}, \{\mathbf{s}, \mathbf{0}, \mathbf{4}\}, \{\mathbf{t}, \mathbf{0}, \mathbf{4}\}, \{\mathbf{u}, \mathbf{0}, \mathbf{4}\}] // \text{Flatten} // \text{Union}$$

$$\text{Out[*]} := \{ \text{True} \}$$

Pairing axioms

$$\text{In[*]} := \left\{ \left(\mathbf{b}\mathbf{m}_{1, 2 \rightarrow 1} \mathbb{E}[\alpha_3 \mathbf{a}_3, \xi_3 \mathbf{x}_3, \mathbf{1}] \right) \sim \mathbf{B}_{1, 3} \sim \mathbf{tP}_{1, 3} \equiv \right.$$

$$\left(\mathbb{E}[\beta_1 \mathbf{b}_1, \eta_1 \mathbf{y}_1, \mathbf{1}] \mathbb{E}[\beta_2 \mathbf{b}_2, \eta_2 \mathbf{y}_2, \mathbf{1}] \mathbf{a}\Delta_{3 \rightarrow 4, 5} \right) \sim \mathbf{B}_{1, 4} \sim \mathbf{tP}_{1, 4} \sim \mathbf{B}_{2, 5} \sim \mathbf{tP}_{2, 5}$$

$$\left. , \left(\mathbf{b}\Delta_{1 \rightarrow 1, 2} \mathbb{E}[\alpha_3 \mathbf{a}_3, \xi_3 \mathbf{x}_3, \mathbf{1}] \mathbb{E}[\alpha_4 \mathbf{a}_4, \xi_4 \mathbf{x}_4, \mathbf{1}] \right) \sim \mathbf{B}_{1, 3} \sim \mathbf{tP}_{1, 3} \sim \mathbf{B}_{2, 4} \sim \mathbf{tP}_{2, 4} \equiv \right.$$

$$\left. \left(\mathbb{E}[\beta_1 \mathbf{b}_1, \eta_1 \mathbf{y}_1, \mathbf{1}] \mathbf{a}\mathbf{m}_{3, 4 \rightarrow 3} \right) \sim \mathbf{B}_{1, 3} \sim \mathbf{tP}_{1, 3} \right\}$$

$$\text{Out[*]} := \{ \text{True}, \text{True} \}$$

$$\text{In[*]} := \left\{ \left(\mathbf{b}\mathbf{S}_1 \mathbb{E}[\alpha_2 \mathbf{a}_2, \xi_2 \mathbf{x}_2, \mathbf{1}] \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{tP}_{1, 2} \equiv \left(\mathbb{E}[\beta_1 \mathbf{b}_1, \eta_1 \mathbf{y}_1, \mathbf{1}] \mathbf{a}\mathbf{S}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{tP}_{1, 2}, \right.$$

$$\left. \left(\mathbf{b}\mathbf{S}\mathbf{i}_1 \mathbb{E}[\alpha_2 \mathbf{a}_2, \xi_2 \mathbf{x}_2, \mathbf{1}] \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{tP}_{1, 2} \equiv \left(\mathbb{E}[\beta_1 \mathbf{b}_1, \eta_1 \mathbf{y}_1, \mathbf{1}] \mathbf{a}\mathbf{S}\mathbf{i}_2 \right) \sim \mathbf{B}_{1, 2} \sim \mathbf{tP}_{1, 2} \right\}$$

$$\text{Out[*]} := \{ \text{True}, \text{True} \}$$

The Double

The double multiplication (should really bind the a's and b's separately)

ybax

$$\text{In[*]:= Block} \left[\{i, j, k\}, \text{dm}_{i,j \rightarrow k} = \left(\mathbb{E} [\beta_i b_i + \alpha_j a_j, \eta_i y_i + \xi_j x_j, \mathbf{1}] (\mathbf{a} \Delta_{i \rightarrow h1, h2} \sim \mathbf{B}_{h2} \sim \mathbf{a} \Delta_{h2 \rightarrow h2, h3}) (\mathbf{b} \Delta_{j \rightarrow t1, t2} \sim \mathbf{B}_{t2} \sim \mathbf{b} \Delta_{t2 \rightarrow t2, t3}) \right) \sim \mathbf{B}_{h3} \sim \mathbf{a} \mathbf{S}_{i, h3} \sim \mathbf{B}_{t1, h3} \sim (\mathbf{t} \mathbf{P}_{t1, h3}) \sim \mathbf{B}_{t3, h1} \sim (\mathbf{t} \mathbf{P}_{t3, h1}) \sim \mathbf{B}_{h2, j, i, t2} \sim (\mathbf{a} \mathbf{m}_{h2, j \rightarrow k} \mathbf{b} \mathbf{m}_{i, t2 \rightarrow k}) \right]$$

ybax

$$\text{Out[*]:= } \mathbb{E} \left[\mathbf{a}_k \alpha_i + \mathbf{a}_k \alpha_j + \mathbf{b}_k \beta_i + \mathbf{b}_k \beta_j, \frac{1}{\mathcal{A}_i \mathcal{A}_j} \right. \\ \left. (\mathbf{y}_k \mathcal{A}_i \mathcal{A}_j \eta_i + \mathbf{y}_k \mathcal{A}_j \eta_j + \mathbf{x}_k \mathcal{A}_i \xi_i + \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i - \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + \mathbf{x}_k \mathcal{A}_i \mathcal{A}_j \xi_j), \mathbf{1} + \right. \\ \left. \frac{1}{4 \mathcal{A}_i \mathcal{A}_j} \left(-4 \mathbf{y}_k \mathcal{A}_j \beta_i \eta_j - 4 \mathbf{x}_k \mathcal{A}_i \beta_j \xi_i + 4 \mathbf{x}_k \mathbf{y}_k \eta_j \xi_i + 4 \mathbf{a}_k \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + 2 \mathbf{y}_k \mathcal{A}_j \eta_j^2 \xi_i - 6 \mathbf{B}_k \mathbf{y}_k \mathcal{A}_j \eta_j^2 \xi_i + \right. \right. \\ \left. \left. 2 \mathbf{x}_k \mathcal{A}_i \eta_j \xi_i^2 - 6 \mathbf{B}_k \mathbf{x}_k \mathcal{A}_i \eta_j \xi_i^2 + \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 - 4 \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 + 3 \mathbf{B}_k^2 \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 \right) \in + \mathbf{O}[\epsilon]^2 \right]$$

ybax

$$\text{In[*]:= Block} \left[\{i\}, \text{dS}_i = \mathbb{E} [\beta_i b_i + \alpha_i a_i, \eta_i y_i + \xi_i x_i, \mathbf{1}] \sim \mathbf{B}_{1,2} \sim (\mathbf{b} \mathbf{S}_{i1} \mathbf{a} \mathbf{S}_{i2}) \sim \mathbf{B}_{1,2} \sim \text{dm}_{2,1 \rightarrow i} \right]$$

ybax

$$\text{Out[*]:= } \mathbb{E} \left[-\mathbf{a}_i \alpha_i - \mathbf{b}_i \beta_i, \frac{-\mathbf{y}_i \mathcal{A}_i \eta_i - \mathbf{B}_i \mathbf{x}_i \mathcal{A}_i \xi_i + \mathcal{A}_i \eta_i \xi_i - \mathbf{B}_i \mathcal{A}_i \eta_i \xi_i}{\mathbf{B}_i}, \right. \\ \left. \mathbf{1} + \frac{1}{4 \mathbf{B}_i^2} \left(4 \mathbf{B}_i \mathbf{y}_i \mathcal{A}_i \eta_i - 4 \mathbf{B}_i \mathbf{y}_i \mathcal{A}_i \beta_i \eta_i - 2 \mathbf{y}_i^2 \mathcal{A}_i^2 \eta_i^2 - 4 \mathbf{a}_i \mathbf{B}_i^2 \mathbf{x}_i \mathcal{A}_i \xi_i - \right. \right. \\ \left. \left. 4 \mathbf{B}_i^2 \mathbf{x}_i \mathcal{A}_i \beta_i \xi_i - 4 \mathbf{B}_i \mathcal{A}_i \eta_i \xi_i + 4 \mathbf{a}_i \mathbf{B}_i \mathcal{A}_i \eta_i \xi_i + 4 \mathbf{B}_i^2 \mathcal{A}_i \eta_i \xi_i - 4 \mathbf{B}_i \mathbf{x}_i \mathbf{y}_i \mathcal{A}_i^2 \eta_i \xi_i + \right. \right. \\ \left. \left. 4 \mathbf{B}_i \mathcal{A}_i \beta_i \eta_i \xi_i - 4 \mathbf{B}_i^2 \mathcal{A}_i \beta_i \eta_i \xi_i + 6 \mathbf{y}_i \mathcal{A}_i^2 \eta_i^2 \xi_i - 2 \mathbf{B}_i \mathbf{y}_i \mathcal{A}_i^2 \eta_i^2 \xi_i - 2 \mathbf{B}_i^2 \mathbf{x}_i^2 \mathcal{A}_i^2 \xi_i^2 + \right. \right. \\ \left. \left. 6 \mathbf{B}_i \mathbf{x}_i \mathcal{A}_i^2 \eta_i \xi_i^2 - 2 \mathbf{B}_i^2 \mathbf{x}_i \mathcal{A}_i^2 \eta_i \xi_i^2 - 3 \mathcal{A}_i^2 \eta_i^2 \xi_i^2 + 4 \mathbf{B}_i \mathcal{A}_i^2 \eta_i^2 \xi_i^2 - \mathbf{B}_i^2 \mathcal{A}_i^2 \eta_i^2 \xi_i^2 \right) \in + \mathbf{O}[\epsilon]^2 \right]$$

ybax

$$\text{In[*]:= Block} \left[\{i, j, k\}, \text{d}\Delta_{i \rightarrow j, k} = (\mathbf{b} \Delta_{i \rightarrow 3, 1} \mathbf{a} \Delta_{i \rightarrow 2, 4}) \sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{d} \mathbf{m}_{3,4 \rightarrow k} \mathbf{d} \mathbf{m}_{1,2 \rightarrow j}) \right]$$

ybax

$$\text{Out[*]:= } \mathbb{E} \left[\mathbf{a}_j \alpha_i + \mathbf{a}_k \alpha_i + \mathbf{b}_j \beta_i + \mathbf{b}_k \beta_i, \mathbf{y}_j \eta_i + \mathbf{B}_j \mathbf{y}_k \eta_i + \mathbf{x}_j \xi_i + \mathbf{x}_k \xi_i, \right. \\ \left. \mathbf{1} + \frac{1}{2} (\mathbf{B}_j \mathbf{y}_j \mathbf{y}_k \eta_i^2 - 2 \mathbf{a}_j \mathbf{x}_k \xi_i + \mathbf{x}_j \mathbf{x}_k \xi_i^2) \in + \mathbf{O}[\epsilon]^2 \right]$$

First check the double formulas on the generators agree with SL2Portfolio.pdf:

```

In[*]:= {
  "[a,y]" -> ((E[0, 0, y2 a1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, y1 a2] ~ B1,2 ~ dm1,2->1) [[3]]),
  "[b,x]" -> ((E[0, 0, x2 b1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, x1 b2] ~ B1,2 ~ dm1,2->1) [[3]]),
  "xy-qyx" -> ((E[0, 0, x1 y2] ~ B1,2 ~ dm1,2->1) [[3]] - (1 + ε) (E[0, 0, y1 x2] ~ B1,2 ~ dm1,2->1) [[3]])
} /. {z_1 -> z} // Expand // Factor

{
  "Δ(a)" -> ((E[0, 0, a1] ~ B1 ~ dΔ1->1,2) [[3]]),
  "Δ(x)" -> ((E[0, 0, x1] ~ B1 ~ dΔ1->1,2) [[3]]),
  "Δ(b)" -> ((E[0, 0, b1] ~ B1 ~ dΔ1->1,2) [[3]]),
  "Δ(y)" -> ((E[0, 0, y1] ~ B1 ~ dΔ1->1,2) [[3]])
} // Simplify

{
  "S(a)" -> ((E[0, 0, a1] ~ B1 ~ dS1) [[3]]),
  "S(x)" -> ((E[0, 0, x1] ~ B1 ~ dS1) [[3]]),
  "S(b)" -> ((E[0, 0, b1] ~ B1 ~ dS1) [[3]]),
  "S(y)" -> ((E[0, 0, y1] ~ B1 ~ dS1) [[3]])
} /. {z_1 -> z} // Simplify

Out[*]:= {[a,y] -> -y + 0[ε]^2, [b,x] -> xε + 0[ε]^2, xy-qyx -> (1 - B) + a Bε + 0[ε]^2}

Out[*]:= {Δ(a) -> (a1 + a2) + 0[ε]^2, Δ(x) -> (x1 + x2) - a1 x2ε + 0[ε]^2,
  Δ(b) -> (b1 + b2) + 0[ε]^2, Δ(y) -> (y1 + B1 y2) + 0[ε]^2}

Out[*]:= {S(a) -> -a + 0[ε]^2, S(x) -> -x - a xε + 0[ε]^2, S(b) -> -b + 0[ε]^2, S(y) -> -y/B + yε/B + 0[ε]^2}

```

Hopf algebra axioms on double

(co)-associativity

```

In[*]:= { (dΔ1->1,2 ~ B2 ~ dΔ2->2,3) ≡ (dΔ1->1,3 ~ B1 ~ dΔ1->1,2),
  (dm1,2->1 ~ B1 ~ dm1,3->1) ≡ (dm2,3->2 ~ B2 ~ dm1,2->1) }

Out[*]:= {True, True}

```

Δ is an algebra morphism

```

In[*]:= dm1,2->1 ~ B1 ~ dΔ1->1,2 ≡ (dΔ1->1,3 dΔ2->2,4) ~ B1,2,3,4 ~ (dm3,4->2 dm1,2->1)

Out[*]:= True

```

S is convolution inverse of id

```

In[*]:= { (dΔ1->1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2->1, (dΔ1->1,2 ~ B2 ~ dS2) ~ B1,2 ~ dm1,2->1 }

Out[*]:= {E[0, 0, 1 + 0[ε]^2], E[0, 0, 1 + 0[ε]^2]}

```

S is a (co)-algebra anti-morphism

```

In[*]:= {dm1,2->1 ~ B1 ~ dS1 ≡ (dS1 dS2) ~ B1,2 ~ dm2,1->1, dS1 ~ B1 ~ dΔ1->1,2 ≡ dΔ1->2,1 ~ B1,2 ~ (dS1 dS2)} // Expand

Out[*]:= {True, True}

```

R-matrix

$$\text{In[*]:= } e_{q-,k-}[x_-] := e^{\sum_{j=1}^{k+1} \frac{(1-q)^j x^j}{j(1-q^j)}}$$

ybax

$$\text{In[*]:= } R_{i-,j-} := \mathbb{E}[b_i a_j, y_i x_j, 1 - \epsilon y_i^2 x_j^2 / 4 + O[\epsilon]^2]$$

$$\text{In[*]:= } \text{Series}[e_{q,1}[z] /. \{z \rightarrow y_i x_j, q \rightarrow 1 + \rho\}, \{\rho, 0, 1\}] /. \{\rho \rightarrow \epsilon\}$$

$$\text{Out[*]:= } e^{x_j y_i} - \frac{1}{4} (e^{x_j y_i} x_j^2 y_i^2) \epsilon + O[\epsilon]^2$$

Quasi-triangular axiom 1:

$$\text{In[*]:= } R_{1,2} \sim B_1 \sim d\Delta_{1 \rightarrow 1,3} \equiv (R_{1,4} R_{3,2}) \sim B_{2,4} \sim dm_{2,4 \rightarrow 2}$$

Out[*]= True

Quasi-triangular axiom 2:

$$\text{In[*]:= } \left((d\Delta_{1 \rightarrow 1,2} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}) \right) \equiv \\ \left((d\Delta_{1 \rightarrow 2,1} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1 \rightarrow 1} dm_{4,2 \rightarrow 2}) \right)$$

Out[*]= True

Reidemeister 3:

$$\text{In[*]:= } \left((R_{1,2} R_{4,3} R_{5,6}) \sim B_{1,4} \sim dm_{1,4 \rightarrow 1} \sim B_{2,5} \sim dm_{2,5 \rightarrow 2} \sim B_{3,6} \sim dm_{3,6 \rightarrow 3} \right) \equiv \\ \left((R_{1,6} R_{2,3} R_{4,5}) \sim B_{1,4} \sim dm_{1,4 \rightarrow 1} \sim B_{2,5} \sim dm_{2,5 \rightarrow 2} \sim B_{3,6} \sim dm_{3,6 \rightarrow 3} \right)$$

Out[*]= True

ybax

$$\text{In[*]:= } \text{Block}[\{i, j\}, \bar{R}_{i-,j-} = \text{Expand} /@ R_{i,j} \sim B_j \sim dS_j]$$

ybax

$$\text{Out[*]:= } \mathbb{E}\left[-a_j b_i, -\frac{x_j y_i}{B_i}, 1 + \frac{(-4 a_j B_i x_j y_i - 3 x_j^2 y_i^2) \epsilon}{4 B_i^2} + O[\epsilon]^2\right]$$

Reidemeister 2

$$\text{In[*]:= } \left\{ (\bar{R}_{1,2} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}), (R_{1,2} \bar{R}_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}) \right\}$$

$$\text{Out[*]:= } \left\{ \mathbb{E}[0, 0, 1 + O[\epsilon]^2], \mathbb{E}[0, 0, 1 + O[\epsilon]^2] \right\}$$

Deriving the Drinfeld element u and its inverse ui

ybax

$$\text{In[*]:= } \text{Block}[\{i\}, \{ \\ u_{i-} = R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1 \rightarrow i}, \\ ui_{i-} := R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1 \rightarrow i} \\ \}]$$

ybax

$$\text{Out[*]:= } \left\{ \mathbb{E}\left[-a_i b_i, -\frac{x_i y_i}{B_i}, B_i + \frac{(-4 a_i B_i^2 - 4 B_i x_i y_i - 4 a_i B_i x_i y_i - 3 x_i^2 y_i^2) \epsilon}{4 B_i} + O[\epsilon]^2\right], \text{Null} \right\}$$

u and ui are inverses

$$\text{In[*]} := (\mathbf{u}_1 \mathbf{u}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1}$$

$$\text{Out[*]} := \mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathcal{O}[\epsilon]^2]$$

The ribbon element v satisfies $v^2 = S(u) u$. The spinner $C = uv^{-1}$.

It is convenient to compute $z = S(u) u^{-1}$ which is something easy.

$$\text{In[*]} := ((\mathbf{u}_1 \sim \mathbf{B}_1 \sim \mathbf{dS}_1) \mathbf{u}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1}$$

$$\text{Out[*]} := \mathbb{E}\left[\mathbf{0}, \mathbf{0}, \frac{1}{\mathbf{B}_1} + \frac{\mathbf{a}_1 \epsilon}{\mathbf{B}_1} + \mathcal{O}[\epsilon]^2\right]$$

(* Needs fixing! *) So in our case $S(u) = u z$ so $S(u)u = u^2 z$ and $v = uz^{\frac{1}{2}}$ and finally $C = uv^{-1} = z^{-\frac{1}{2}} = e^{t_1/2}(1 - \epsilon a_1)$.

ybax

```
In[*]:= Block[{i},
  {CC_i_ = E[0, 0, B_i^{1/2} e^{-\epsilon a_i/2} + O[epsilon]^2],
  CC_i_ = E[0, 0, B_i^{-1/2} e^{\epsilon a_i/2} + O[epsilon]^2]
}]
```

ybax

$$\text{Out[*]} := \left\{ \mathbb{E}\left[\mathbf{0}, \mathbf{0}, \sqrt{\mathbf{B}_i} - \frac{1}{2} \left(\mathbf{a}_i \sqrt{\mathbf{B}_i}\right) \epsilon + \mathcal{O}[\epsilon]^2\right], \mathbb{E}\left[\mathbf{0}, \mathbf{0}, \frac{1}{\sqrt{\mathbf{B}_i}} + \frac{\mathbf{a}_i \epsilon}{2 \sqrt{\mathbf{B}_i}} + \mathcal{O}[\epsilon]^2\right] \right\}$$

ybax

```
In[*]:= Block[{i, j}, {
  Kink_i_ = (R_{1,3} CC_2) ~ B_{1,2} ~ dm_{1,2 \to 1} ~ B_{1,3} ~ dm_{1,3 \to i},
  Kink_j_ = (R_{1,3} CC_2) ~ B_{1,2} ~ dm_{1,2 \to 1} ~ B_{1,3} ~ dm_{1,3 \to j}
}]
```

ybax

$$\text{Out[*]} := \left\{ \mathbb{E}\left[\mathbf{a}_i \mathbf{b}_i, \mathbf{x}_i \mathbf{y}_i, \frac{1}{\sqrt{\mathbf{B}_i}} + \frac{(2 \mathbf{a}_i - \mathbf{x}_i^2 \mathbf{y}_i^2) \epsilon}{4 \sqrt{\mathbf{B}_i}} + \mathcal{O}[\epsilon]^2\right], \right. \\ \left. \mathbb{E}\left[-\mathbf{a}_j \mathbf{b}_j, -\frac{\mathbf{x}_j \mathbf{y}_j}{\mathbf{B}_j}, \sqrt{\mathbf{B}_j} + \frac{(-2 \mathbf{a}_j \mathbf{B}_j^2 - 4 \mathbf{a}_j \mathbf{B}_j \mathbf{x}_j \mathbf{y}_j - 3 \mathbf{x}_j^2 \mathbf{y}_j^2) \epsilon}{4 \mathbf{B}_j^{3/2}} + \mathcal{O}[\epsilon]^2\right] \right\}$$

$$\text{In[*]} := \mathbf{k2} = (\mathbf{R}_{3,1} \mathbf{CC}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow i} /. \mathbf{e} \rightarrow \mathbf{E};$$

$$\mathbf{k4} = (\overline{\mathbf{R}}_{3,1} \overline{\mathbf{CC}}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow j} /. \mathbf{e} \rightarrow \mathbf{E};$$

$$\text{Simplify}@\{\mathbf{Kink}_i \equiv \mathbf{k2}, \overline{\mathbf{Kink}}_j \equiv \mathbf{k4}, (\mathbf{Kink}_i \overline{\mathbf{Kink}}_j) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow 1}\}$$

$$\text{Out[*]} := \{\text{True}, \text{True}, \mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathcal{O}[\epsilon]^2]\}$$

Reidemeister 2:

$$\text{In[*]} := (\mathbf{R}_{1,2} \overline{\mathbf{R}}_{3,4}) \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow 1} \sim \mathbf{B}_{2,4} \sim \mathbf{dm}_{2,4 \rightarrow 2}$$

$$\text{Out[*]} := \mathbb{E}[\mathbf{0}, \mathbf{0}, \mathbf{1} + \mathcal{O}[\epsilon]^2]$$

Cyclic Reidemeister 2:

In[*]:= $(R_{1,4} \overline{R}_{5,2} \overline{CC}_3) \sim B_{2,4} \sim dm_{2,4 \rightarrow 2} \sim B_{1,3} \sim dm_{1,3 \rightarrow 1} \sim B_{1,5} \sim dm_{1,5 \rightarrow 1} \equiv \overline{CC}_1$

Out[*]= True

Trefoil

ybax

In[*]:= $Z = R_{1,5} R_{6,2} R_{3,7} \overline{CC}_4 \overline{Kink}_8 \overline{Kink}_9 \overline{Kink}_{10};$
Do[Z = Z ~ B_{1,r} ~ dm_{1,r→1}, {r, 2, 10}];
Simplify /@ Z

ybax

Out[*]= $E \left[\theta, \theta, \frac{B_1}{1 - B_1 + B_1^2} + \right.$
 $\left. (B_1 (-B_1 + 2 B_1^2 + 2 B_1^4 + a_1 (-1 + B_1 - B_1^3 + B_1^4) - 2 x_1 y_1 - B_1^3 (3 + 2 x_1 y_1)) \epsilon) / (1 - B_1 + B_1^2)^3 + O[\epsilon]^2 \right]$

Timing [

$Z = R_{1,5} R_{6,2} R_{3,7} \overline{CC}_4 \overline{Kink}_8 \overline{Kink}_9 \overline{Kink}_{10};$
Do[Z = Z ~ B_{1,r} ~ dm_{1,r→1}, {r, 2, 10}];
Simplify /@ Z]

ybax

In[*]:= $b2t_{i-} := E [\alpha_i a_i - \beta_i t_i, \xi_i x_i + \eta_i y_i, 1 + \epsilon \beta_i a_i + O[\epsilon]^2]$
 $t2b_{i-} := E [\alpha_i a_i - \tau_i b_i, \xi_i x_i + \eta_i y_i, 1 + \epsilon \tau_i a_i + O[\epsilon]^2]$

In[*]:= $R_{1,5} R_{6,2} R_{3,7} \overline{CC}_4 \overline{Kink}_8 \overline{Kink}_9 \overline{Kink}_{10}$

Out[*]= $E [a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10},$
 $x_5 y_1 + x_7 y_3 + x_2 y_6 - \frac{x_8 y_8}{B_8} - \frac{x_9 y_9}{B_9} - \frac{x_{10} y_{10}}{B_{10}}, \frac{\sqrt{B_8} \sqrt{B_9} \sqrt{B_{10}}}{\sqrt{B_4}} +$
 $\left(\sqrt{B_{10}} \left(\sqrt{B_9} \left(\sqrt{B_8} \left(\frac{a_4}{2 \sqrt{B_4}} - \frac{x_5^2 y_1^2}{4 \sqrt{B_4}} - \frac{x_7^2 y_3^2}{4 \sqrt{B_4}} - \frac{x_2^2 y_6^2}{4 \sqrt{B_4}} \right) + \frac{-2 a_8 B_8^2 - 4 a_8 B_8 x_8 y_8 - 3 x_8^2 y_8^2}{4 \sqrt{B_4} B_8^{3/2}} \right) + \right.$
 $\left. \frac{\sqrt{B_8} (-2 a_9 B_9^2 - 4 a_9 B_9 x_9 y_9 - 3 x_9^2 y_9^2)}{4 \sqrt{B_4} B_9^{3/2}} \right) +$
 $\left. \frac{1}{4 \sqrt{B_4} B_{10}^{3/2}} \sqrt{B_8} \sqrt{B_9} (-2 a_{10} B_{10}^2 - 4 a_{10} B_{10} x_{10} y_{10} - 3 x_{10}^2 y_{10}^2) \right) \epsilon + O[\epsilon]^2]$

In[*]:= $(R_{1,5} R_{6,2} R_{3,7} \overline{CC_4} \overline{Kink_8} \overline{Kink_9} \overline{Kink_{10}}) \sim B_{Range[10]} \sim Product[b2t_i, \{i, 10\}]$

$$Out[*]:= E \left[-a_5 t_1 - a_7 t_3 - a_2 t_6 + a_8 t_8 + a_9 t_9 + a_{10} t_{10}, \frac{1}{T_8 T_9 T_{10}} \right. \\ \left. (T_8 T_9 T_{10} x_5 y_1 + T_8 T_9 T_{10} x_7 y_3 + T_8 T_9 T_{10} x_2 y_6 - T_9 T_{10} x_8 y_8 - T_8 T_{10} x_9 y_9 - T_8 T_9 x_{10} y_{10}), \right. \\ \left. \frac{\sqrt{T_8} \sqrt{T_9} \sqrt{T_{10}}}{\sqrt{T_4}} + \frac{1}{4 \sqrt{T_4} T_8^{3/2} T_9^{3/2} T_{10}^{3/2}} \right. \\ \left. (4 a_4 T_8^2 T_9^2 T_{10}^2 + 4 a_1 a_5 T_8^2 T_9^2 T_{10}^2 + 4 a_2 a_6 T_8^2 T_9^2 T_{10}^2 + 4 a_3 a_7 T_8^2 T_9^2 T_{10}^2 - \right. \\ \left. 4 a_8 T_8^2 T_9^2 T_{10}^2 - 4 a_8^2 T_8^2 T_9^2 T_{10}^2 - 4 a_9 T_8^2 T_9^2 T_{10}^2 - 4 a_9^2 T_8^2 T_9^2 T_{10}^2 - 4 a_{10} T_8^2 T_9^2 T_{10}^2 - 4 a_{10}^2 T_8^2 T_9^2 T_{10}^2 - \right. \\ \left. T_8^2 T_9^2 T_{10}^2 x_5^2 y_1^2 - T_8^2 T_9^2 T_{10}^2 x_7^2 y_3^2 - T_8^2 T_9^2 T_{10}^2 x_2^2 y_6^2 - 8 a_8 T_8 T_9 T_{10} x_8 y_8 - 3 T_9^2 T_{10}^2 x_8^2 y_8^2 - \right. \\ \left. 8 a_9 T_8 T_9 T_{10} x_9 y_9 - 3 T_8^2 T_{10}^2 x_9^2 y_9^2 - 8 a_{10} T_8 T_9 T_{10} x_{10} y_{10} - 3 T_8^2 T_9^2 x_{10}^2 y_{10}^2) \in + O[\epsilon]^2 \right]$$

In[*]:= $Z = \left(\left((R_{1,5} R_{6,2} R_{3,7} \overline{CC_4} \overline{Kink_8} \overline{Kink_9} \overline{Kink_{10}}) \sim B_{Range[10]} \sim Product[b2t_i, \{i, 10\}] \right) / \cdot T_- \rightarrow T_1 \right) \sim B_{Range[10]} \sim Product[t2b_i, \{i, 10\}]$

$$Out[*]:= E \left[a_5 b_1 + a_7 b_3 + a_2 b_6 - a_8 b_8 - a_9 b_9 - a_{10} b_{10}, \frac{1}{B_1} \right. \\ \left. (B_1 x_5 y_1 + B_1 x_7 y_3 + B_1 x_2 y_6 - x_8 y_8 - x_9 y_9 - x_{10} y_{10}), B_1 + \frac{1}{4 B_1} \right. \\ \left. (4 a_1 B_1^2 + 4 a_4 B_1^2 - 4 a_8 B_1^2 - 4 a_9 B_1^2 - 4 a_{10} B_1^2 - B_1^2 x_5^2 y_1^2 - B_1^2 x_7^2 y_3^2 - B_1^2 x_2^2 y_6^2 + 4 a_1 B_1 x_8 y_8 - 8 a_8 B_1 x_8 y_8 - \right. \\ \left. 3 x_8^2 y_8^2 + 4 a_1 B_1 x_9 y_9 - 8 a_9 B_1 x_9 y_9 - 3 x_9^2 y_9^2 + 4 a_1 B_1 x_{10} y_{10} - 8 a_{10} B_1 x_{10} y_{10} - 3 x_{10}^2 y_{10}^2) \in + O[\epsilon]^2 \right]$$

Timing[

Do[Z = Z ~ B1,r ~ dm1,r-1, {r, 2, 10}];
Simplify@Z[[3]]]

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

$$Out[*]:= \left\{ 5.39063, \frac{B_1}{1 - B_1 + B_1^2} + \right. \\ \left. (B_1 (-B_1 + 2 B_1^2 + 2 B_1^4 + a_1 (-1 + B_1 - B_1^3 + B_1^4) - 2 x_1 y_1 - B_1^3 (3 + 2 x_1 y_1)) \in) / (1 - B_1 + B_1^2)^3 + O[\epsilon]^2 \right\}$$

In[*]:= Timing[

Z = R1,5 R6,2 R3,7 CC4 Kink8 Kink9 Kink10 / . B_ -> B1;
Do[Print["doing ", r]; Z = Z ~ B1,r ~ dm1,r-1 / . B_ -> B1, {r, 2, 10}];
Simplify@Z[[3]]]

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

$$\text{Out[]=} \left\{ 5.3125, \frac{B_1}{1 - B_1 + B_1^2} + \frac{1}{2 B_1 (1 - B_1 + B_1^2)^3} \left(2 a_1 B_1^2 (-1 + B_1 - B_1^3 + B_1^4) - 6 x_1^2 y_1^2 + 4 B_1^7 x_1^2 y_1^2 - 2 B_1^8 x_1^2 y_1^2 + B_1^2 x_1 y_1 (5 - 6 x_1 y_1) + 3 B_1 x_1 y_1 (-1 + 2 x_1 y_1) + B_1^6 (3 + 3 x_1 y_1 - 6 x_1^2 y_1^2) - B_1^5 (4 + 13 x_1 y_1 + 2 x_1^2 y_1^2) + B_1^4 (2 + 15 x_1 y_1 + 4 x_1^2 y_1^2) - B_1^3 (1 + 15 x_1 y_1 + 6 x_1^2 y_1^2) \right) \in + O[\epsilon]^2 \right\}$$