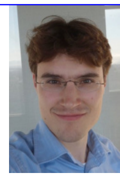
A Fast, Strong, Topologically Meaningful and Fun Knot Invariant

Abstract. The title covers all the good. The bad is that we don't really understand this invariant Θ. Wait, is that just part of the fun?
Continues Rozansky, Kricker, Garoufalidis, and Ohtsuki [Ro1, Ro2, Ro3, Kr, GR, Oh], joint with van der Veen [BV3].


van der Veen

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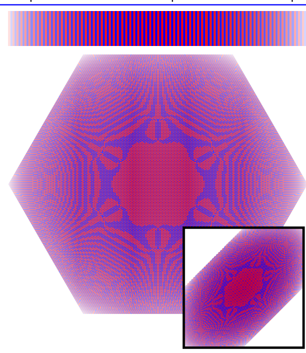
Strong. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (Vol is approximate, $H \supset \Delta, J$)

	knots	$(H, Kh, Vol, \sigma_{LT})$	$\Theta = (\Delta, \theta)$	together
xing ≤ 10	249	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(84)	(19)	(10)
xing ≤ 13	12,965	(911)	(194)	(169)
xing ≤ 14	59,937	(5,917)	(1,118)	(972)
xing ≤ 15	313,230	(41,434)	(6,758)	(6,304)

Fast. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.

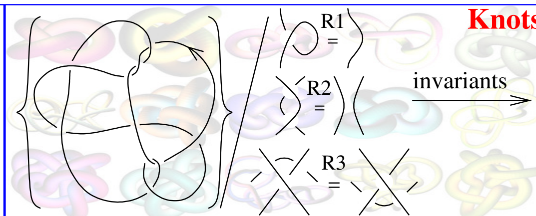


Fun. There's so much more to see in 2D pictures than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. θ gives a genus bound (with confidence and with a near proof). θ seems to give a criterion for a knot to be fibered (conjectured with a large scale verification). There are "safe" conjectured characterizations of θ as "the two loop invariant" and as "the one cobracket invariant". We hope (with reason) θ will say something about ribbon knots.

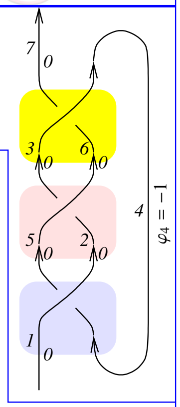
Conventions. T, T_1, and T_2 are indeterminates and $T_3 := T_1T_2$.

Knots.



Tell them apart? Alternating? Bound a genus 7 surface? Complement is fibered over S^1? Complement is hyperbolic? Bounds a disk with only ribbon singularities? Bounds a topological / smooth non-singular disk in B^4? ...

Preparation. Draw an n-crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k.



Model T Traffic Rules. Cars always drive forward. When a car crosses over a sign-s bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0$. At the very end, cars fall off and disappear. On various edges *traffic counters* are placed. See also [Jo, LTW].

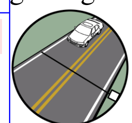


image credits: diamondtraffic.com


$p = 1 - T^s$

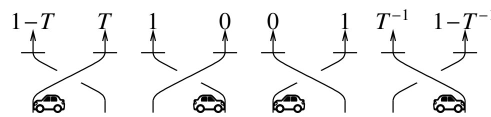
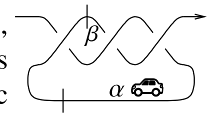


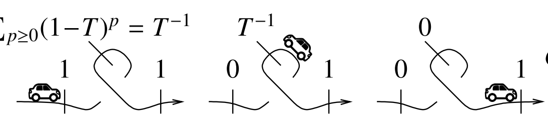

image credits: Dall-E

Definition. The *traffic function* $G = (g_{\alpha\beta})$ (also, the *Green function* or the *two-point function*) is the reading of a traffic counter at β, if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point). There are also model-T_ν traffic functions $G_\nu = (g_{\nu\alpha\beta})$ for $\nu = 1, 2, 3$.

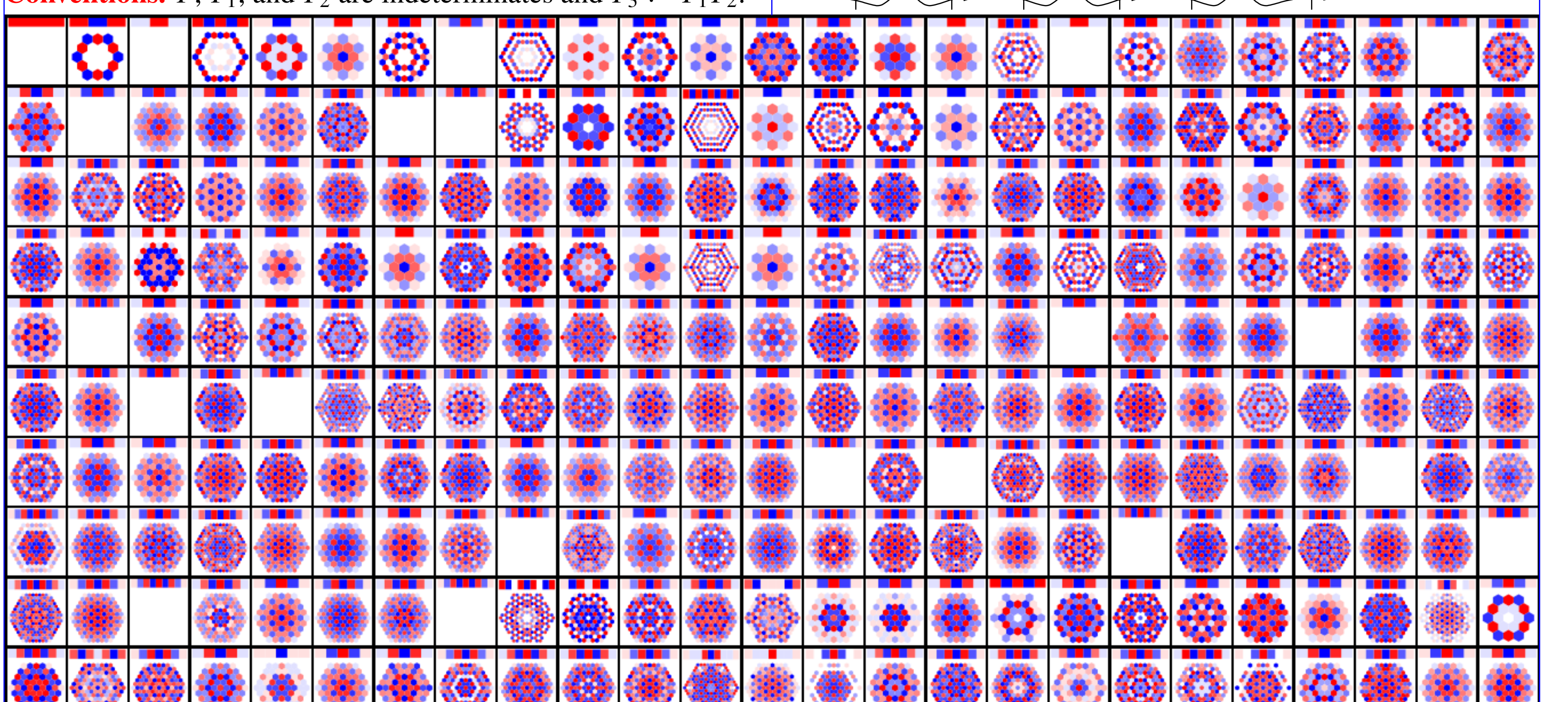


Example.

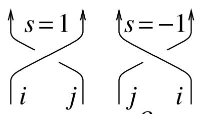
$\sum_{p\geq 0}(1-T)^p = T^{-1}$



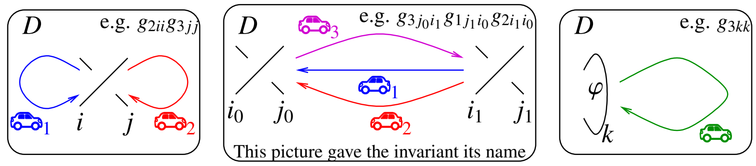
$$G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



Theorem [BV3]. With $c=(s,i,j)$, $c_0=(s_0,i_0,j_0)$, and $c_1=(s_1,i_1,j_1)$ denoting crossings, there is a quadratic $F_1(c)\in\mathbb{Q}(T_\nu)[g_{\nu\alpha\beta}:\alpha,\beta\in\{i,j\}]$, a cubic $F_2(c_0,c_1)\in\mathbb{Q}(T_\nu)[g_{\nu\alpha\beta}:\alpha,\beta\in\{i_0,j_0,i_1,j_1\}]$, and a linear $F_3(\varphi,k)$ such that θ is a knot invariant:

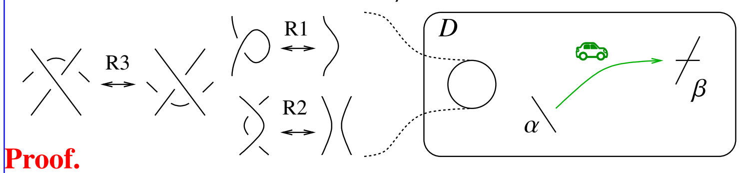


$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\text{normalization, see later}}\left(\sum_c F_1(c)+\sum_{c_0,c_1}F_2(c_0,c_1)+\sum_k F_3(\varphi_k,k)\right),$$

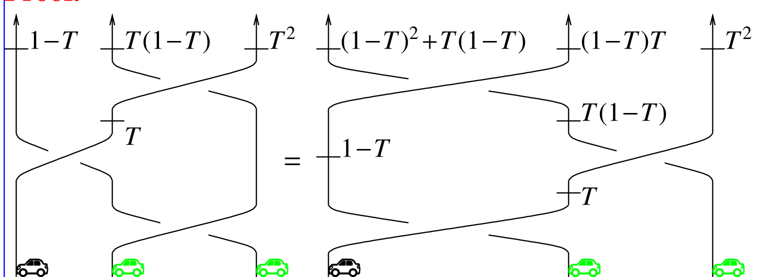


If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

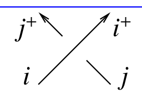
Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Proof.



Lemma 2. With $k^+ := k+1$, the "g-rules" hold near a crossing $c=(s,i,j)$:



$$g_{j\beta}=g_{j^+\beta}+\delta_{j\beta}\qquad g_{i\beta}=T^s g_{i^+\beta}+(1-T^s)g_{j^+\beta}+\delta_{i\beta}\qquad g_{2n^+,\beta}=\delta_{2n^+,\beta}$$

$$g_{\alpha i^+}=T^s g_{\alpha i}+\delta_{\alpha i^+}\qquad g_{\alpha j^+}=g_{\alpha j}+(1-T^s)g_{\alpha i}+\delta_{\alpha j^+}\qquad g_{\alpha,1}=\delta_{\alpha,1}$$

Corollary 1. G is easily computable, for $AG=I\ (=GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$$c=(s,i,j)\mapsto\begin{array}{c|cc} A & \text{col } i^+ & \text{col } j^+\\ \hline \text{row } i & -T^s & T^s-1\\ \text{row } j & 0 & -1\end{array}$$

For the trefoil example, we have:

$$A=\begin{pmatrix}1&-T&0&0&T-1&0&0\\0&1&-1&0&0&0&0\\0&0&1&-T&0&0&T-1\\0&0&0&1&-1&0&0\\0&0&T-1&0&1&-T&0\\0&0&0&0&0&1&-1\\0&0&0&0&0&0&1\end{pmatrix},$$

$$G=\begin{pmatrix}1&T&1&T&1&T&1\\0&1&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T^2}{T^2-T+1}&1\\0&0&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T}{T^2-T+1}&\frac{T^2}{T^2-T+1}&1\\0&0&\frac{1-T}{T^2-T+1}&\frac{1}{T^2-T+1}&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&1\\0&0&\frac{1-T}{T^2-T+1}&-\frac{(T-1)T}{T^2-T+1}&\frac{1}{T^2-T+1}&\frac{T}{T^2-T+1}&1\\0&0&0&0&0&1&1\\0&0&0&0&0&0&1\end{pmatrix}$$

Note. The Alexander polynomial Δ is given by

$$\Delta=T^{(-\varphi-w)/2}\det(A),\qquad\text{with }\varphi=\textstyle\sum_k\varphi_k,\ w=\sum_c s.$$

We also set $\Delta_\nu:=\Delta(T_\nu)$ for $\nu=1,2,3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b},b,\epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

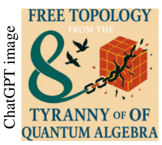
Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K)\geq\deg_{T_1}\theta(K)$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

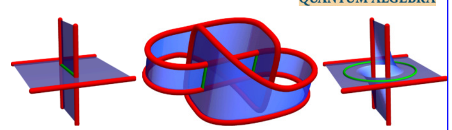
Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.



ChatGPT image

Dream 10. With Conjecture 7 in mind, θ will have something to say about ribbon knots.



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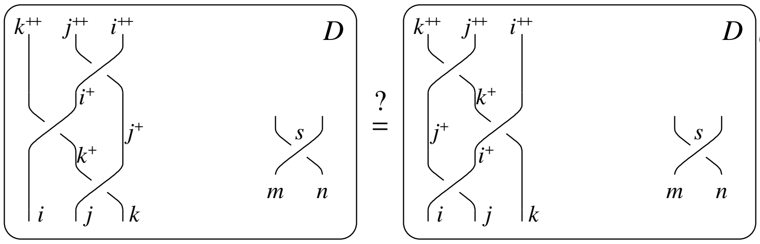
[Ro3] —, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

[Sch] S. Schaveling, *Expansions of Quantum Group Invariants,* Ph.D. thesis, Universiteit Leiden, September 2020, ωεβ/Scha.

Corollary 2. Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/v25/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
T3 = T1 T2;
```

```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

```
F1[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
       (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
       g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
       ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
         (T3^s - 1) g3ji
          (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
            (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
    ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

```
F3[φ_, k_] = -φ/2 + φ g3kk;
```

```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
   gν_jβ_ ⧴ gνj⁺β + δjβ,
   gν_iβ_ ⧴ Tν^s gνi⁺β + (1 - Tν^s) gνj⁺β + δiβ,
   gν_α_i⁺ ⧴ Tν^s gναi + δαi⁺,
   gν_α_j⁺ ⧴ gναj + (1 - Tν^s) gναi + δαj⁺
  }
```

```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] +
   Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺},
     {s, m, n}] //. gR1,j,k ⋃ gR1,i,k⁺ ⋃ gR1,i⁺,j⁺;
rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺},
     {s, m, n}] //. gR1,i,j ⋃ gR1,i⁺,k ⋃ gR1,j⁺,k⁺;
Simplify[lhs == rhs]
```

```
True
```

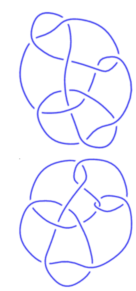
The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴
     (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                       0     -1   ))];
   Δ = T^((-Total[φ] - Total[Cs⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] :=
    Factor[ℰ /. gν_,α_,β_ ⧴ (G⟦α, β⟧ /. T → Tν)];
   θ = ev[Σ_{k=1}^n F1[Cs⟦k⟧]];
   θ += ev[Σ_{k1=1}^n Σ_{k2=1}^n F2[Cs⟦k1⟧, Cs⟦k2⟧]];
   θ += ev[Σ_{k=1}^{2 n} F3[φ⟦k⟧, k]];
   Factor@
    {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```

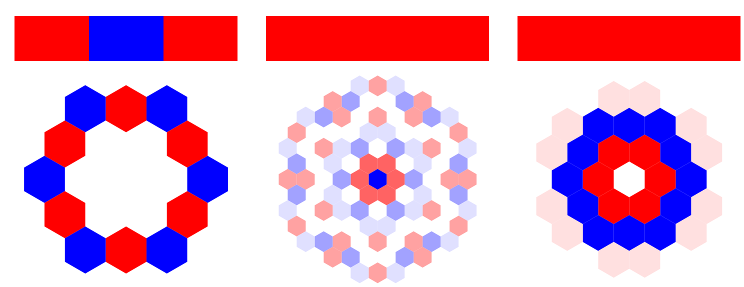
The Trefoil, Conway, and Kinoshita-Terasaka

```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$



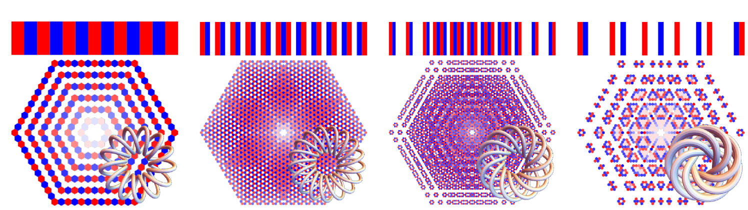
```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
  {"3_1", "K11n34", "K11n42"}]
```



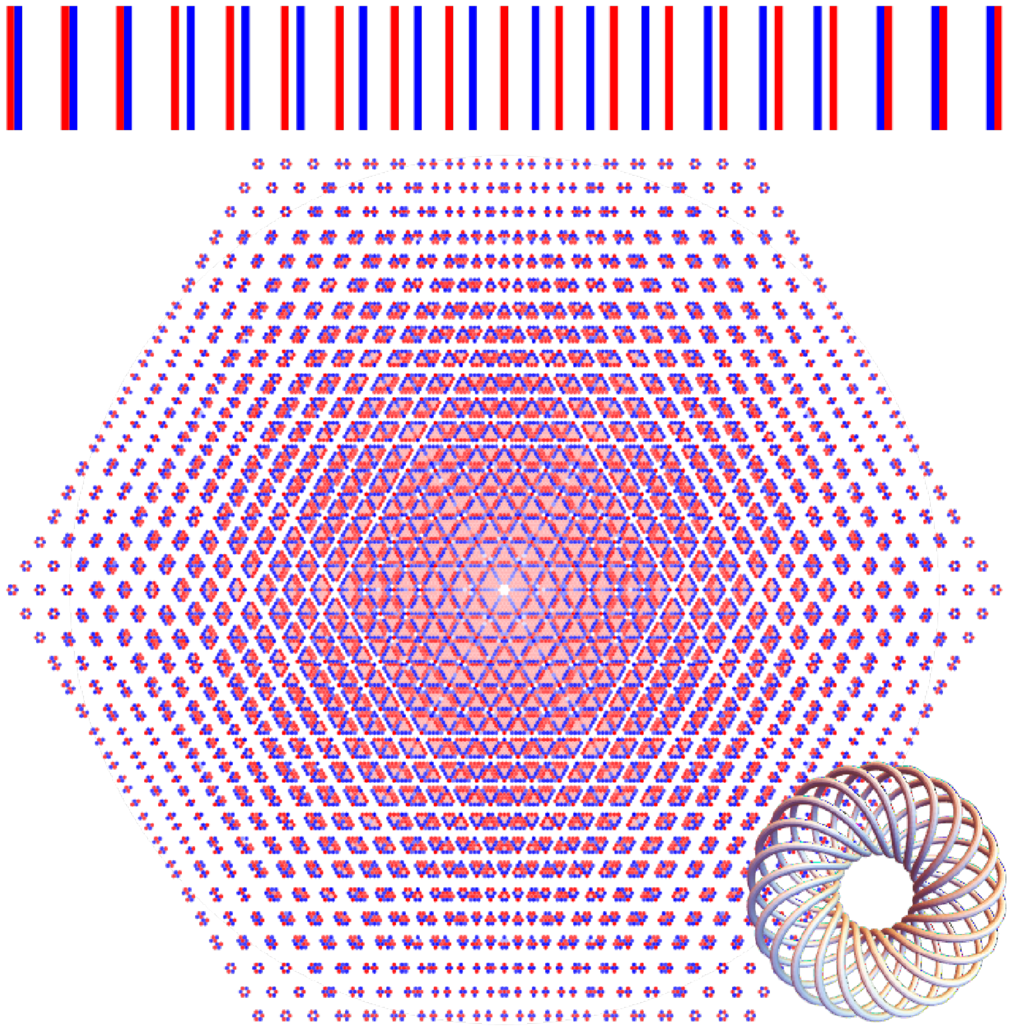
(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

Some Torus Knots

```
GraphicsRow[ImageCompose[
     PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
     TubePlot[TorusKnot @@ #, ImageSize → 240],
     {Right, Bottom}, {Right, Bottom}
    ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```



The 132-crossing torus knot $T_{22/7}$: (many more at ωεβ/TK)



Random knots from [DHOEBL] with 51 – 75 crossings: (many more at ωεβ/DK)