Cheat Sheet OneCo

Deriving Gassner. $\mathcal{L}^{2Dw}$ is $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo locality, $[a_{ij},a_{ik}]=0$, $[a_{ik},a_{jk}]=-[a_{ij},a_{jk}]=b_ja_{ik}-b_ia_{jk}$, and (mod $\langle a_{ii}\rangle$) $[a_{ij},a_{ji}]=b_ia_{ji}-b_ja_{ij}$. Acts on $V=\mathbb{Q}[\![b_i]\!]\langle x_i=a_{i\infty}\rangle$ by $[a_{ij},x_i]=0$, $[a_{ij},x_j]=b_ix_j-b_jx_i$. Hence $e^{\operatorname{ad}a_{ij}}x_i=x_i$, $e^{\operatorname{ad}a_{ij}}x_j=e^{b_i}x_j+\frac{b_j}{b_i}(1-e^{b_i})x_i$. Renaming $\bar{x}_i=x_i/b_i$, $t_i=e^{b_i}$, get

$$[e^{\operatorname{ad}a_{ij}}]_{\bar{x}_i,\bar{x}_j}=\begin{pmatrix}1 & 1-t_i\\ 0 & t_i\end{pmatrix}.$$

The $\mathcal{L}^{2Dw}$ Adjoint representation. $e^{\operatorname{ad}a_{ij}}$ acts by

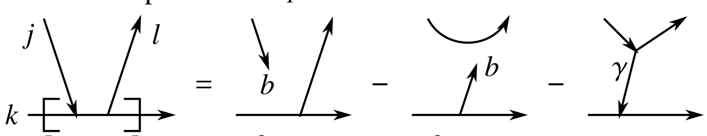
$$a_{kl}\mapsto a_{kl},\qquad a_{ik}\mapsto a_{ik},\qquad a_{kj}\mapsto e^{-b_i}a_{kj}+\frac{b_k}{b_i}(1-e^{-b_i})a_{ij},$$
$$a_{ki}\mapsto a_{ki}+(1-e^{-b_i})a_{kj}+b_k\frac{e^{-b_i}-1}{b_i}a_{ij},$$
$$a_{jk}\mapsto e^{b_i}a_{jk}+\frac{b_j}{b_i}(1-e^{b_i})a_{ik},\qquad a_{ji}\mapsto e^{b_i}a_{ji}+\frac{b_j}{b_i}(1-e^{b_i})a_{ij}.$$

Implementation/verification: pensieve://2015-04/nb/ZeroCo.pdf, pensieve://2016-04/nb/BurauAndAd.pdf.

Adjoint Gassner. Renaming $\bar{a}_{ij}=a_{ij}/b_i$ and $t_i=e^{b_i}$, get $[\bar{a}_{ij},\bar{a}_{ik}]=0$, $[\bar{a}_{ik},\bar{a}_{jk}]=-[\bar{a}_{ij},\bar{a}_{jk}]=\bar{a}_{ik}-\bar{a}_{jk}$, and (mod $\langle\bar{a}_{ii}\rangle$) $[\bar{a}_{ij},\bar{a}_{ji}]=\bar{a}_{ji}-\bar{a}_{ij}$, so

$$\bar{a}_{kj}\mapsto t_i^{-1}\bar{a}_{kj}+(1-t_i^{-1})\bar{a}_{ij},$$
$$\bar{a}_{ki}\mapsto\bar{a}_{ki}+(1-t_i^{-1})\bar{a}_{kj}+(t_i^{-1}-1)\bar{a}_{ij},$$
$$\bar{a}_{jk}\mapsto t_i\bar{a}_{jk}+(1-t_i)\bar{a}_{ik},\qquad\bar{a}_{ji}\mapsto t_i\bar{a}_{ji}+(1-t_i)\bar{a}_{ij}.$$

Question. Interpretation? π_T-Artin?



$$[a_{jk},a_{kl}]\quad=\quad b_ja_{kl}\quad-\quad b_ka_{jl}\quad-\quad\gamma_{jkl}$$

2Dv. b: bracket trace; c: cobracket trace; $\langle b,c\rangle=\delta\in\{0,1\}$; $\deg b_i=\deg c_j=\deg a_{ij}=\deg\delta=1$. Implementation/verification: pensieve://2015-08/nb/abc.pdf.

$\mathcal{A}^{2Dv}$ is $\mathbb{Q}[\![\delta]\!]FA(b_i,c_j,a_{ij})$ (so $\mathcal{L}^v=\{f+f^{ij}a_{ij}\}$) modulo locality,

tt. $[a_{jk},a_{jl}]=c_la_{jk}-c_ka_{jl}$,

hh. $[a_{jk},a_{ik}]=b_ia_{jk}-b_ja_{ik}$,

Swinging. $\delta a_{ij}a_{kl}-\delta a_{il}a_{kj}=b_kc_la_{ij}-b_ic_la_{kj}-b_kc_ja_{il}+b_ic_ja_{kl}$

ht. $[a_{jk},a_{kl}]=b_ja_{kl}-b_ka_{jl}-c_la_{jk}+c_ka_{jl}$,

ab,ac. $\operatorname{ad}a_{jk}\colon b_j,-b_k,-c_j,c_k\mapsto\gamma_{jk}:=\delta a_{jk}-b_jc_k$,

Backie. $[a_{jk},a_{kj}]=(b_j+c_k)a_{kj}-(b_k+c_j)a_{jk}+(b_j-c_j)a_{kk}-(b_k-c_k)a_{jj}+\gamma_{jk}-\gamma_{kj}$, with $\gamma_{jk}:=\delta a_{jk}-b_jc_k$,

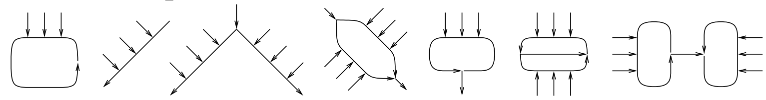
bc. $[b_i,c_j]=0$.

So $a_{ij}f=f^\delta a_{ij}-\frac{b_ic_j}{\delta}(f^\delta-f)$, $[a_{ij},f]=(f^\delta-f)\left(a_{ij}-\frac{b_ic_j}{\delta}\right)$, with $f^\delta:=f/\!\!/\begin{pmatrix}b_i\to b_i+\delta & b_j\to b_j-\delta\\ c_i\to c_i-\delta & c_j\to c_j+\delta\end{pmatrix}$.

The Ascending Algebra $\mathcal{A}_+^{2Dv}$. Same but with only a_{ij}, $i<j$.

The OneCo Sub-Quotient is $\langle a_{ij}\rangle$ modulo $\delta^2=\delta c_i=c_jc_k=0$, so $\mathcal{L}^{1co}$ is (coefficient functions non-central, in $\mathbb{Q}[\![b_i]\!]$)

The 1co Graphs.



In abc.nb: $R^{jk}=e^{a_{jk}}\rho$ with $\rho:=$

$$\psi(b_j)\left(-c_k+\frac{c_ka_{jk}}{b_j}-\frac{\delta a_{jk}a_{jk}}{b_j^2}\right)+\frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_ka_{kk}-\frac{\delta a_{jk}a_{kk}}{b_j}\right),$$

and with $\phi(x):=e^{-x}-1=-x+x^2/2-\dots$, and $\psi(x):=((x+2)e^{-x}-2+x)/(2x)=x^2/12-x^3/24+\dots$.

In MostGeneralR.nb:

```
rule2 = {gg3|4|5|7|8[_] → 0, gg2[x_] ⧴ e^x/x, ff_[__] → 0};
ρ0[j, k] /. rule2 // S
```

$$c\left[-\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j},k\right]+ca\left[\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^2},k,j,k\right]+ca\left[\frac{e^{b_k}}{b_k},j,k,k\right]+$$
$$ca\left[-\frac{e^{-b_j}\left(-1+e^{b_j}\right)(2+b_k)}{2b_k^2},k,k,k\right]+\delta a\left[\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^2},j,k\right]+$$
$$\delta aa\left[-\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^3},j,k,j,k\right]+\delta aa\left[\frac{e^{-b_j}\left(-1+e^{b_j}\right)(2+b_k)}{2b_jb_k^2},j,k,k,k\right]$$

```
R[1, 2]@a[1, 1, ∞] /. rule2 // S
```

$$a[1,1,\infty]+c\left[-e^{b_2}b_1,\infty\right]+ca\left[1-e^{b_2},\infty,1,2\right]+$$
$$ca\left[e^{b_2}+\frac{-1+e^{-b_1}}{b_1},2,1,\infty\right]+ca\left[-\frac{e^{b_2}b_1}{b_2},2,2,\infty\right]+\delta a\left[e^{b_2},1,\infty\right]+$$
$$\delta aa\left[-\frac{-1+e^{-b_1}+b_1}{b_1^2},1,2,1,\infty\right]+\delta aa\left[\frac{e^{b_2}}{b_2},1,2,2,\infty\right]$$

```
R[1, 2]@a[1, 2, ∞] /. rule2 // S
```

$$a\left[e^{b_1},2,\infty\right]+a\left[-\frac{\left(-1+e^{b_1}\right)b_2}{b_1},1,\infty\right]+$$
$$c\left[\frac{-\left(-1+e^{b_1}\right)b_2+b_1\left(-1+e^{b_1}+\left(e^{b_1}-e^{b_2}+e^{b_1+b_2}\right)b_2\right)}{b_1},\infty\right]+$$
$$ca\left[\frac{e^{b_2}\left(-1+e^{b_1}\right)b_2}{b_1},\infty,1,2\right]+ca\left[\frac{\left(-1+e^{b_1}\right)\left(1+e^{b_2}b_2\right)}{b_2},2,2,\infty\right]+$$
$$ca\left[-\frac{e^{-b_1}\left(-1+e^{b_1}\right)\left(-1+e^{b_1}+e^{b_1+b_2}b_2\right)}{b_1},2,1,\infty\right]+$$
$$\delta a\left[\frac{\left(-1+e^{b_1}\right)b_2-b_1\left(-1+e^{b_1}+\left(e^{b_1}-e^{b_2}+e^{b_1+b_2}\right)b_2\right)}{b_1^2},1,\infty\right]+$$
$$\delta aa\left[\frac{e^{-b_1}\left(-1+e^{b_1}\right)^2}{b_1^2},1,2,1,\infty\right]+\delta aa\left[-\frac{\left(-1+e^{b_1}\right)\left(1+e^{b_2}b_2\right)}{b_1b_2},1,2,2,\infty\right]$$

OneCo Monoblog.

(160510) The next few steps: • Full adjoint scattering in 1-co. • Solve again for R. • Find a manifestly polynomial formula for R.

(160508) How would I present the TS stitching formula?

(160505) A faithful representation for $\mathcal{A}^{2,2}$? Ado suggests existence.

(1504) If $S_n:=\sum_{k=0}^{n-1}A^kCB^{n-1-k}$ then $AS_n-S_nB=A^nC-CB^n$ so $S_n=(L_A-R_B)^{-1}(A^nC-CB^n)$.

(160317) To do: For 0-co a and b, compute the 1-co part of $e^{-a}be^a$.

(151019d) Perhaps I should switch to a circuit algebra perspective, plus meta-monoid ops.

(151019c) Make the braid representation presentable?

(151019b) Switch to an EK basis?

(151019a) To do: Find and implement the group-like condition.

ZeroCo Implementation Showcase

http://drorbn.net/AcademicPensieve/Projects/OneCo-1604/Zero
Started 2016-04-16; modified 2016-05-15 11:02:22-04:00

In "T before H" conventions.

Generalities.

```
Simp[γ_] := Expand[γ];
CF[γ_] := γ /. (λ_β | λ_a) :> MapAt[Simp, λ, 1];
AutoCollecting[λ_] := (λ /: λ[0, ___] = 0;
   λ /: λ[f_, r___] + λ[g_, r___] := λ[Simp[f + g], r];
   λ /: g_ * λ[f_, r___] := λ[Simp[g f], r]);
AutoCollecting /@ {β, a};
UU /: UU[x_] + UU[y_] := UU[x + y];
UU /: a_ * UU[x_] := UU[Expand[a x]];
UU /: D[u_UU, vs__] :=
  CF[u /. (λ_β | λ_a) :> MapAt[D[#, vs] &, λ, 1]];
UU /: Coefficient[u_UU, λ_[js___]] :=
  Total[Cases[u, λ[f_, js] :> f, ∞]];
Kδ /: Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
```

Definition of *tm*.

```
UU[γ_] // tm[x_, y_, z_] := CF[UU[
    Expand[γ /. a[f_, x | y, j_] :> a[f, z, j] /. b_{x|y} → b_z]]];
```

Definition of *hm*.

```
UU[γ_] // hm[x_, y_, z_] :=
  CF[UU[Expand[γ /. a[f_, i_, x | y] :> a[f, i, z]]]];
```

Definition of *hts*.

```
UU[γ_] // hts[y_, x_] := CF[UU[Expand[γ /.
      a[f_, i_, j_] :> a[f, i, j] - Kδ_ix Kδ_jy β[f b_x]]]];
```

Definition of *dm*.

```
dm[x_, y_, z_][γ_] :=
 γ // hts[x, y] // tm[x, y, z] // hm[x, y, z]
```

Renaming operations.

```
tσ[x_List, y_List][γ_] := (rr = Replace[Thread[x → y]];
   CF[γ /. b_i_ :> b_rr@i /. a[f_, i_, j_] :> a[f, rr@i, j]]);
tσ[x_, y_][γ_] := tσ[{x}, {y}][γ];
hσ[x_List, y_List][γ_] :=
  CF[
   γ /. a[f_, i_, j_] :> a[f, i, Replace[Thread[x → y]]@j]];
hσ[x_, y_][γ_] := hσ[{x}, {y}][γ];
dσ[x_, y_][γ_] := γ // tσ[x, y] // hσ[x, y];
```

Definition of *tb*.

```
tb[x_][UU[L_], UU[R_]] := UU[0];
```

Definition of *thb*.

```
thb[x_, y_][UU[L_], UU[R_]] :=
  CF[UU[Expand[Distribute[pp[L, R]] /. {
       pp[0, _] → 0, pp[_, 0] → 0, pp[_β, _] → 0,
       pp[_, _β] → 0,
       pp[a[f_, i_, j_], a[g_, k_, l_]] :>
        Kδ_yl Kδ_xi (-a[b_k f g, i, j] + a[b_i f g, k, j])
      }]]];
htb[x_, y_][L_UU, R_UU] := -thb[y, x][R, L];
```

Definition of *hb*.

```
hb[y_][UU[L_], UU[R_]] :=
  CF[UU[Expand[Distribute[pp[L, R]] /. {
        pp[0, _] → 0, pp[_, 0] → 0,
        pp[_β, _] → 0, pp[_, _β] → 0
       } /. {
       pp[a[f_, i_, y], u_] :>
        (u /. a[g_, j_, k_] :>
           Kδ_yk (a[b_j f g, i, y] - a[b_i f g, j, k])),
        _pp → 0
      }]]];
```

Definition of *db*.

Using $h_1h_2t_1t_2 \to h_1h_2t_1t_2 \to h_1h_2t_2t_1 \to h_2h_1t_2t_1 \to h_2h_1t_2t_1$:

```
db[x_][u_UU, v_UU] := Module[{t, h}, Plus[
    htb[x, x][u // tσ[x, t], v // hσ[x, h]] // tm[t, x, x] //
     hm[x, h, x],
    tb[x][u, v // hσ[x, h]] // hm[x, h, x],
    hb[x][u, v // tσ[x, t]] // tm[t, x, x],
    thb[x, x][u // hσ[x, h], v // tσ[x, t]] //
      tm[t, x, x] // hm[x, h, x] ]];
```

The bracket.

```
bb[S_List] := Module[{w, bar, t, n = 0, i, k},
   w = #2 // dσ[S, bar /@ S];
   Sum[t = db[S⟦k⟧][#1, w // dσ[bar[S⟦k⟧], S⟦k⟧]];
    Do[t = t // dm[bar[S⟦i⟧], S⟦i⟧, S⟦i⟧], {i, 1, k - 1}];
    Do[t = t // dm[S⟦i⟧, bar[S⟦i⟧], S⟦i⟧],
     {i, k + 1, Length@S}];
    t, {k, Length@S}] ] &
bb[S___] := bb[{S}]
```

Definition of *ct*.

```
ct[s_] := ct[s, s]; ct[] = ct[0, 0];
ct[h_, t_][UU[L_], UU[R_]] :=
  UU[Distribute[pp[L, R]] /. {
      pp[_β, _] → 0,
      pp[a[f_, i_, h], β[g_]] :> β[f b_i g / b_t],
      pp[a[f_, i_, h], a[g_, t, j_]] :> a[f g, i, j],
      pp[a[f_, i_, h], a[g_, j_, k_]] :> a[f b_i g / b_t, j, k],
      pp[a[__], _] → 0}] // CF;
```

Global Generalities.

(TSD for "Tail Scattering Data")

```
TSD[λ_]_j_ := Lookup[λ, j, UU@a[1, j, h∞]];
UU[u_] // γ_TSD := CF[u /. λ_a :> γ@λ];
TSD /: (γ_TSD)^-1 := Module[{S = Keys @@ γ, m},
   m = Table[Coefficient[γ_i, a[j, h∞]], {i, S}, {j, S}] //
     Inverse;
   TSD@<|Table[S⟦α⟧ →
       CF@UU@Sum[a[m⟦α, β⟧, S⟦β⟧, h∞], {β, Length@S}],
      {α, Length@S}]|>
  ];
a[f_, j_, k_] // γ_TSD := Module[{S = Keys @@ γ, γi},
   Switch[{MemberQ[S, j], MemberQ[S, k]},
    {False, False}, UU@a[f, j, k],
    {True, False}, γ_j /. a[g_, i_, h∞] :> a[f g, i, k],
    {False, True}, (γi = γ^-1;
     CF@Sum[
       γ[bb[S ⋃ {j}][γi_i, UU@a[f, j, k]]] /. {
         a[_, i, h∞] :> 0, a[g_, l_, h∞] :> a[g / b_i, l, i]},
       {i, S}]),
    {True, True}, ct[h∞, t∞][γ@a[f, j, h∞],
     γ@a[1, t∞, k]]
   ]];
```

Exponentiating an arrow.

```
Ea[t_, j_, k_] := TSD[⟨|
    j → CF@UU[a[1, j, h∞]],
    k → CF@UU[a[e^(t b_j), k, h∞] + a[-((-1 + e^(t b_j)) b_k)/b_j, j, h∞]]|⟩];
R[j_, k_] := Ea[1, j, k]
```