Pensieve header: A Demo for Gauss Diagram fromulas and Gauss-Gassner-Alexander computations.. More at http://www.-math.toronto.edu/~drorbn/Talks/NCSU-1604/

The Gauss-Gassner-Alexander Demo

Initialization

```
<< KnotTheory`
```

Initialization

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

GaussDiagrams

```
GD[g_GD] := g;
GD[L_] := GD @@ PD[L] /.
   X[i_, j_, k_, l_] :> If[PositiveQ@X[i, j, k, l], Ap[l, i], Am[j, i]];
Draw[g_GD] := Module[{n = Max@Cases[g, _Integer, ∞]}, Graphics[{
    Line[{{0, 0}, {n + 1, 0}}],
    List @@ g /. (ah_)[i_, j_] :> {
       Arrow[BezierCurve[{{i, 0}, {i + j, Abs[j - i]} / 2, {j, 0}}]],
       Text[ah /. {Ap → "+", Am → "-"}, {i, 0.3}]},
    Table[Text[i, {i, -0.5}], {i, n}]}]]
```

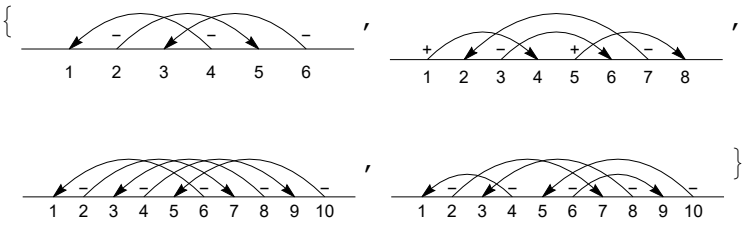
DrawGDsTo5

```
Draw /@ GD /@ AllKnots@{3, 5}
```

DrawGDsTo5

KnotTheory::loading : Loading precomputed data in PD4Knots`.

DrawGDsTo5



GDsTo5

```
GD /@ AllKnots@{3, 5}
```

GDsTo5

```
{GD[Am[4, 1], Am[6, 3], Am[2, 5]], GD[Ap[1, 4], Ap[5, 8], Am[3, 6], Am[7, 2]],
 GD[Am[6, 1], Am[8, 3], Am[10, 5], Am[2, 7], Am[4, 9]], GD[Am[4, 1], Am[8, 3], Am[10, 5], Am[6, 9], Am[2, 7]]}
```

V2

```
CF[g_GD] := Sort[
   g /. Thread[Sort@Cases[g, _Integer, ∞] → Range[2 Length[g]]]];
PV[F_GD, g_GD] /; Length[F] > Length[g] := 0;
PV[F_GD, g_GD] /; Length[F] < Length[g] := Sum[
   PV[F, y], {y, Subsets[g, {Length[F]}]}];
PV[F_GD, g_GD] /; Length[F] == Length[g] := If[
   CF[F] === CF[g /. Ap | Am → A], (-1)^Count[g, Am__], 0];
V2[g_] := V2[g] = PV[GD[A3,1, A2,4], GD[g]];
```

V2Test

```
Format[Knot[n_, k_]] := nk;
Table[K → V2[K], {K, AllKnots@{3, 7}}]
```

V2Test

```
{3[1] → 1, 4[1] → -1, 5[1] → 3, 5[2] → 2, 6[1] → -2, 6[2] → -1, 6[3] → 1, 7[1] → 6, 7[2] → 3, 7[3] → 5, 7[4] → 4, 7[5] → 4, 7[6] → 1, 7[7] → -1}
```

V3

```
PV[F1_ + F2_, g_] := PV[F1, g] + PV[F2, g];
PV[c_ * F_GD, g_] := c PV[F, g];
ρk_[g_] := g /. i_Integer :> Mod[i - k, 2 Length@g, 1];
F3 = Sum[(3 ρk@GD[A1,5, A4,2, A6,3] + 2 ρk@GD[A1,4, A5,2, A3,6]), {k, 0, 5}];
V3[K_] := V3[K] = PV[F3, GD@K] / 6;
```

V3Test

```
Table[K → V3[K], {K, AllKnots@{3, 7}}]
```

V3Test

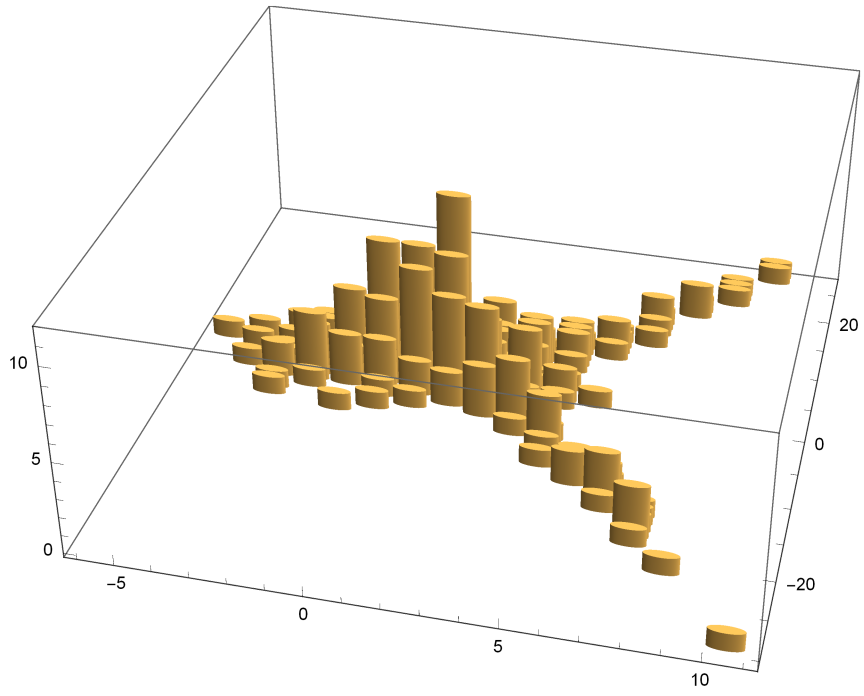
```
{3_1 → -1, 4_1 → 0, 5_1 → -5, 5_2 → -3, 6_1 → 1, 6_2 → 1,
 6_3 → 0, 7_1 → -14, 7_2 → -6, 7_3 → 11, 7_4 → 8, 7_5 → -8, 7_6 → -2, 7_7 → -1}
```

```
SetOptions[Histogram3D,
  ImageSize → 4 × 72 / 0.65,
  ViewPoint → {0.67099, -2.74409, 1.86273},
  ChartElements → Graphics3D[Cylinder[]]
 ];
```

WillertonFish

```
Histogram3D[
 Table[{V2[K], V3[K]}, {K, AllKnots@{3, 10}}], {1}]
```

WillertonFish




```
/. -(a_Plus) b_ :> (-a) b
```

Formatting

```
G[λ_]a_,b_ := ∂ta,hb λ;
G /: Factor[G[λ_]] := G[Collect[λ, h_, Collect[#, t_, Factor] &]];
Format@γ_G := Module[{S = Union@Cases[γ, (h | t)a_ :> a, ∞]},
   Table[γa,b, {a, S}, {b, S}] // MatrixForm] ;
```

GProgram

```
G /: G[λ1_] G[λ2_] := G[λ1 + λ2];

m_{a_,b_→c_}[G[λ_]] := Module[{α, β, γ, δ, ϵ, ϕ, ψ, Ξ, μ},
    {{α, β, Θ}, {γ, δ, ϵ}, {ϕ, ψ, Ξ}} = {{∂_{t_a,h_a} λ, ∂_{t_a,h_b} λ, ∂_{t_a} λ},
      {∂_{t_b,h_a} λ, ∂_{t_b,h_b} λ, ∂_{t_b} λ}, {∂_{h_a} λ, ∂_{h_b} λ, λ}} /. (t | h)_{a|b} → 0; μ = 1 - β;
    G[Tr[{{t_c}, {1}}^T . {{γ + α δ / μ, ϵ + δ Θ / μ}, {ϕ + α ψ / μ, Ξ + ψ Θ / μ}} . {{h_c}, {1}}]] /. T_{a|b} → T_c // Factor];

Rp_{a_,b_} := G[Tr[{{t_a}, {t_b}}^T . {{1, 1 - T_a}, {0, T_a}} . {{h_a}, {h_b}}]];

Rm_{a_,b_} := Rp_{a,b} /. T_a → 1 / T_a;
```

GGProgram

```
GG[g_GD, k_, F_, BB_] := Module[{n = 2 Length@g + Length@BB, y, cuts, rr, γ0, γ},
    γ0 = G[t_{n+1} h_{n+1}] Times @@ g /. {Ap → Rp, Am → Rm};
    γ0 *= G[Sum[β_{a,b} t_a h_b, {a, BB}, {b, BB}]];
    Sum[γ = γ0;
     cuts = Cases[y, _Integer, ∞] ⋃ {n + 1};
     rr = Thread[cuts → Range[Length@cuts]];
     Do[If[! MemberQ[cuts, j], γ = γ // m_{j,j+1→j+1}], {j, n}];
     F[y /. rr, γ /. (v_)_a_ :↦ v_{a/.rr}],
     (*over*) {y, Subsets[List @@ g, k]}]];
GG[g_GD, k_, F_] := GG[g, k, F, {}];
```

GG441

```
GG[GD@Knot[4, 1], {1}, F]
```

GG441

$$F\left[\{Am_{1,2}\}, \begin{pmatrix} -\frac{-1+T_2-T_1T_2+T_3-T_1T_3-T_2T_3+T_1T_2T_3}{T_1T_3} & \frac{(-1+T_1)(1-T_2+T_1T_2)(-1+T_3)}{T_1T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1T_3} & \frac{-1+T_1+T_2-T_1T_2+T_3-T_2T_3+T_1T_2T_3}{T_1T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1)T_2(-1+T_3)}{T_3} & T_2 \end{pmatrix}\right] +$$

$$F\left[\{Am_{2,1}\}, \begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2T_1-T_2+T_1T_2}{-T_1-T_2+T_1T_2} & -\frac{(-1+T_2)(-1+T_1+T_2-2T_1T_2-T_2^2+T_1T_2^2)}{T_2(-T_1-T_2+T_1T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right] +$$

$$F\left[\{Ap_{1,2}\}, \begin{pmatrix} -\frac{1-2T_1-T_2+T_1T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2T_2+T_1T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] + F\left[\{Ap_{1,2}\}, \begin{pmatrix} 1 & \frac{(-1+T_1)(1-2T_2-T_3+T_2T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2T_2-T_3+T_2T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right]$$

AlexanderFunctional

```
FA[{x_}, γ_] := Simplify[
    Switch[x, Ap__, 1, Am__, -1] * Switch[x, _1,2, (γ2,2 γ3,3 - γ2,3 γ3,2) / (γ3,3 + γ1,3 γ3,2 - γ1,2 γ3,3),
      _2,1, (γ1,3 γ3,2 - γ1,2 γ3,3) / (γ3,2 - γ2,3 γ3,2 + γ2,2 γ3,3)] /. T_ → T];
GGA[K_, bb___] := GG[GD@K, {1}, FA, bb];
```

GGA441

```
Simplify@With[{K = Knot[4, 1]},
  {GGA[K], Alexander[K][T], T ∂_T Log[Alexander[K][T]]}]
```

GGA441

$$\left\{\frac{T(-3+2T)}{1-3T+T^2}, 3-\frac{1}{T}-T, \frac{-1+T^2}{1-3T+T^2}\right\}$$

GGATesting

```
Table[
 K → Simplify[GGA[K] - T ∂T Log[Alexander[K][T]]],
 {K, AllKnots@{3, 7}}]
```

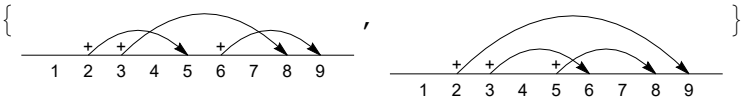
GGATesting

```
{3_1 → -1, 4_1 → 1, 5_1 → -2, 5_2 → -2, 6_1 → 0, 6_2 → 0,
 6_3 → 0, 7_1 → -3, 7_2 → -3, 7_3 → 4, 7_4 → 4, 7_5 → -3, 7_6 → -1, 7_7 → 2}
```

Invariance

```
Draw /@ {R3L = GD[Ap2,5, Ap3,8, Ap6,9], R3R = GD[Ap5,8, Ap2,9, Ap3,6]}
```

Invariance



Invariance

```
Simplify[GGA[R3L, {1, 4, 7, 10}] == GGA[R3R, {1, 4, 7, 10}] /. β10,b_ :> 1 - β1,b - β4,b - β7,b]
```

Invariance

```
True
```

OneCo441

```
GG[GD@Knot[4, 1], {1, 2}, F] /. F[y_List, γ_G] :> F[Column@y, γ]
```

OneCo441

$$F\left[\mathrm{Am}_{1,2},\begin{pmatrix} -\frac{-1+T_2-T_1\,T_2+T_3-T_1\,T_3-T_2\,T_3+T_1\,T_2\,T_3}{T_1\,T_3} & \frac{(-1+T_1)\,(1-T_2+T_1\,T_2)\,(-1+T_3)}{T_1\,T_3} & -\frac{(-1+T_1)\,(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)\,(-1+T_3)}{T_1\,T_3} & \frac{-1+T_1+T_2-T_1\,T_2+T_3-T_2\,T_3+T_1\,T_2\,T_3}{T_1\,T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2\,(-1+T_3)}{T_3} & -\frac{(-1+T_1)\,T_2\,(-1+T_3)}{T_3} & T_2 \end{pmatrix}\right]+$$

$$F\left[\mathrm{Am}_{2,1},\begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1\,T_2} & -\frac{(-1+T_1)\,(-1+T_2)^2}{T_2\,(-T_1-T_2+T_1\,T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2\,T_1-T_2+T_1\,T_2}{-T_1-T_2+T_1\,T_2} & -\frac{(-1+T_2)\,(-1+T_1+T_2-2\,T_1\,T_2-T_2^2+T_1\,T_2^2)}{T_2\,(-T_1-T_2+T_1\,T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right]+$$

$$F\left[\mathrm{Ap}_{1,2},\begin{pmatrix} -\frac{1-2\,T_1-T_2+T_1\,T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2\,(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1\,(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1\,(1-T_1-2\,T_2+T_1\,T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]+F\left[\mathrm{Ap}_{1,2},\begin{pmatrix} 1 & \frac{(-1+T_1)\,(1-2\,T_2-T_3+T_2\,T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)\,(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1\,(1-2\,T_2-T_3+T_2\,T_3)}{-1+T_2+T_3} & \frac{T_1\,(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2\,(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}\mathrm{Am}_{2,3}\\ \mathrm{Am}_{4,1}\end{matrix},\begin{pmatrix} \frac{1}{T_4} & 0 & -\frac{-1+T_1}{T_4} & 0 & 0 \\ 0 & 1 & \frac{T_1\,(-1+T_2)}{T_2} & 0 & -\frac{(-1+T_2)\,(-1+T_3)}{T_2} \\ 0 & 0 & \frac{T_1}{T_2} & 0 & -\frac{-1+T_3}{T_2} \\ \frac{-1+T_4}{T_4} & 0 & -\frac{(-1+T_1)\,(-1+T_4)}{T_4} & 1 & 0 \\ 0 & 0 & 0 & 0 & T_3 \end{pmatrix}\right]+F\left[\begin{matrix}\mathrm{Ap}_{1,2}\\ \mathrm{Ap}_{3,4}\end{matrix},\begin{pmatrix} 1 & -\frac{-1+T_1}{T_4} & 0 & -\frac{(-1+T_1)\,(-1+T_2)}{T_2} & 0 \\ 0 & \frac{T_1}{T_4} & 0 & \frac{T_1\,(-1+T_2)}{T_2} & 0 \\ 0 & -\frac{(-1+T_3)\,(-1+T_4)}{T_4} & 1 & -\frac{-1+T_3}{T_2} & 0 \\ 0 & \frac{T_3\,(-1+T_4)}{T_4} & 0 & \frac{T_3}{T_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}\mathrm{Ap}_{1,3}\\ \mathrm{Am}_{2,4}\end{matrix},\begin{pmatrix} 1 & 0 & 1-T_1 & 0 & 0 \\ 0 & -\frac{-1+T_4-T_2\,T_4+T_5-T_2\,T_5-T_4\,T_5+T_2\,T_4\,T_5}{T_2\,T_5} & 0 & \frac{-1+T_2}{T_2} & -\frac{(-1+T_2)\,(-1+T_4)}{T_2} \\ 0 & 0 & T_1 & 0 & 0 \\ 0 & -\frac{(-1+T_4)\,(-1+T_5)}{T_2\,T_5} & 0 & \frac{1}{T_2} & -\frac{-1+T_4}{T_2} \\ 0 & \frac{T_4\,(-1+T_5)}{T_5} & 0 & 0 & T_4 \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}\mathrm{Ap}_{1,3}\\ \mathrm{Am}_{4,2}\end{matrix},\begin{pmatrix} 1 & 0 & 1-T_1 & -\frac{(-1+T_1)\,(-1+T_3)}{T_3} & \frac{(-1+T_1)\,(-1+T_3)\,(-1+T_4)}{T_3} \\ 0 & \frac{1}{T_4} & 0 & 0 & 0 \\ 0 & 0 & T_1 & \frac{T_1\,(-1+T_3)}{T_3} & -\frac{T_1\,(-1+T_3)\,(-1+T_4)}{T_3} \\ 0 & \frac{-1+T_4}{T_4} & 0 & \frac{1}{T_3} & -\frac{-1+T_4}{T_3} \\ 0 & 0 & 0 & 0 & T_4 \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}\mathrm{Ap}_{2,4}\\ \mathrm{Am}_{1,3}\end{matrix},\begin{pmatrix} \frac{1}{T_4} & -\frac{-1+T_1}{T_4} & \frac{-1+T_1}{T_1} & 0 & 0 \\ -\frac{(-1+T_2)\,(-1+T_4)}{T_4} & \frac{-1+T_1+T_2-T_1\,T_2+T_4-T_2\,T_4+T_1\,T_2\,T_4}{T_4} & 0 & 1-T_2 & 0 \\ 0 & 0 & \frac{1}{T_1} & 0 & 0 \\ \frac{T_2\,(-1+T_4)}{T_4} & -\frac{(-1+T_1)\,T_2\,(-1+T_4)}{T_4} & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]+$$

$$F\left[\begin{matrix}\mathrm{Ap}_{2,4}\\ \mathrm{Am}_{3,1}\end{matrix},\begin{pmatrix} \frac{1}{T_3} & -\frac{-1+T_1}{T_3} & -\frac{(-1+T_1)\,(-1+T_2)}{T_2\,T_3} & 0 & 0 \\ 0 & T_1 & \frac{T_1\,(-1+T_2)}{T_2} & 1-T_2 & 0 \\ \frac{-1+T_3}{T_3} & -\frac{(-1+T_1)\,(-1+T_3)}{T_3} & -\frac{-1+T_1+T_2-T_1\,T_2-T_1\,T_3-T_2\,T_3+T_1\,T_2\,T_3}{T_2\,T_3} & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

Exporting the above as PDF files

The below is adapted from pensieve://Projects/WKO4/CellExport.nb.

```
SetDirectory["C:/drorbn/AcademicPensieve/Talks/NCSU-1604/"];
```



```
ConditionalExport[fname_String, rest___] := Module[{temp, exists},
```

```
    temp = "ConditionalExportTemporary" <> "." <> FileExtension[fname];
    exists = FileExistsQ[fname];
    Export[temp, rest];
    If[exists && FileByteCount[fname] === FileByteCount[temp],
     DeleteFile[temp],
     (* else *) Print["Exporting " <> fname <> "..."];
     If[exists, DeleteFile[fname]];
     RenameFile[temp, fname]
    ];
    fname
   ]

  Button["Export",
   SetOptions[$FrontEndSession, PrintingStyleEnvironment → "Working"];
   TagProperties[_] := {};
   TagProperties["GG441"] = {PageWidth → 8 / 0.65};
   TagProperties["OneCo441"] = {PageWidth → 11 / 0.65};
   Options[CellExport] = {
     PageWidth → 4 / 0.65, CellFilter → Identity, ExportBaseFilename → Automatic,
     ExportFormat → ".pdf", ExportOptions → {}, Split → False
    };
   CellExport[tag_String, opts___Rule] := CellExport[
     NotebookGet[EvaluationNotebook[]],
     tag, opts
    ];
   CellExport[nb_Notebook, tag_String] := CellExport[nb, tag, TagProperties[tag]];
   CellExport[nb_Notebook, tag_String, OptionsPattern[]] := Module[
     {cells, cell, filename, format},
     filename = OptionValue[ExportBaseFilename] /. Automatic → tag;
     format = OptionValue[ExportFormat];
     cells = OptionValue[CellFilter][Cases[
        nb, c_Cell /; FreeQ[List @@ c, Cell] && ! FreeQ[c, CellTags → tag],
        Infinity
       ]];
     If[! OptionValue[Split],
      If[Length[cells] ≥ 1,
       If[Length[cells] == 1,
        cells = Append[First[cells], PageWidth → 1.2 × 72 OptionValue[PageWidth]],
        cells = Cell[CellGroup[cells], PageWidth → 72 OptionValue[PageWidth]]
       ];
       ConditionalExport[
        filename <> format, cells,
        ImageResolution → 300,
        OptionValue[ExportOptions]
       ]
      ],
      k = 0;
      Table[
       ++k;
       ConditionalExport[
        filename <> "-" <> ToString[k] <> format,
```

```
        Append[cell, PageWidth → 72 OptionValue[PageWidth]],
        ImageResolution → 300,
        OptionValue[ExportOptions]
       ],
       {cell, cells}
      ]
     ]
    ];
  nb = NotebookGet[EvaluationNotebook[]];
  tags = Cases[nb, (CellTags → tag_) :> tag, Infinity] // Union;
  CellExport /@ tags;
  Print["Done."]
]
```

Export

```
Exporting V2.pdf...
Exporting V3.pdf...
```