5. Some Problems in Heaven

Unfortunately, $\dim \mathcal{A}(\mathcal{X}, X) = \dim \Lambda(\mathcal{X}, X) = 4^{|X|}$ is big. Fortunately, we have the following theorem, a version of one of the main results in Halacheva's thesis, [Ha1, Ha2]:

Theorem. Working in $\Lambda(\mathcal{X} \cup X)$, if $w = \omega \mathbb{e}^\lambda$ is a balanced Gaussian (namely, a scalar ω times the exponential of a quadratic $\lambda = \sum_{\zeta \in \mathcal{X}, z \in X} \alpha_{\zeta,z} \zeta z$), then generically so is $c_{x,\xi}\mathbb{e}^\lambda$.

(This is great news! The space of balanced quadratics is only $|\mathcal{X}||X|$-dimensional!)

Proof. Recall that $c_{x,\xi}\colon (1, \xi, x, x\xi)w' \mapsto (1, 0, 0, 1)w'$, write $\lambda = \mu + \eta x + \xi y + \alpha \xi x$, and ponder $\mathbb{e}^\lambda =$

$\eta x \xi y = q x \xi \eta y$

$$\ldots + \frac{1}{k!}\underbrace{(\mu + \eta x + \xi y + \alpha\xi x)(\mu + \eta x + \xi y + \alpha\xi x)\cdots(\mu + \eta x + \xi y + \alpha\xi x)}_{k \text{ factors}} + \ldots.$$

Aside: $x \eta y \xi = q^{-1} x \xi \eta y = -x \xi y \eta$

Then $c_{x,\xi}\mathbb{e}^\lambda$ has three contributions:

- ✓ $\mathbb{e}^\mu$, from the term proportional to 1 (namely, independent of ξ and x) in $\mathbb{e}^\lambda$
- ✓ ~~$+\alpha\mathbb{e}^\mu$~~ $-q\alpha e^\mu$, from the term proportional to $x\xi$, where the x and the ξ come from the same factor above.
- $q\eta y \mathbb{e}^\mu$, from the term proportional to $x\xi$, where the x and the ξ come from different factors above.

So $c_{x,\xi}\mathbb{e}^\lambda = \mathbb{e}^\mu(1 - q\alpha + q\eta y) = (1 - q\alpha)\mathbb{e}^\mu(1 + q\eta y/(1 - q\alpha)) = (1 - q\alpha)\mathbb{e}^\mu \mathbb{e}^{q\eta y/(1-q\alpha)} = (1 - q\alpha)\mathbb{e}^{\mu + q\eta y/(1-q\alpha)}$.

∎