Pensieve header: A talk and a program about Archibald- ($\mathcal A$-) and Γ-calculus and the Halacheva map between them; the Γ part. Continues pensieve://2021-03/

Γ-Calculus

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\begin{frame}{\LARGE 6. An Implementation of Γ.}
If I didn't implement I wouldn't believe myself.
\vskip 2mm
Written in Mathematica~\cite{Wolfram:Mathematica}, available as the notebook {\tt Gamma.nb} at \url{http://drorbn.net/mo21/ap}. Code lines are highlighted in grey, demo lines are plain.
We start with canonical forms for quadratics with rational function coefficients:

pdf

```
In[∘]:= CCF[ℰ_] := Factor[ℰ];
CF[ℰ_] := Module[{vs = Union@Cases[ℰ, (ξ | x)_, ∞]},
   Total[(CCF[#[[2]]] (Times @@ vs^#[[1]])) & /@ CoefficientRules[ℰ, vs]]];
```

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\end{frame}
\begin{frame}\null
Multiplying and comparing Γ objects:

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```
In[∘]:= Γ /: Γ[is1_, os1_, cs1_, ω1_, λ1_] Γ[is2_, os2_, cs2_, ω2_, λ2_] :=
  Γ[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], ω1 ω2, λ1 + λ2]
Γ /: Γ[is1_, os1_, _, ω1_, λ1_] ≡ Γ[is2_, os2_, _, ω2_, λ2_] := TrueQ[
  (Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧ Simplify[ω1 == ω2] ∧ CF@λ1 == CF@λ2]
```

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No rules for linear operations!
\end{frame}
\begin{frame}\null
Contractions:

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```
In[∘]:= c_{h_,t_}@Γ[is_, os_, cs_, ω_, λ_] := Module[{α, η, y, μ},
   α = ∂_{ξ_t,x_h} λ; μ = λ /. ξ_t | x_h → 0;
   η = ∂_{x_h} λ /. ξ_t → 0; y = ∂_{ξ_t} λ /. x_h → 0;
   Γ[
     DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}],
     CCF[(1 - α) ω], CF[μ + η y / (1 - α)]
    ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]]];
c@Γ[is_, os_, cs_, ω_, λ_] := Fold[c_{#2,#2}[#1] &, Γ[is, os, cs, ω, λ], is ⋂ os]
```

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\end{frame}

\begin{frame}\null
The crossings and the point:

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```
In[∘]:= Γ[X_{i_,j_,k_,l_}[S_, T_]] := Γ[{l, i}, {j, k},
    ⟨|ξ_i → S, x_j → T, x_k → S, ξ_l → T|⟩, T^{-1/2}, CF[{ξ_l, ξ_i}.(1 1-T; 0 T).{x_j, x_k}]];
  Γ[X̄_{i_,j_,k_,l_}[S_, T_]] := Γ[{i, j}, {k, l},
    ⟨|ξ_i → S, ξ_j → T, x_k → S, x_l → T|⟩, T^{1/2}, CF[{ξ_i, ξ_j}.(T^-1 0; 1-T^-1 1).{x_k, x_l}]];
  Γ[X_{i_,j_,k_,l_}] := Γ[X_{i,j,k,l}[τ_i, τ_l]];
  Γ[X̄_{i_,j_,k_,l_}] := Γ[X̄_{i,j,k,l}[τ_i, τ_j]];
```

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```
In[∘]:= Γ[P_{i_,j_}[T_]] := Γ[{i}, {j}, ⟨|ξ_i → T, x_j → T|⟩, 1, ξ_i x_j];
  Γ[P_{i_,j_}] := Γ[P_{i,j}[τ_i]];
```

tex

\end{frame}
\begin{frame}\null
Automatic intelligent contractions:

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```
In[∘]:= Γ[{γ_Γ}] := c[γ];
  Γ[{γ1_Γ, γs__Γ}] := Module[{γ2},
    γ2 = First@MaximalBy[{γs}, Length[γ1⟦1⟧ ⋂ #⟦2⟧] + Length[γ1⟦2⟧ ⋂ #⟦1⟧] &];
    Γ[Join[{c[γ1 γ2]}, DeleteCases[{γs}, γ2]]] ]
  Γ[Os_List] := Γ[Γ /@ Os]
```

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\end{frame}
\begin{frame}\null
Conversions $\calA\leftrightarrow\Gamma$:

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```
In[∘]:= Γ@𝒜[is_, os_, cs_, w_] := Module[{i, j, ω = Coefficient[w, Wedge[]]},
     Γ[is, os, cs, ω, Sum[Cancel[-Coefficient[w, x_j ∧ ξ_i] ξ_i x_j / ω], {i, is}, {j, os}]]
    ];
  𝒜@Γ[is_, os_, cs_, ω_, λ_] := 𝒜[is, os, cs, Expand[ω WExp[Expand[λ] /. ξ_a_ x_b_ :> ξ_a ∧ x_b]]];
```

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The conversions are inverses of each other:

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In[∘]:= $$\gamma = \Gamma[\{1, 2, 3\}, \{1, 2, 3\}, \{x_1 \to \tau_1, x_2 \to \tau_2, x_3 \to \tau_3, \xi_1 \to \tau_1, \xi_2 \to \tau_2, \xi_3 \to \tau_3\}, \omega, a_{11}\, x_1\, \xi_1 + a_{12}\, x_2\, \xi_1 + a_{13}\, x_3\, \xi_1 + a_{21}\, x_1\, \xi_2 + a_{22}\, x_2\, \xi_2 + a_{23}\, x_3\, \xi_2 + a_{31}\, x_1\, \xi_3 + a_{32}\, x_2\, \xi_3 + a_{33}\, x_3\, \xi_3];$$
$$\Gamma@\mathcal{A}@\gamma == \gamma$$

Out[∘]=
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True

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The conversions commute with contractions:

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In[∘]:= $$\Gamma@c_{3,3}@\mathcal{A}@\gamma \equiv c_{3,3}@\gamma$$

Out[∘]=
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True

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\end{frame}

The Naik-Stanford Double Delta Move (again)

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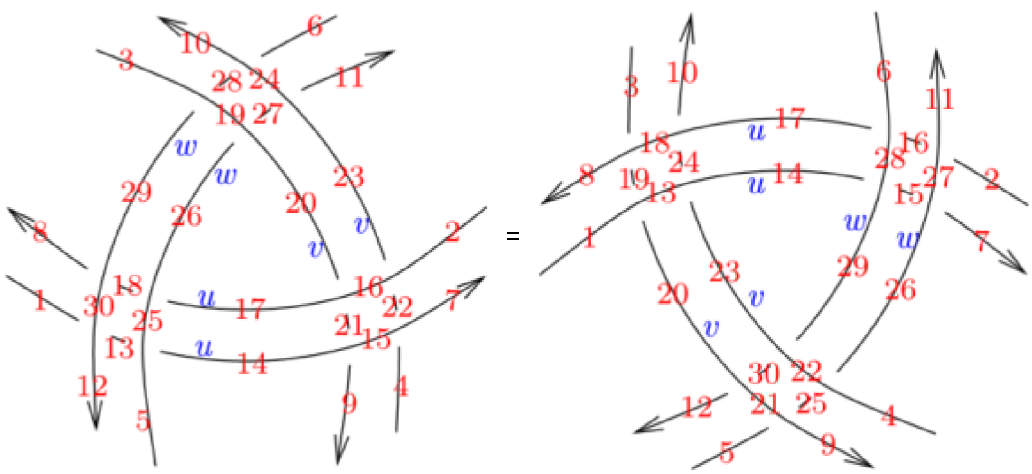
\begin{frame} {\large The Naik-Stanford Double Delta Move (again)}
\[\scalebox{0.8}{\input{figs/NaikStanford.pdf_t}} \]



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In[∘]:= $$\text{Timing}\big[\Gamma@\{X_{6,10,28,24}[w, v], \overline{X}_{28,3,29,19}[w, v], X_{26,20,27,19}[w, v], \overline{X}_{27,23,11,24}[w, v], X_{1,12,13,30}[u, w], \overline{X}_{13,5,14,25}[u, w], X_{17,26,18,25}[u, w], \overline{X}_{18,29,8,30}[u, w], X_{4,7,22,15}[v, u], \overline{X}_{22,2,23,16}[v, u], X_{20,17,21,16}[v, u], \overline{X}_{21,14,9,15}[v, u]\} \equiv$$
$$\Gamma@\{X_{5,9,25,21}[w, v], \overline{X}_{25,4,26,22}[w, v], X_{29,23,30,22}[w, v], \overline{X}_{30,20,12,21}[w, v], X_{2,11,16,27}[u, w], \overline{X}_{16,6,17,28}[u, w], X_{14,29,15,28}[u, w], \overline{X}_{15,26,7,27}[u, w], X_{3,8,19,18}[v, u], \overline{X}_{19,1,20,13}[v, u], X_{23,14,24,13}[v, u], \overline{X}_{24,17,10,18}[v, u]\}\big]$$

Out[∘]=
pdf

{0.75, True}

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\end{frame}

Aside added pot-mortem:

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In[∘]:= $\texttt{Timing}[\mathcal{A}@\Gamma@\{X_{6,10,28,24}[w,v], \overline{X}_{28,3,29,19}[w,v], X_{26,20,27,19}[w,v], \overline{X}_{27,23,11,24}[w,v],$
$X_{1,12,13,30}[u,w], \overline{X}_{13,5,14,25}[u,w], X_{17,26,18,25}[u,w], \overline{X}_{18,29,8,30}[u,w],$
$X_{4,7,22,15}[v,u], \overline{X}_{22,2,23,16}[v,u], X_{20,17,21,16}[v,u], \overline{X}_{21,14,9,15}[v,u]\} \equiv$
$\mathcal{A}@\Gamma@\{X_{5,9,25,21}[w,v], \overline{X}_{25,4,26,22}[w,v], X_{29,23,30,22}[w,v], \overline{X}_{30,20,12,21}[w,v],$
$X_{2,11,16,27}[u,w], \overline{X}_{16,6,17,28}[u,w], X_{14,29,15,28}[u,w], \overline{X}_{15,26,7,27}[u,w],$
$X_{3,8,19,18}[v,u], \overline{X}_{19,1,20,13}[v,u], X_{23,14,24,13}[v,u], \overline{X}_{24,17,10,18}[v,u]\}]$

Out[∘]=
pdf

{185.094, True}

Conway’s Third Identity (again)

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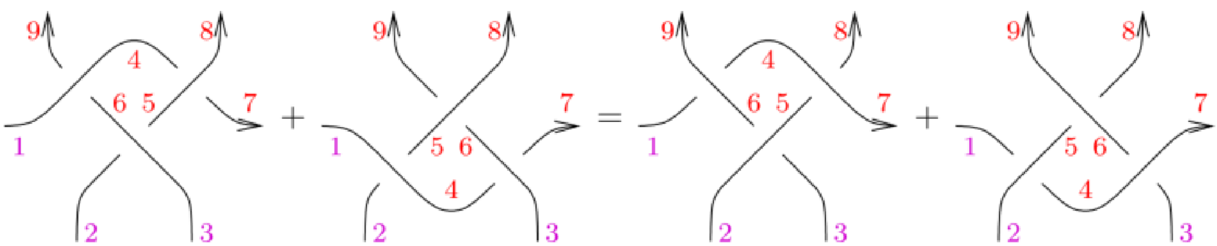
\begin{frame}{\large Conway’s Third Identity}
\[\input{figs/C3.pdf_t} \]



tex

Sorry, Γ has nothing to say about that...’
\end{frame}

Post-Mortem Additions

Yet,

In[∘]:= $\texttt{Timing}[\mathcal{A}@\{X_{6,4,9,1}, \overline{X}_{4,5,7,8}, \overline{X}_{2,3,5,6}\} + \mathcal{A}@\{X_{2,4,5,1}, \overline{X}_{4,3,7,6}, X_{6,8,9,5}\} \equiv$
$\mathcal{A}@\{\overline{X}_{1,6,4,9}, X_{5,7,8,4}, X_{3,5,6,2}\} + \mathcal{A}@\{\overline{X}_{1,2,4,5}, X_{3,7,6,4}, \overline{X}_{5,6,8,9}\}]$

Out[∘]=

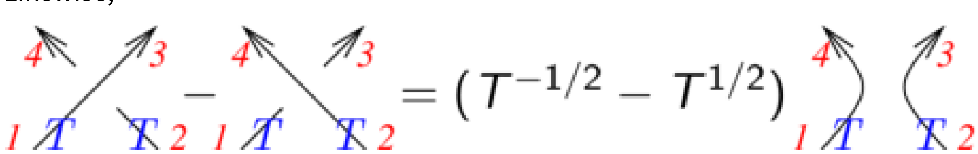
{0.109375, True}

In[∘]:= $\texttt{Timing}[\mathcal{A}@\Gamma@\{X_{6,4,9,1}, \overline{X}_{4,5,7,8}, \overline{X}_{2,3,5,6}\} + \mathcal{A}@\Gamma@\{X_{2,4,5,1}, \overline{X}_{4,3,7,6}, X_{6,8,9,5}\} \equiv$
$\mathcal{A}@\Gamma@\{\overline{X}_{1,6,4,9}, X_{5,7,8,4}, X_{3,5,6,2}\} + \mathcal{A}@\Gamma@\{\overline{X}_{1,2,4,5}, X_{3,7,6,4}, \overline{X}_{5,6,8,9}\}]$

Out[∘]=

{0.0625, True}

Likewise,



In[∘]:= $\text{Timing}\left[\mathcal{A}@\{X_{2,3,4,1}[T,T]\} - \mathcal{A}@\{\overline{X}_{1,2,3,4}[T,T]\} \equiv \left(T^{-1/2} - T^{1/2}\right) \mathcal{A}@\{P_{1,4}[T], P_{2,3}[T]\}\right]$

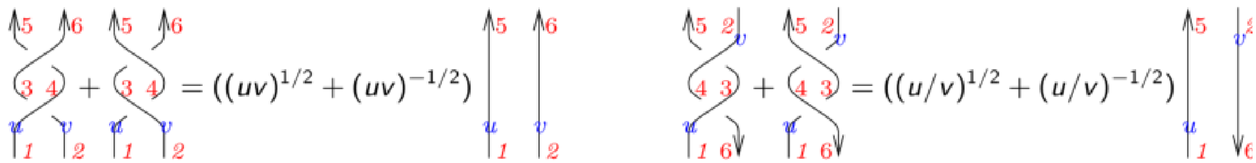
Out[∘]=

{0.015625, True}

In[∘]:= $\text{Timing}\left[\mathcal{A}@\Gamma@\{X_{2,3,4,1}[T,T]\} - \mathcal{A}@\Gamma@\{\overline{X}_{1,2,3,4}[T,T]\} \equiv \left(T^{-1/2} - T^{1/2}\right) \mathcal{A}@\Gamma@\{P_{1,4}[T], P_{2,3}[T]\}\right]$

Out[∘]=

{0., True}



In[∘]:= $\text{Timing}\Big[\Big\{$

$\mathcal{A}@\{X_{2,4,3,1}[v,u], X_{4,6,5,3}\} + \mathcal{A}@\{\overline{X}_{1,2,4,3}[u,v], \overline{X}_{3,4,6,5}\} \equiv$
$\left(u^{1/2} v^{1/2} + u^{-1/2} v^{-1/2}\right) \mathcal{A}@\{P_{1,5}[u], P_{2,6}[v]\},$

$\mathcal{A}@\{\overline{X}_{4,1,6,3}[v,u], \overline{X}_{3,2,5,4}\} + \mathcal{A}@\{X_{1,6,3,4}[u,v], X_{2,5,4,3}\} \equiv$
$\left(u^{1/2} v^{-1/2} + u^{-1/2} v^{1/2}\right) \mathcal{A}@\{P_{1,5}[u], P_{2,6}[v]\}\Big\}\Big]$

Out[∘]=

{0.015625, {True, True}}

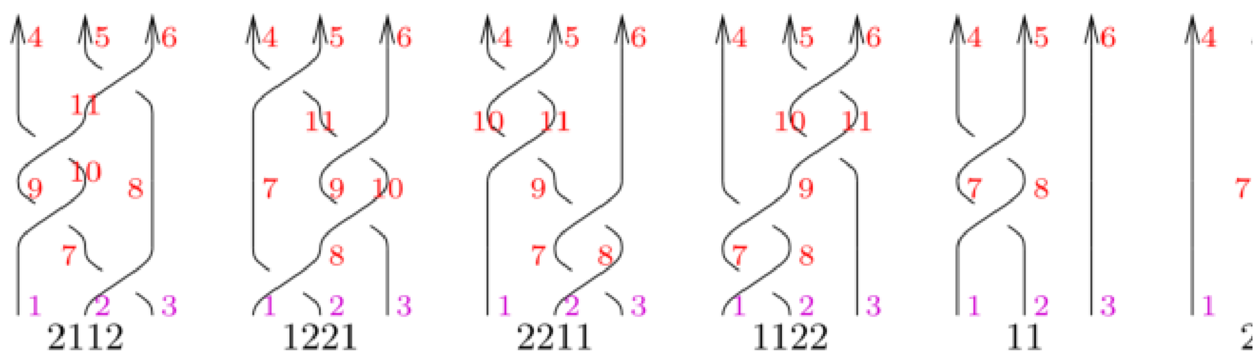
In[∘]:= $\text{Timing}\Big[\Big\{$

$\mathcal{A}@\Gamma@\{X_{2,4,3,1}[v,u], X_{4,6,5,3}\} + \mathcal{A}@\Gamma@\{\overline{X}_{1,2,4,3}[u,v], \overline{X}_{3,4,6,5}\} \equiv$
$\left(u^{1/2} v^{1/2} + u^{-1/2} v^{-1/2}\right) \mathcal{A}@\Gamma@\{P_{1,5}[u], P_{2,6}[v]\},$

$\mathcal{A}@\Gamma@\{\overline{X}_{4,1,6,3}[v,u], \overline{X}_{3,2,5,4}\} + \mathcal{A}@\Gamma@\{X_{1,6,3,4}[u,v], X_{2,5,4,3}\} \equiv$
$\left(u^{1/2} v^{-1/2} + u^{-1/2} v^{1/2}\right) \mathcal{A}@\Gamma@\{P_{1,5}[u], P_{2,6}[v]\}\Big\}\Big]$

Out[∘]=

{0.03125, {True, False}}



In[ ]:=
```
Timing[
 𝒜2112 = 𝒜@{X3,8,7,2, X7,10,9,1, X10,11,4,9, X8,6,5,11};
 𝒜1221 = 𝒜@{X2,8,7,1, X3,10,9,8, X10,6,11,9, X11,5,4,7};
 𝒜2211 = 𝒜@{X3,8,7,2, X8,6,9,7, X9,11,10,1, X11,5,4,10};
 𝒜1122 = 𝒜@{X2,8,7,1, X8,9,4,7, X3,11,10,9, X11,6,5,10};
 𝒜11 = 𝒜@{X2,8,7,1, X8,5,4,7, P3,6};  𝒜22 = 𝒜@{X3,8,7,2, X8,6,5,7, P1,4};
 𝒜ϕ = 𝒜@{P1,4, P2,5, P3,6};
 g+[z_] := z^(1/2) + z^(-1/2); g-[z_] := z^(1/2) - z^(-1/2);
 g+[τ1] g-[τ2] 𝒜2112 - g-[τ2] g+[τ3] 𝒜1221 - g-[τ3 / τ1] (𝒜2211 + 𝒜1122) +
   g-[τ2 τ3 / τ1] g+[τ3] 𝒜11 - g+[τ1] g-[τ1 τ2 / τ3] 𝒜22 ≡ g-[τ3^2 / τ1^2] 𝒜ϕ
]
```

Out[ ]=
```
{0.3125, True}
```

In[ ]:=
```
Timing[
 𝒜2112 = 𝒜@Γ@{X3,8,7,2, X7,10,9,1, X10,11,4,9, X8,6,5,11};
 𝒜1221 = 𝒜@Γ@{X2,8,7,1, X3,10,9,8, X10,6,11,9, X11,5,4,7};
 𝒜2211 = 𝒜@Γ@{X3,8,7,2, X8,6,9,7, X9,11,10,1, X11,5,4,10};
 𝒜1122 = 𝒜@Γ@{X2,8,7,1, X8,9,4,7, X3,11,10,9, X11,6,5,10};
 𝒜11 = 𝒜@Γ@{X2,8,7,1, X8,5,4,7, P3,6};  𝒜22 = 𝒜@Γ@{X3,8,7,2, X8,6,5,7, P1,4};
 𝒜ϕ = 𝒜@Γ@{P1,4, P2,5, P3,6};
 g+[z_] := z^(1/2) + z^(-1/2); g-[z_] := z^(1/2) - z^(-1/2);
 g+[τ1] g-[τ2] 𝒜2112 - g-[τ2] g+[τ3] 𝒜1221 - g-[τ3 / τ1] (𝒜2211 + 𝒜1122) +
   g-[τ2 τ3 / τ1] g+[τ3] 𝒜11 - g+[τ1] g-[τ1 τ2 / τ3] 𝒜22 ≡ g-[τ3^2 / τ1^2] 𝒜ϕ
]
```

Out[ ]=
```
{0.21875, True}
```