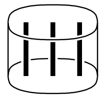
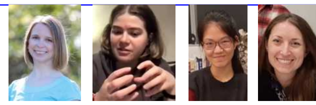
Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef

Preliminary Definitions. Fix $p \in \mathbb{N}$ and $\mathbb{F} = \mathbb{Q}/\mathbb{C}$. Let $D_p := D^2 \setminus (p \text{ pts})$, and let the Pole Dance Studio be $PDS_p := D_p \times I$.



PDS_3

Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau [Ma] and Alekseev and Naef [AN1].



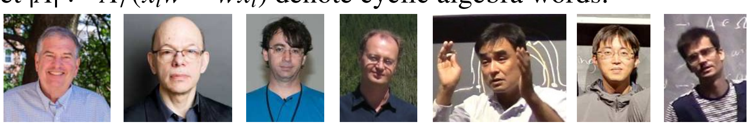
Dancso Hogan Liu Scherich

We study the pole-strand and strand-strand double filtration on the space of tangles in a pole dance studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and simplify substantial parts of the work of the aforementioned authors on expansions for the Goldman-Turaev Lie bi-algebra.

Jessica, Nancy, Tamara, Zsuzsi, & Dror in PDS_4

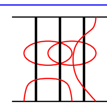
⟹ Expansions $W\colon FG\langle X_i\rangle \to FA\langle x_i\rangle$:
Magnus: $X_i \mapsto 1 + x_i$, $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$
Exponential: $X_i^{\pm 1} \mapsto e^{\pm x_i}$.

Definitions. Let $\pi := FG\langle X_1, \dots, X_p\rangle$ be the free group (of deformation classes of based curves in D_p), $\bar\pi$ be the framed free group (deformation classes of based immersed curves), $|\pi|$ and $|\bar\pi|$ denote $\mathbb{F}$-linear combinations of cyclic words ($|x_i w| = |w x_i|$, unbased curves), $A := FA\langle x_1, \dots, x_p\rangle$ be the free associative algebra, and let $|A| := A/(x_i w = w x_i)$ denote cyclic algebra words.



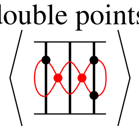
Theorem 1 (Goldman, Turaev, Massuyeau, Alekseev, Kawazumi, Kuno, Naef). $|\bar\pi|$ and $|A|$ are Lie bialgebras, and there is a "homomorphic expansion" $W\colon |\bar\pi| \to |A|$: a morphism of Lie bialgebras with $W(|X_i|) = 1 + |x_i| + \dots$.

Further Definitions. • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{F}\langle\text{framed tangles in } PDS_p\rangle$.



• $\mathcal{K}_t^s$:=(the image via (double point) $\to$ (overcrossing) $-$ (undercrossing) of tangles in PDS_p that have t double points, of which s are strand-strand).

E.g.,



$$\mathcal{K}_5^2(\bigcirc) = \langle \dots \rangle / . \quad (\text{double point}) \to (\text{overcrossing}) - (\text{undercrossing})$$

• $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. Most important, $\mathcal{K}^{/1}(\bigcirc) = |\bar\pi|$, and there is $P\colon \mathcal{K}(\bigcirc) \to |\bar\pi|$.

• $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

Fact 1. The Kontsevich Integral is an "expansion" $Z\colon \mathcal{K} \to \mathcal{A}$, compatible with several noteworthy structures.

Fact 2 (Le-Murakami, [LM1]). Z satisfies the strand-strand HOMFLY-PT relations: It descends to $Z_H\colon \mathcal{K}_H \to \mathcal{A}_H$, where

$$\mathcal{K}_H := \mathcal{K} \Big/ \Big((\text{overcrossing}) - (\text{undercrossing}) = (e^{\hbar/2} - e^{-\hbar/2}) \cdot (\text{smoothing}) \Big)$$

$$\mathcal{A}_H := \mathcal{A}/\big((\text{chord}) = \hbar\, (\text{smoothed chord}) \text{ or } (\text{chord}) = \hbar\, (\text{crossed smoothing}) \big)$$

and $\deg \hbar = (1, 1)$.

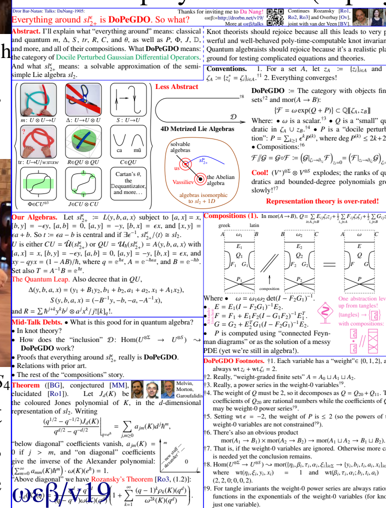
Proof of Fact 2. $Z(\text{overcrossing}) - Z(\text{undercrossing}) = (\text{crossing}) \cdot \left(e^{\mathsf{H}/2} - e^{-\mathsf{H}/2}\right)$

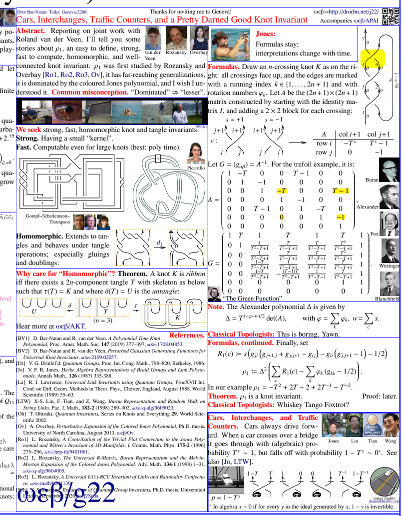
$$= (\text{crossing}) \cdot \left(e^{\hbar \times/2} - e^{-\hbar\times/2}\right) = \left(e^{\hbar/2} - e^{-\hbar/2}\right) (\text{smoothing}). \quad \square$$

Le, Murakami

Other Passions. With Roland van der Veen, I use "solvable approximation" and "Perturbed Gaussian Differential Operators" to unveil simple, strong, fast to compute, and topologically meaningful knot invariants near the Alexander polynomial. ($\subset$ polymath!)

van der Veen





Key 1. $W\colon |\bar\pi| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar\pi|$, **Lemma 1.** "Division by $\hbar$" is well-defined.

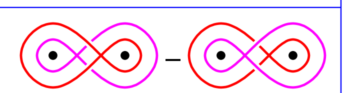
$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

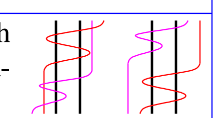
$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

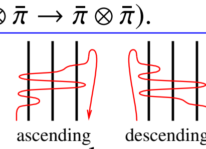
Example 1. With $\gamma_1, \gamma_2 \in |\pi|$ (or $|\bar\pi|$) set $\lambda_0(\gamma_1, \gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1, \gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of γ_i. Then η_1 is the Goldman bracket! Note that here λ_0 and λ_1 are not well-defined, yet η_1 is.

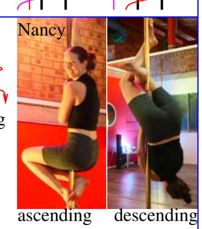


Example 2. With $\gamma_1, \gamma_2 \in \pi$ (or $\bar\pi$) and with λ_0, λ_1 as on the right, we get the "double bracket" $\eta_2\colon \pi \otimes \pi \to \pi \otimes \pi$ (or $\bar\pi \otimes \bar\pi \to \bar\pi \otimes \bar\pi$).

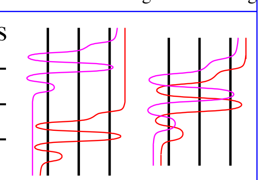


Example 3. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar\pi \to \bar\pi \otimes |\bar\pi|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.





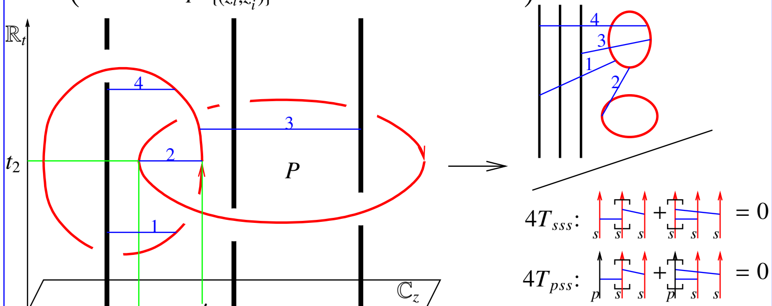
Example 4 [Ma]. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar\pi \to \bar\pi \otimes \bar\pi$. After some massaging, it too becomes the Turaev cobracket.



The rest is essentially **Exercises: 1.** Lemma 1? **2.** $\mathcal{A}$? **3.** Fact 2? **4.** $\mathcal{A}^{/1}$? Especially, $\mathcal{A}^{/1}(\bigcirc) \cong |A|$! **5.** Explain why Kontsevich likes our λ's. **6.** Figure out η_i^a, $i = 1, \dots, 4$.

Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z = \left(\sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \oint_{\substack{t_1<\ldots<t_m \\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \right)^{\sim} \in \mathcal{A}$$



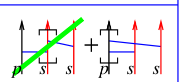
graded by the number of chords
filtered by the number of ss chords

Comments on the Kontsevich Integral.

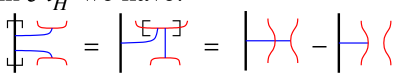
1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never pp chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. Z is an "expansion".
5. Z respects the ss filtration and so descends to $Z^{/s}\colon \mathcal{K}^{/s} \to \mathcal{A}^{/s}$.

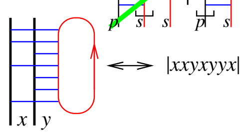
Comments on $\mathcal{A}$.

In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc) = |A|$!

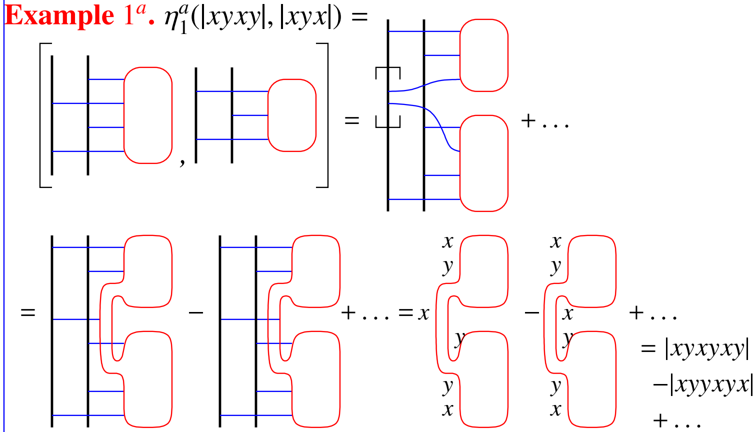


In $\mathcal{A}_H^{/2}$ we have:

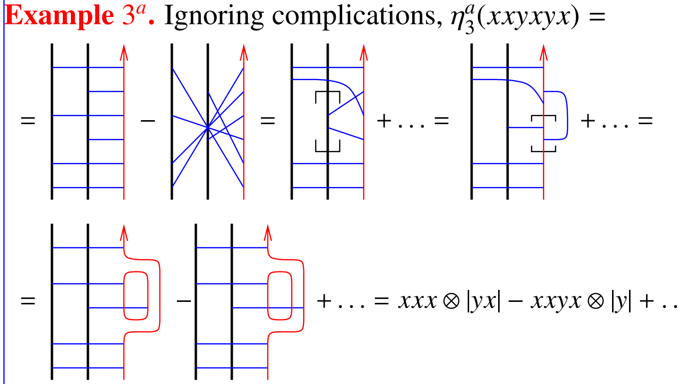




Example 1^a. $\eta_1^a(|xyxy|, |xyx|) =$



Example 3^a. Ignoring complications, $\eta_3^a(xxyxyx) =$



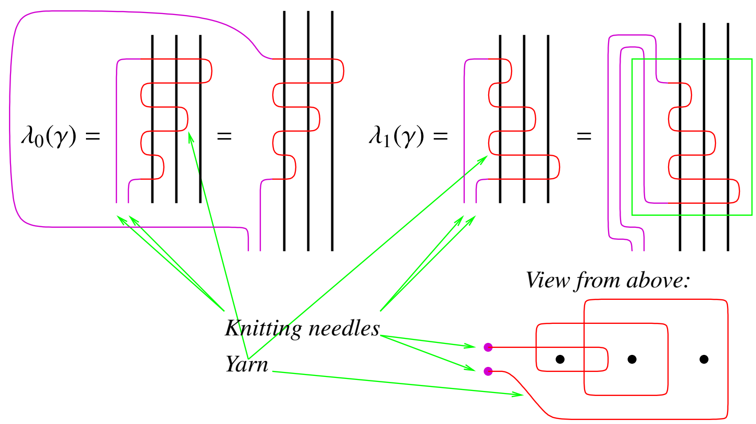
Proof of Lemma 1. We partially prove Theorem 2 instead:

Theorem 2. $\operatorname{gr}^\bullet \mathcal{K}_H \cong \mathbb{F}[\![\hbar]\!] \otimes (\mathcal{K}^{/1})_0$.

Proof mod $\hbar^2$. The map $\leftarrow$ is obvious. To go $\to$, map $\mathcal{K}_H \to \mathbb{F}[\![\hbar]\!] \otimes \mathcal{K}^{/1}$ using $\overcrossing \mapsto \times + \frac{\hbar}{2})($ and $\undercrossing \mapsto \times - \frac{\hbar}{2})($ and apply the functor $\operatorname{gr}^\bullet$.

Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.



Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara Vergne

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