Dror Bar-Natan: Talks: LearningSeminarOnCategorification-2006:

The Alexander Polynomial is a Quantum Invariant in a Different Way

Thanks for inviting me to speak in my basement!

ωεβ:=http://drorbn.net/cat20/

▶ On a chat window here I saw a comment “Alexander is the quantum $gl(1|1)$ invariant”. I have an opinion about this, and I’d like to share it. First, some stories.

I left the wonderful subject of Categorification nearly 15 years ago. It got crowded, lots of very smart people had things to say, and I feared I will have nothing to add. Also, clearly the next step was to categorify all other “quantum invariants”. Except it was not clear what “categorify” means. Worse, I felt that I (perhaps “we all”) didn’t understand “quantum invariants” well enough to try to categorify them, whatever that might mean.

I still feel that way! I learned a lot since 2006, yet I’m still not comfortable with quantum algebra, quantum groups, and quantum invariants. I still don’t feel that I know what God had in mind when She created this topic.

Yet I’m not here to rant about my philosophical quandaries, but only about things that I learned about the Alexander polynomial after 2006.

Yes, the Alexander polynomial fits within the Dogma, “one invariant for every Lie algebra and representation” (it’s $gl(1|1)$, I hear). But it’s better to think of it as a quantum invariant arising by other means, outside the Dogma.

Alexander comes from (or in) practically any non-Abelian Lie algebra. Foremost from the not-even-semi-simple 2D “$ax+b$” algebra. You get a polynomially-sized extension to tangles using some lovely formulas (can you categorify them?). It generalizes to higher dimensions and it has an organized family of siblings. (There are some questions too, beyond categorification).

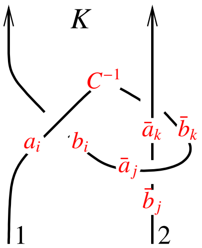
I note the spectacular existing categorification of Alexander by Ozsváth and Szabó. The theorems are proven and a lot they say, the programs run and fast they run. Yet if that’s where the story ends, She has abandoned us. Or at least abandoned me: a simpleton will never be able to catch up.

If you care only about categorification, the take-home from my talk will be a challenge: Categorify what I believe is the best Alexander invariant for tangles.

The Yang-Baxter Technique. Given an algebra U (typically some $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and suitable elements R, C,

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{with}\quad R^{-1}=\sum \bar{a}_i\otimes\bar{b}_i \quad\text{and}\quad C,C^{-1}\in U,$$

form
$$Z(K)=\sum_{i,j,k} a_iC^{-1}\bar{b}_k\bar{a}_jb_i\otimes\bar{b}_j\bar{a}_k.$$



Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

Gentle’s Agreement. Everything converges!

Example 1. Let $\mathfrak{a}:=L\langle a,x\rangle/([a,x]=x)$, $\mathfrak{b}:=\mathfrak{a}^\star=\langle b,y\rangle$, and $\mathfrak{g}:=\mathfrak{b}\rtimes\mathfrak{a}=\mathfrak{b}\oplus\mathfrak{a}$ with $[a,x]=x$, $[a,y]=-y$, $[b,\cdot]=0$, and $[x,y]=b$ and with $\deg(y,b,a,x)=(1,1,0,0)$. Let $U=\hat{\mathcal{U}}(\mathfrak{g})$ and

$$R:=\mathbb{e}^{b\otimes a+y\otimes x}\in U\otimes U \quad\text{or better}\quad R_{ij}:=\mathbb{e}^{b_ia_j+y_ix_j}\in U_i\otimes U_j, \quad\text{and}\quad C_i=\mathbb{e}^{-b_i/2}.$$

Theorem 1. With “scalars”:=power series in $\{b_i\}$ which are rational functions in $\{b_i\}$ and $\{B_i:=\mathbb{e}^{b_i}\}$,

a tangle w/o closed components

the “i over j” linking numbers (integers)

categorify us! scalars

$$Z(K)=\mathbb{O}_{ybax}\left(\omega^{-1}\mathbb{e}^{\underbrace{l^{ij}b_ia_j}_{L}+\underbrace{q^{ij}y_ix_j}_{Q}}(1+\epsilon P_1+\epsilon^2P_2+\dots)\right)$$

“normal ordering” at *ybax* order

a scalar; if K is a long knot, the Alexander poly $\Delta(T)$ categorify me!

a docile perturbation for other Lie algebras; semisimple algebras have a hidden parameter ϵ!

With Roland van der Veen

Continues Lev Rozansky

Example 2. Let $\mathfrak{h}:=A\langle p,x\rangle/([p,x]=1)$ be the Heisenberg algebra, with $C_i=\mathbb{e}^{t/2}$ and $R_{ij}=\mathbb{e}^{t/2}\mathbb{e}^{t(p_i-p_j)x_j}$.

I just told you the whole Alexander story! Everything else is details.

Claim. $R_{ij}=\mathbb{O}_{px}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right)$.

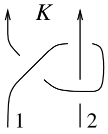
Theorem 2. $Z(K)=\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ where ω and the q^{ij} are rational functions in $T=\mathbb{e}^t$. In fact ω and ωq^{ij} are Laurent polynomials (categorify us!). When K is a long knot, ω is the Alexander polynomial.

Packaging. Write $\mathbb{O}_{px}\left(\omega^{-1}\mathbb{e}^{q^{ij}p_ix_j}\right)$ as

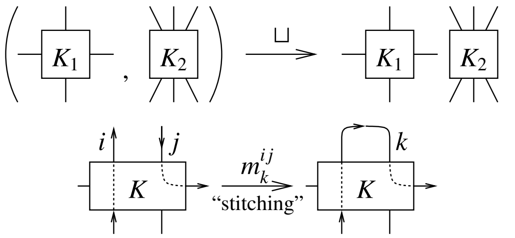
$$\mathbb{E}_{p_1,\dots,x_1,\dots}[\omega,Q]\leftrightarrow \begin{array}{c|cccc} \omega & x_1 & x_2 & \cdots \\ \hline p_1 & q^{11} & q^{12} & \cdots \\ p_2 & q^{21} & q^{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

The “First Tangle”. $Z(K)=$

$$\mathbb{E}_{12}\left[\frac{2T-1}{T},\frac{(T-1)(p_1-p_2)(Tx_1-x_2)}{2T-1}\right] = \begin{array}{c|cc} 2-T^{-1} & x_1 & x_2 \\ \hline p_1 & \frac{T(T-1)}{2T-1} & \frac{1-T}{2T-1} \\ p_2 & \frac{T(1-T)}{2T-1} & \frac{T-1}{2T-1} \end{array}$$



(v-)Tangles. Generated by $\{\nwarrow\!\!\!\!\nearrow, \nwarrow\!\!\!\!\nearrow\}$!



There’s also strand doubling and reversal…

Theorem 3. Full evaluation via

$$\left({}_i\nearrow_j,\ {}_j\nwarrow_i\right)\to\begin{array}{c|cc} 1 & x_i & x_j \\ \hline p_i & 0 & T^{\pm1}-1 \\ p_j & 0 & 1-T^{\pm1} \end{array} \qquad (1)\square$$

$$K_1\sqcup K_2\to\begin{array}{c|cc} \omega_1\omega_2 & X_1 & X_2 \\ \hline P_1 & A_1 & 0 \\ P_2 & 0 & A_2 \end{array} \qquad (2)\square$$

$$\begin{array}{c|ccc} \omega & x_i & x_j & \cdots \\ \hline p_i & \alpha & \beta & \theta \\ p_j & \gamma & \delta & \epsilon \\ \vdots & \phi & \psi & \Xi \end{array} \xrightarrow{hm_k^{ij}} \qquad (3)$$

$$\begin{array}{c|cc} (1+\gamma)\omega & x_k & \cdots \\ \hline p_k & 1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\epsilon}{1+\gamma} \\ \vdots & \psi+\frac{(1-\delta)\phi}{1+\gamma} & \Xi-\frac{\phi\epsilon}{1+\gamma} \end{array}$$

“Γ-calculus” relates via $A\leftrightarrow I-A^T$ and has slightly simpler formulas: $\omega\to(1-\beta)\omega$,

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}\to\begin{pmatrix} \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta} \\ \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta} \end{pmatrix}$$

Why Should You Categorify This? The simplest and fastest Alexander for tangles, easily generalizes to the multi-variable case, generalizes to v-tangles and w-tangles, generalizes to other Lie algebras. In fact, it’s in almost any Lie algebra, and you don’t even need to know what is $gl(1|1)$! But you’ll have to deal with denominators and/or divisions!

Note. Example 1 ⇝ Example 2 via $\mathfrak{g}\hookrightarrow\mathfrak{h}(t)$ via $(y,b,a,x)\mapsto(-tp,t,px,x)$.

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(p,x)^* = (\pi,\xi)$

The Generating Series $\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])\to\mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \underset{\mathcal{G}}{\xrightarrow{\sim}} \mathbb{Q}[z_B][\![\zeta_A]\!]\ni\mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0}\qquad\text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\mathrm{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\mathrm{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Examples. • $\mathcal{G}(id\colon\mathbb{Q}[p,x]\to\mathbb{Q}[p,x]) = \mathbb{e}^{\pi p+\xi x}$.
• Consider $R_{ij}\in(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]\cong\mathrm{Hom}\left(\mathbb{Q}[]\to\mathbb{Q}[p_i,x_i,p_j,x_j]\right)[\![t]\!]$. Then $\mathcal{G}(R_{ij}) = \mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j} = \mathbb{e}^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p,x\rangle/([p,x]=1)$, let $\mathbb{O}_i\colon\mathbb{Q}[p_i,x_i]\to\mathfrak{h}_i$ is the "p before x" PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i,x_i,p_j,x_j]\xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j}\mathfrak{h}_i\otimes\mathfrak{h}_j\xrightarrow{m_k^{ij}}\mathfrak{h}_k\xrightarrow{\mathbb{O}_k^{-1}}\mathbb{Q}[p_k,x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$.
Proof. Recall the "Weyl CCR" $\mathbb{e}^{\xi x}\mathbb{e}^{\pi p} = \mathbb{e}^{-\xi\pi}\mathbb{e}^{\pi p}\mathbb{e}^{\xi x}$, and find

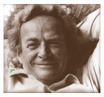
$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\pi_i p_i+\xi_i x_i+\pi_j p_j+\xi_j x_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1}\\ &= \mathbb{e}^{\pi_i p_i}\mathbb{e}^{\xi_i x_i}\mathbb{e}^{\pi_j p_j}\mathbb{e}^{\xi_j x_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\pi_i p_k}\mathbb{e}^{\xi_i x_k}\mathbb{e}^{\pi_j p_k}\mathbb{e}^{\xi_j x_k}/\!\!/\mathbb{O}_k^{-1}\\ &= \mathbb{e}^{-\xi_i\pi_j}\mathbb{e}^{(\pi_i+\pi_j)p_k}\mathbb{e}^{(\xi_i+\xi_j)x_k}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}.\end{aligned}$$

GDO := The category with objects finite sets and

$$\mathrm{mor}(A\to B) = \left\{\mathcal{L} = \omega\mathbb{e}^Q\right\}\subset\mathbb{Q}[\![\zeta_A,z_B]\!],$$

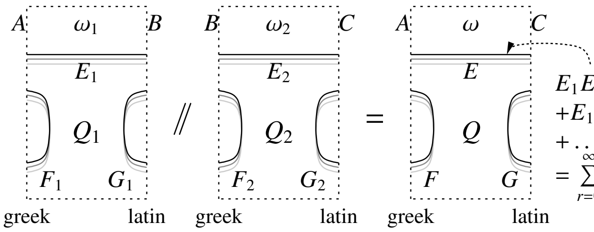
where: • ω is a scalar. • Q is a "small" quadratic in $\zeta_A\cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

Compositions. In mor$(A\to B)$,


R. Feynman

$$Q = \sum_{i\in A,j\in B}E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A}F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B}G_{ij}z_i z_j,$$

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \ldots$)



$$E_1E_2 + E_1F_2G_1E_2 + E_1F_2G_1F_2G_1E_2 + \ldots = \sum_{r=0}^{\infty}E_1(F_2G_1)^rE_2$$

where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^TG_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2\det(I-F_2G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := \mathbb{e}^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{O}_{p_jx_j}\left(\mathbb{e}^{(\mathbb{e}^t-1)(p_i-p_j)x_j}\right) =: \mathbb{O}(\Psi)$. We show that $\Phi_1 = \Phi_2$ in $(\mathfrak{h}_i\otimes\mathfrak{h}_j)[\![t]\!]$ by showing that both solve the ODE $\partial_t\Phi = (p_i-p_j)x_j\Phi$ with $\Phi|_{t=0} = 1$. For Φ_1 this is trivial. $\Phi_2|_{t=0} = 1$ is trivial, and

$$\partial_t\Phi_2 = \mathbb{O}(\partial_t\Psi) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi)$$

$$(p_i-p_j)x_j\Phi_2 = (p_i-p_j)x_j\mathbb{O}(\Psi) = (p_i-p_j)\mathbb{O}(x_j\Psi - \partial_{p_j}\Psi)$$
$$= \mathbb{O}\left((p_i-p_j)(x_j\Psi + (\mathbb{e}^t-1)x_j\Psi)\right) = \mathbb{O}(\mathbb{e}^t(p_i-p_j)x_j\Psi)\qquad\square$$

Implementation.

Without, don't trust!

```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

```
𝔼_{A1_→B1_}[ω1_, Q1_] 𝔼_{A2_→B2_}[ω2_, Q2_] ^:= 𝔼_{A1⋃A2→B1⋃B2}[ω1 ω2, Q1 + Q2]
```

```
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂_{i,j}Q1, {i, A1}, {j, B1}]; E2 = M[∂_{i,j}Q2, {i, A2}, {j, B2}];
  F1 = M[∂_{i,j}Q1, {i, A1}, {j, A1}]; F2 = M[∂_{i,j}Q2, {i, A2}, {j, A2}];
  G1 = M[∂_{i,j}Q1, {i, B1}, {j, B1}]; G2 = M[∂_{i,j}Q2, {i, B2}, {j, B2}];
  𝔼_{A1→B2}[CF[ω1 ω2 Det[I - F2.G1]^{1/2}], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).B2]]]]
```

```
A_ \ B_ := Complement[A, B];
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼_{A1⋃(A2\B1*)→B1⋃A2*}[ω1, Q1 + Sum[ζ* ζ, {ζ, A2 \ B1*}]] //
  𝔼_{B1*⋃A2→B2⋃(B1\A2*)}[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

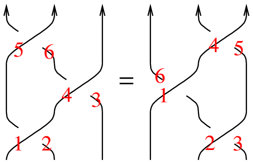
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)_i;
l_List* := #* & /@ l;
```

```
R_{i_,j_} := 𝔼_{{}→{p_i,x_i,p_j,x_j}}[T^{-1/2}, (1 - T) p_j x_j + (T - 1) p_i x_j];
R̄_{i_,j_} := 𝔼_{{}→{p_i,x_i,p_j,x_j}}[T^{1/2}, (1 - T^{-1}) p_j x_j + (T^{-1} - 1) p_i x_j];
C_{i_} := 𝔼_{{}→{p_i,x_i}}[T^{-1/2}, 0];
C̄_{i_} := 𝔼_{{}→{p_i,x_i}}[T^{1/2}, 0];
```

```
hm_{i_,j_→k_} := 𝔼_{{π_i,ξ_i,π_j,ξ_j}→{p_k,x_k}}[1, -ξ_i π_j + (π_i + π_j) p_k + (ξ_i + ξ_j) x_k]
```

```
𝔼_{{}→vs_}[ωi_, Q_]_h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M⟦2 ;;, 2 ;;⟧ = Table[CF[∂_{i,j}Q], {i, ps}, {j, xs}];
  M⟦2 ;;, 1⟧ = ps; M⟦1, 2 ;;⟧ = xs;
  MatrixForm[M]_h]
```

Proof of Reidemeister 3.

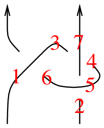


```
(R_{1,2} R_{4,3} R_{5,6} // hm_{1,4→1} hm_{2,5→2} hm_{3,6→3}) ==
 (R_{2,3} R_{1,6} R_{4,5} // hm_{1,4→1} hm_{2,5→2} hm_{3,6→3})

True
```

$\square$

The "First Tangle".



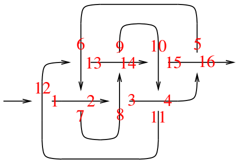
```
Factor /@
 (z = R_{1,6} C̄_3 R̄_{7,4} R̄_{5,2} // hm_{1,3→1} // hm_{1,4→1} // hm_{1,5→1} // hm_{1,6→1} // hm_{2,7→2})
```

$$\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2T}{T},\ \frac{(-1+T)\ (p_1-p_2)\ (T x_1 - x_2)}{-1+2T}\right]$$

```
z_h
```

$$\begin{pmatrix}\frac{-1+2T}{T} & x_1 & x_2\\ p_1 & \frac{-T+T^2}{-1+2T} & \frac{1-T}{-1+2T}\\ p_2 & \frac{T-T^2}{-1+2T} & \frac{-1+T}{-1+2T}\end{pmatrix}_h$$

The knot 8_{17}.



```
z = R̄_{12,1} R̄_{27} R̄_{83} R̄_{4,11} R_{16,5} R_{6,13} R_{14,9} R_{10,15};
Table[z = z // hm_{1k→1}, {k, 2, 16}] // Last
```

$$\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3},\ 0\right]$$

Proof of Theorem 3, (3).

```
{(γ1 = 𝔼_{{}→{p1,x1,p2,x2,p3,x3}}[ω, {p1, p2, p3}.(α β θ; γ δ ϵ; ϕ ψ Ξ).{x1, x2, x3}])_h,
 (γ1 // hm_{1,2→0})_h}
```

$$\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3\\ p_1 & \alpha & \beta & \theta\\ p_2 & \gamma & \delta & \epsilon\\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h,\ \begin{pmatrix}\omega+\gamma\,\omega & x_0 & x_3\\ p_0 & \frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\gamma} & \frac{\epsilon-\alpha\,\epsilon+\theta+\gamma\,\theta}{1+\gamma}\\ p_3 & \frac{\phi-\delta\,\phi+\psi+\gamma\,\psi}{1+\gamma} & \frac{\Xi+\gamma\,\Xi-\epsilon\,\phi}{1+\gamma}\end{pmatrix}_h\right\}$$

$\square$

References.

On ωεβ=http://drorbn.net/cat20

1. (2m) Thanks, technicalities.
2. (4m) Read the sidebar.
3. (4m) Quantum invariants in an algebra and the read-out issue.
4. (2m) The Dogma and the exp-issue.
5. (5m) For $ax + b$, get Gaussians! (these are easily computable as we shall see),
6. (3m) In general, get "docile perturbed Gaussians"; the meaning of ϵ (still efficiently computable!).
7. (4m) Packaging.
8. (5m) The "Gold Standard" theorem.
9. (7m) Ending discussion.
10. (24m) Full computability.