Greece handout on July 14, 2016

July 14, 2016 8:50 AM

Work in Progress!

The Brute and the Hidden Paradise

Abstract. There is expected to be a hidden paradise of poly-time computable knot polynomials lying just beyond the Alexander polynomial. I will describe my brute attempts to gain entry.

Why "expected"? Gauss diagram formulas [PV, GPV] show that $v_{d,f}(K) = \sum_{Y \subset X(K),\, |Y|=d} f(Y)$ finite-type invariants are all poly-time, and tempt to conjecture that there are no others. But Alexander shows it nonsense:

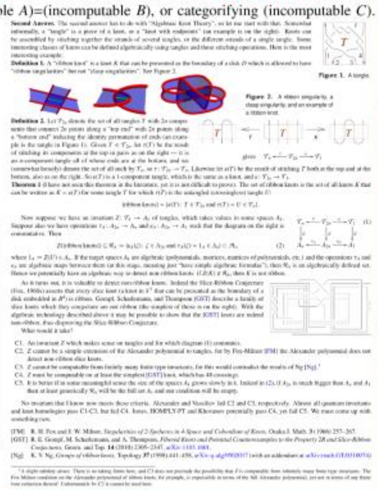
d	2	3	4	5	6	7	8	$\cdots$
known invts in* $O(n^d)$	1	1	∞	3	4	8	11	$\cdots$

This is an unreasonable picture! **Fresh, numerical, no cheating.*
So there ought to be further poly-time invariants.

Also. • The diagonal above the Alexander diagonal in the Melvin-Morton-Rozansky [MM, Ro] of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.

Why "paradise"? Foremost answer: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

ωεβ/K17: (extend to tangles, perhaps detect non-slice ribbon knots)



Moral. Need "stitching":

TODO: a schematic picture of stitching

Why "brute"? Cause it's the only thing I know, for now. There may be better ways in, and it's fair to hope that sooner or later they will be found.

The Gold Standard is set by the formulas [BNS, BN] for Alexander. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left({}_a\!\nearrow\!{}_b,\ {}_b\!\nwarrow\!{}_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm1} \\ b & 0 & t_a^{\pm1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a,t_b\to t_c,\ \mu:=1-\beta]{m_c^{ab}} \left(\begin{array}{c|cc} \mu\omega & c & S \\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu \\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu \end{array} \right)$$

$\mu \to 1-\beta$?

Help Needed!

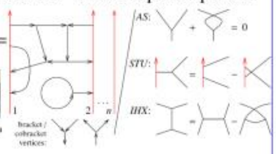
I'm slow and feeble-minded.

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

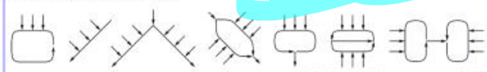
$Z: \left\{ \begin{array}{l} S\text{-component} \\ v/b\text{-tangles} \end{array} \right\} \to \mathcal{A}^v_S :=$



(it is enough to know Z on $\times$ and have disjoint union and stitching formulas) *...exponential and too hard!*

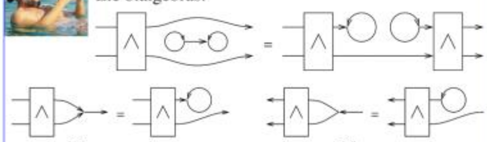
Idea. Look for "ideal" quotients of $\mathcal{A}^v_S$ that have poly-sized descriptions; *...specifically, limit the co-brackets.*

1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:



...manageable but still exponential!

The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and $2D$, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed,

Claim. $R_{jk} = e^{a_{jk}} e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp \mathcal{P}^{2,2}$, with $\rho_{jk} :=$

$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk} a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k \phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk} a_{kk}}{b_j}\right),$$

and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) := ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$.

A

This already gives some new v-braid group reps, (see below)

Problem. How do you multiply in $\exp(L)$ (think $L = \mathcal{P}^{2,2}$)? BCH is a theoretical dream. Instead, ~~• use the adjoint representation, MORE, • and the Euler trick: MORE.~~

use "scatter and glow" and "feed back loops"

A: How do you stitch?

Dror Bar-Natan: Talks: Greece-1607: ωεβ:=http://drorbn.net/Greece-1607/

Work in Progress!

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The adjoint-Gassner example; also, display an implementation.

Schematic stitching.

. . . by a miracle, this leads to Γ-calculus.

Some new (v)braid group reps.

Local Algebra

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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