GWU-1612 handout on 161208

December 8, 2016 9:37 AM

way to ugly! should have use the pre-pragmatic program?

slicing

On Elves and Invariants

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen.

$\omega\epsilon\beta$:=http://drorbn.net/GWU-1612/

Work in Progress! Fluid! Help Needed!

Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1:

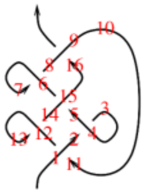
1. **Preparation.** Given K, results $\langle$*long word*$\|$ *simple formulas*$\rangle$.

2. **Rewrite rules.** Make the *word* simpler and the *formulas* more complicated, until the *word* "*elf*" is reached.

3. **Readout.** The invariant ρ_1 is read from the last *formulas*.

$$\text{Knot } K \xrightarrow{\text{preparation}} \langle elf\dots elf \| \omega_0; L_0; Q_0; P_0\rangle \xrightarrow{\text{rewrite rules}} \langle elf \| \omega; -; -; P\rangle \xrightarrow{\text{readout}} \rho_1(K) = \rho_1(\omega, P)$$

Preparation. Draw K using a 0-framed planar diagram D where all crossings are pointing up. Walk along D labeling features by $1, \dots, m$ in order: over-passes, under-passes, and right-heading cups and caps ("$\pm$-cuaps"). If x is a xing, let i_x and j_x be the labels on its over/under strands, and let s_x be its sign. If c is a cuap, let i_c be its labels and s_c be its sign. With $\epsilon^2 = 0$, form



$$\prod_{x:\text{ xing}} R^{s_x}_{i_x j_x} \prod_{c:\text{ cuap}} U^{s_c}_{i_c} = \omega_0^{-1} e^{L_0 \log t + Q_0/\omega_0}(1 + \epsilon\omega_0^{-4} P_0)$$

where $(R^+_{ij}, R^-_{ij}) =$

$$\left(e^{l_j \log t + e_i f_j + \epsilon(e_i l_i f_j + l_i l_j + e_i^2 f_j^2/4)},\ e^{-l_j \log t - e_i f_j/t + \epsilon(e_i l_i f_j/t - l_i l_j - e_i^2 f_j^2/4t^2)}\right),$$

where $U^\pm_i = t^{\mp 1/2} e^{\pm\epsilon l_i t^{\mp 2}}$, and where P_0 collects all the terms proportional to ϵ and (ω_0, L_0, Q_0) are ϵ-free and L_0 has the l_j terms in the exponential and Q_0 the $e_i f_j$ terms. This done, output

$$\langle e_1 l_1 f_1 e_2 l_2 f_2 \cdots e_m l_m f_m \| \omega_0; L_0; Q_0; P_0\rangle.$$

a direct route to ω_0, L_0, Q_0, P_0?

In *formulas*, L is always $\mathbb{Z}$-linear in $\{l_i\}$, Q is an R-linear combination of $\{e_i f_j\}$ where $R := \mathbb{Z}[t^{\pm 1}]$, and P is an R-linear combination of $\{1,\ l_i,\ l_i l_j,\ e_i f_j,\ e_i l_j f_k,\ e_i e_j f_k f_l\}$.

Rewrite Rules. Manipulate $\langle$*word*$\|$ *formulas*$\rangle$ expressions using the rewrite rules below, until you come to the form $\langle e_1 l_1 f_1 \| \omega; -; -; P\rangle$. Output (ω, P).

Rule 1, Deletions. If a letter appears in *word* but not in *formulas*, you can delete it.

Rule 2, Merges. In *word*, you can replace *adjacent* $v_i v_j$ with v_k (for $v \in \{e, l, f\}$) while making the same changes in *formulas* (provided k creates no naming clashes). E.g.,

$$\langle \dots e_i e_j \dots \| Z\rangle \to \langle \dots e_k \dots \| Z|_{e_i, e_j \to e_k}\rangle.$$

Rule 3, *le* Corrections. Provided k introduces no clashes, given $\langle \dots l_j e_i \dots \| \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha e_i + Q'$, write $P = P(e_i, l_j)$ (with messy coefficients), set $q = e^\gamma \beta e_k + \gamma l_k$, and output

$$\left\langle \dots e_k l_k \dots \| \omega; L|_{l_j \to l_k}; t^\lambda \alpha e_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q|_{\beta \to \alpha/\omega,\, \gamma \to \lambda \log t}\right\rangle.$$

Rule 4, *fl* Corrections. Provided k introduces no clashes, given $\langle \dots f_i l_j \dots \| \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha f_i + Q'$, write $P = P(f_i, l_j)$ (with messy coefficients), set $q = e^\gamma \beta f_k + \gamma l_k$, and output

$$\left\langle \dots l_k f_k \dots \| \omega; L|_{l_j \to l_k}; t^\lambda \alpha f_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q|_{\beta \to \alpha/\omega,\, \gamma \to \lambda \log t}\right\rangle.$$

Rule 5, *fe* Corrections. Provided k introduces no clashes, given $\langle \dots f_i e_j \dots \| \omega; L; Q; P\rangle$, decompose $Q = Q_{fe} f_i e_j + Q_f f_i + Q_e e_j + Q'$ write $P = P(f_i, e_j)$ (with messy coefficients), set $\mu = 1 + (t-1)\delta$ and $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$, and output

$$\left\langle \dots e_k f_k \dots \left\| \begin{array}{c} \mu\omega; L; \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + e^{-q} P(\partial_\alpha, \partial_\beta)(e^q) \end{array} \right.\right\rangle\Bigg|_{\substack{\alpha \to Q^f/\omega,\, \beta \to Q^e/\omega, \\ \delta \to Q^{fe}/\omega}},$$

where Λ_k is the Λόγος, "a principle of order and knowledge":

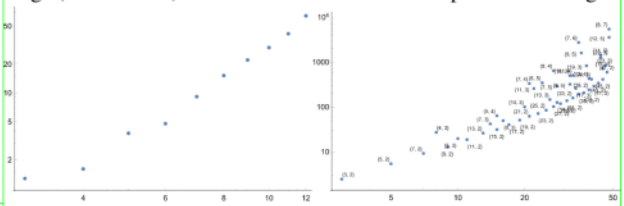
$$\begin{aligned}\Lambda_k = \frac{t+1}{4}\Big(&-\delta(\mu+1)\left(\beta^2 e_k^2 + \alpha^2 f_k^2\right) - \delta^3(3\mu+1) e_k^2 f_k^2 \\ &- 2(\beta e_k + \alpha f_k)\left(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1) e_k f_k + 2\delta\mu^2 l_k\right) \\ &- 4(\alpha\beta + \delta\mu)\left(\delta(\mu+1) e_k f_k + \mu^2 l_k\right) - 4\delta^2\mu^2 e_k f_k l_k \\ &+ (t-1)\left(2(\alpha\beta + \delta\mu)^2 - \alpha^2\beta^2\right)\Big).\end{aligned}$$

Readout. Given $\langle elf \| \omega; -; -; P\rangle$, output

$$\rho_1(K) := \frac{t(t\omega'\omega^3 - P|_{e,l,f \to 0})}{(t-1)^2\omega^2}.$$

(ω is the Alexander polynomial, L and Q are not interesting).

Experimental Analysis ($\omega\epsilon\beta$/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

include Kaufman poly?

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \le 2g - 1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representaion and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

diagram	n_k^t Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral
	0_1^a 1 0	0 / ✔ 0 / ✔		3_1^a $t-1$ t	1 / ✘ 1 / ✘
	4_1^a $3-t$ 0	1 / ✘ 1 / ✔		5_1^a t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘
	5_2^a $2t-3$ $5t-4$	1 / ✘ 1 / ✘		6_1^a $5-2t$ $t-4$	1 / ✔ 1 / ✘
	6_2^a $-t^2+3t-3$ t^3-4t^2+4t-4	2 / ✘ 1 / ✘		6_3^a t^2-3t+5 0	2 / ✘ 1 / ✔
	7_1^a t^3-t^2+t-1 $3t^5+5t^3+6t$	3 / ✘ 3 / ✘		7_2^a $3t-5$ $14t-16$	1 / ✘ 1 / ✘
	7_3^a $2t^2-3t+3$ $-9t^3+8t^2-16t+12$	2 / ✘ 2 / ✘		7_4^a $4t-7$ $32-24t$	1 / ✘ 2 / ✘
	7_5^a $2t^2-4t+5$ $9t^3-16t^2+29t-28$	2 / ✘ 2 / ✘		7_6^a $-t^2+5t-7$ $t^3-8t^2+19t-20$	2 / ✘ 1 / ✘
	7_7^a t^2-5t+9 $8-3t$	2 / ✘ 1 / ✘		8_1^a $7-3t$ $5t-16$	1 / ✘ 1 / ✘
	8_2^a $-t^3+3t^2-3t+3$ $2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘ 2 / ✘		8_3^a $9-4t$ 0	1 / ✘ 2 / ✔
	8_4^a $-2t^2+5t-5$ $3t^3-8t^2+6t-4$	2 / ✘ 2 / ✘		8_5^a $-t^3+3t^2-4t+5$ $-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘ 2 / ✘
	8_6^a $-2t^2+6t-7$ $5t^3-20t^2+28t-32$	2 / ✘ 2 / ✘		8_7^a t^3-3t^2+5t-5 $-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘ 1 / ✘
	8_8^a $2t^2-6t+9$ $-t^3+4t^2-12t+16$	2 / ✔ 2 / ✘		8_9^a $-t^3+3t^2-5t+7$ 0	3 / ✔ 1 / ✔
	8_{10}^a t^3-3t^2+6t-7 $-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✘ 2 / ✘		8_{11}^a $-2t^2+7t-9$ $5t^3-24t^2+39t-44$	2 / ✘ 1 / ✘
	8_{12}^a $t^2-7t+13$ 0	2 / ✘ 2 / ✔		8_{13}^a $2t^2-7t+11$ $-t^3+4t^2-14t+20$	2 / ✘ 1 / ✘
	8_{14}^a $-2t^2+8t-11$ $5t^3-28t^2+57t-68$	2 / ✘ 1 / ✘		8_{15}^a $3t^2-8t+11$ $21t^3-64t^2+120t-140$	2 / ✘ 2 / ✘
	8_{16}^a t^3-4t^2+8t-9 $t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✘ 2 / ✘		8_{17}^a $-t^3+4t^2-8t+11$ 0	3 / ✘ 1 / ✔
	8_{18}^a $-t^3+5t^2-10t+13$ 0	3 / ✘ 2 / ✔		8_{19}^n t^3-t^2+1 $-3t^5-4t^2-3t$	3 / ✘ 3 / ✘
	8_{20}^n t^2-2t+3 $4t-4$	2 / ✔ 1 / ✘		8_{21}^n $-t^2+4t-5$ $t^3-8t^2+16t-20$	2 / ✘ 1 / ✘
	9_1^a $t^4-t^3+t^2-t+1$ $4t^7+7t^5+9t^3+10t$	4 / ✘ 4 / ✘		9_2^a $4t-7$ $30t-40$	1 / ✘ 1 / ✘
	9_3^a $2t^3-3t^2+3t-3$ $-13t^5+12t^4-25t^3+20t^2-32t+24$	3 / ✘ 3 / ✘		9_4^a $3t^2-5t+5$ $23t^3-28t^2+46t-44$	2 / ✘ 2 / ✘
	9_5^a $6t-11$ $100-65t$	1 / ✘ 2 / ✘		9_6^a $2t^3-4t^2+5t-5$ $13t^5-24t^4+45t^3-52t^2+68t-64$	3 / ✘ 3 / ✘
	9_7^a $3t^2-7t+9$ $23t^3-56t^2+99t-108$	2 / ✘ 2 / ✘		9_8^a $-2t^2+8t-11$ $3t^3-16t^2+29t-28$	2 / ✘ 2 / ✘
	9_9^a $2t^3-4t^2+6t-7$ $13t^5-24t^4+55t^3-72t^2+98t-96$	3 / ✘ 3 / ✘		9_{10}^a $4t^2-8t+9$ $-40t^3+72t^2-114t+120$	2 / ✘ 2,3 / ✘