Pensieve header: Examples for the Da-Nang talk: Double Integration and the trefoil.

Startup

```
In[•]:=   SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\DaNang-1905"];
          << "Engine-Speedy.m";
          << "Objects.m";
```

cm

```
In[•]:=   Λ0 = HoldForm[ (η_i + (e^{-α_i-ε β_i} η_j)/(1 + ε η_j ξ_i)) y_k + (β_i + β_j + Log[1 + ε η_j ξ_i]/ε) b_k +
              (α_i + α_j + Log[1 + ε η_j ξ_i]) a_k + ((e^{-α_j-ε β_j} ξ_i)/(1 + ε η_j ξ_i) + ξ_j) x_k];

          TeXForm[Λ0]
          Λ = ReleaseHold[Λ0]
```

$$Out[•]= \; a_k \left(\text{Log}[1 + \epsilon\, \eta_j\, \xi_i] + \alpha_i + \alpha_j\right) +$$
$$b_k \left(\frac{\text{Log}[1 + \epsilon\, \eta_j\, \xi_i]}{\epsilon} + \beta_i + \beta_j\right) + y_k \left(\eta_i + \frac{e^{-\alpha_i - \epsilon\, \beta_i}\, \eta_j}{1 + \epsilon\, \eta_j\, \xi_i}\right) + x_k \left(\frac{e^{-\alpha_j - \epsilon\, \beta_j}\, \xi_i}{1 + \epsilon\, \eta_j\, \xi_i} + \xi_j\right)$$

```
\left(\eta _i+\frac{e^{-\alpha _i-\epsilon  \beta _i} \eta _j}{1+\epsilon  \eta _j \xi _i}
  _j+\frac{\log \left(1+\epsilon  \eta _j \xi _i\right)}{\epsilon }\right) b_k+\left(\alp
  _j \xi _i\right)\right) a_k+\left(\frac{e^{-\alpha _j-\epsilon  \beta _j} \xi _i}{1+\ep
```

```
rho
In[•]:=   HL[ε_] := Style[ε, Background → If[TrueQ@ε, ▮, ▮]];
          {ρy = (0 0 / ε 0), ρb = (0 0 / 0 -ε), ρa = (1 0 / 0 0), ρx = (0 1 / 0 0)};
          HL /@ {ρa.ρx - ρx.ρa == ρx,  ρa.ρy - ρy.ρa == -ρy,
             ρb.ρy - ρy.ρb == -ε ρy,  ρb.ρx - ρx.ρb == ε ρx,  ρx.ρy - ρy.ρx == ρb + ε ρa}
```

```
rho
Out[•]=   {True, True, True, True, True}
```

```
rho
In[•]:=   HL@Simplify@With[{E = MatrixExp},
             E[η_i ρy].E[β_i ρb].E[α_i ρa].E[ξ_i ρx].E[η_j ρy].E[β_j ρb].E[α_j ρa].E[ξ_j ρx] ==
                E[∂_{y_k} Λ ρy].E[∂_{b_k} Λ ρb].E[∂_{a_k} Λ ρa].E[∂_{x_k} Λ ρx]]
```

```
rho
Out[•]=   True
```

```
rho
In[•]:=   Series[Λ, {ε, 0, 1}]
```

```
rho
```
$$Out[•]= \; \left(a_k\,(\alpha_i + \alpha_j) + y_k\,(\eta_i + e^{-\alpha_i}\,\eta_j) + b_k\,(\beta_i + \beta_j + \eta_j\,\xi_i) + x_k\,(e^{-\alpha_j}\,\xi_i + \xi_j)\right) +$$
$$\left(a_k\,\eta_j\,\xi_i - \frac{1}{2}\,b_k\,\eta_j^2\,\xi_i^2 - e^{-\alpha_i}\,y_k\,\eta_j\,(\beta_i + \eta_j\,\xi_i) - e^{-\alpha_j}\,x_k\,\xi_i\,(\beta_j + \eta_j\,\xi_i)\right)\,\epsilon + O[\epsilon]^2$$

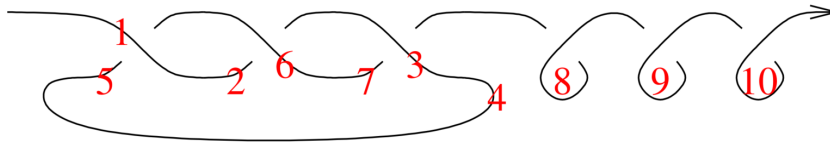## Double Integration

Integrals

```
In[ ]:= PP_ := Identity; $k = 1; ℏ = γ = 1;
       inp = 𝔼_{}→{1} [3 a₁ b₁, 5 x₁ y₁, 1] // dm_{i,1→i};
       Table[
         HL@TrueQ[
           (inp // (sϒ_{i→1,1,2,2} RR) // BM // AM // P_{1,2}) dϵ_j ≡
           (inp // ΔΔ // (sϒ_{i→1,1,2,2} RR) // BM // AM // P_{1,2})],
         {ΔΔ, {dΔ_{i→i,j}, dΔ_{i→j,i}}}, {AM, {dm_{2,4→2}, dm_{4,2→2}}}, {BM, {dm_{1,3→1}, dm_{3,1→1}}},
         {RR, {R_{3,4}, R_{3,4} // dS₃ // dS₃, R_{3,4} // dS₄ // dS₄}}
       ] // MatrixForm
```

Out[ ]//MatrixForm=
Integrals

$$\left(\begin{pmatrix} \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{True} \end{pmatrix} \begin{pmatrix} \text{False} & \text{False} & \text{True} \\ \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{False} \end{pmatrix} \right)$$

The Trefoil

Trefoil

Trefoil

```
In[ ]:= $k = 2;
       Simplify[R_{1,5} R_{6,2} R_{3,7} C̄₄ Kink̄₈ Kink̄₉ Kink̄₁₀ // dm_{1,2→1} // dm_{1,3→1} // dm_{1,4→1} // dm_{1,5→1} // dm_{1,6→1} //
         dm_{1,7→1} // dm_{1,8→1} // dm_{1,9→1} // dm_{1,10→1}] /. v_{-1} :→ v
```

Trefoil

$$Out[ ]= \ \mathbb{E}_{\{\}→\{1\}} \Big[0, 0,$$

$$\frac{B}{1 - B + B^2} + \left(B\left(-B + 2B^2 + 2B^4 + a\left(-1 + B - B^3 + B^4\right) - 2xy - B^3\left(3 + 2xy\right)\right)\right) \Big/ \left(1 - B + B^2\right)^3 +$$

$$\frac{1}{2\left(1 - B + B^2\right)^5} B\left(4B^8 + a^2\left(1 - B + B^2\right)^2\left(1 + B - 6B^2 + B^3 + B^4\right) + 6B^5 x^2 y^2 + 2xy\left(-2 + 3xy\right) -\right.$$

$$B^7\left(11 + 4xy\right) - 2B^2\left(1 + 6x^2 y^2\right) - 2B^4\left(1 - 2xy + 6x^2 y^2\right) + B\left(1 + 8xy + 6x^2 y^2\right) +$$

$$B^6\left(6 + 8xy + 6x^2 y^2\right) + B^3\left(4 + 4xy + 30x^2 y^2\right) + 2a\left(1 - B + B^2\right)\left(2B^6 + 2xy + 8B^3\left(1 + xy\right) -\right.$$

$$\left.\left. 5B^2\left(1 + 2xy\right) - 2B^5\left(1 + 2xy\right) - B^4\left(7 + 2xy\right) + B\left(2 + 4xy\right)\right)\right) \epsilon^2 + O[\epsilon]^3\Big]$$