Dror Bar-Natan: Talks: Buffalo-170916:

The Dogma is Wrong

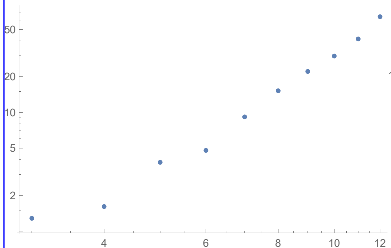
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

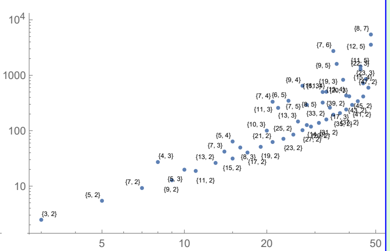
ωεβ:=http://drorbn.net/ub17/ Full version: ωεβ/ld17

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:

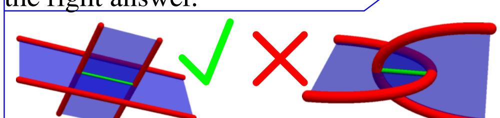




Power. On the 250 knots with at most 10 crossings, the pair (ω,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

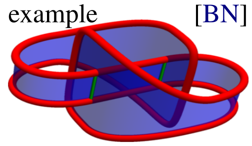
Genus. Up to 12 xings, always ρ_1 is symmetric under $t\leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg\rho_1^+\leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

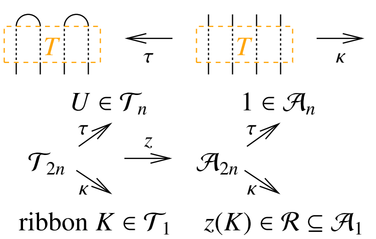
Ribbon Knots.

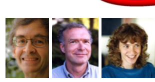


a ribbon singularity a clasp singularity

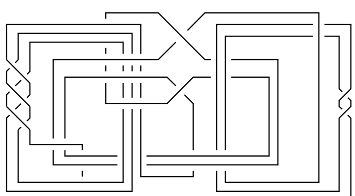
example [BN]







Gompf, Scharlemann, Thompson [GST]



[Vo]: Works for Alexander!

with $\mathcal{R}:=\kappa(\tau^{-1}(1))$

$A^+=-t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_1^+=5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $S(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(a_1^3y_1a_2e^{y_3}x_3^9\mid x_3a_1\otimes y_1y_3a_2\right)=x^9a^3\otimes ye^ya\in\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

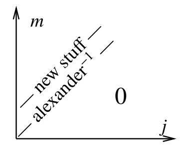
Theorem ([BNG], conjectured [MM], elucidated [Ro1]).

Melvin, Morton, Garoufalidis

Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

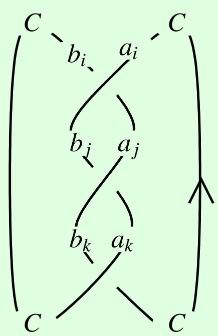
$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q)=\frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U\quad\text{and}\quad C\in U,$$

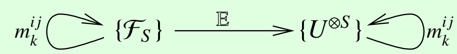
form

$$Z=\sum_{i,j,k}Ca_ib_ja_kC^2b_ia_jb_kC.$$

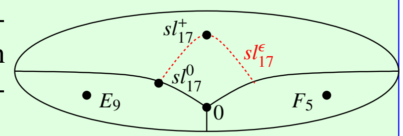
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*. (also, ruins strand doubling)

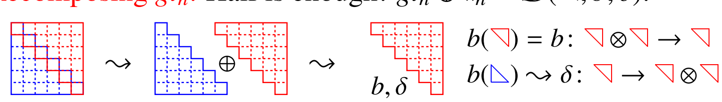
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".



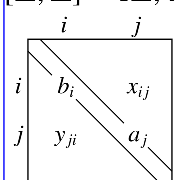
The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl_{17}^k:=sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.



Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(\nabla,b,\delta)$:



Now define $gl_n^\epsilon:=\mathcal{D}(\nabla,b,\epsilon\delta)$. Schematically, this is $[\nabla,\nabla]=\nabla$, $[\Delta,\Delta]=\epsilon\Delta$, and $[\nabla,\Delta]=\Delta+\epsilon\nabla$. In detail, it is



$$[x_{ij},x_{kl}]=\delta_{jk}x_{il}-\delta_{li}x_{kj}\qquad[y_{ij},y_{kl}]=\epsilon\delta_{jk}y_{il}-\epsilon\delta_{li}y_{kj}$$

$$[x_{ij},y_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}x_{il}+\delta_{il}(b_i+\epsilon a_i)/2+\delta_{i>l}y_{il})$$
$$-\delta_{li}(\epsilon\delta_{k<j}x_{kj}+\delta_{kj}(b_j+\epsilon a_j)/2+\delta_{k>j}y_{kj})$$

$$[a_i,x_{jk}]=(\delta_{ij}-\delta_{ik})x_{jk}\qquad[b_i,x_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})x_{jk}$$

$$[a_i,y_{jk}]=(\delta_{ij}-\delta_{ik})y_{jk}\qquad[b_i,y_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})y_{jk}$$

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon=\langle t,y,a,x\rangle/([t,\cdot]=0,[a,x]=x,[a,y]=-y,[x,y]=t-2\epsilon a)$ and let $\mathfrak{g}_k=\mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K)=\mathbb{O}\left(\omega\mathbb{e}^{L+Q+P}:\ \bigotimes_{i\in S}y_ia_ix_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i:=\mathbb{e}^{t_i}$), where $L=\sum l_{ij}t_ia_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q=\sum q_{ij}y_ix_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon,y_i,a_i,x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i,\sqrt{a_i},x_i\}$. Furthermore, after setting $t_i=t$ and $T_i=T$ for all i, the invariant $Z(K)$ is poly-time computable.

The PBW Problem. In $\mathcal{U}(\mathfrak{g}^\epsilon)$, bring $Z = y^3a^2x^2 \cdot y^2a^2x$ to yax-order. In other words, find $g \in \mathbb{Z}[\epsilon,t,y,a,x]$ such that $Z = \mathbb{O}(f = y_1^3y_2^2a_1^2a_2^2x_1^2x_2\colon y_1a_1x_1y_2a_2x_2) = \mathbb{O}(g\colon yax)$.

Solution, Part 1. In $\hat{\mathcal{U}}(\mathfrak{g}^\epsilon)$ we have

$$X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} := e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x}e^{\tau_2 t}e^{\eta_2 y}e^{\alpha_2 a}e^{\xi_2 x} = e^{\tau t}e^{\eta y}e^{\alpha a}e^{\xi x} =: Y_{\tau,\eta,\alpha,\xi},$$

where τ,η,α,ξ are ugly functions of $\tau_1,\eta_i,\alpha_i,\xi_i$:

$$\tau = \tau_1+\tau_2-\frac{\log(1-\epsilon\eta_2\xi_1)}{\epsilon} = \tau_1+\tau_2+\eta_2\xi_1+\frac{\epsilon}{2}\eta_2^2\xi_1^2+\dots,$$

$$\eta = \eta_1+\frac{e^{-\alpha_1}\eta_2}{(1-\epsilon\eta_2\xi_1)} = \eta_1+e^{-\alpha_1}\eta_2+\epsilon e^{-\alpha_1}\eta_2^2\xi_1+\dots,$$

$$\alpha = \alpha_1+\alpha_2+2\log(1-\epsilon\eta_2\xi_1) = \alpha_1+\alpha_2-2\epsilon\eta_2\xi_1+\dots,$$

$$\xi = \frac{e^{-\alpha_2}\xi_1}{(1-\epsilon\eta_2\xi_1)}+\xi_2 = e^{-\alpha_2}\xi_1+\xi_2+\epsilon e^{-\alpha_2}\eta_2\xi_1^2+\dots.$$

Note 1. This defines a mapping $\Phi\colon \mathbb{R}^8_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} \to \mathbb{R}^4_{\tau,\eta,\alpha,\xi}$.

Proof. $\mathfrak{g}^\epsilon$ has a 2D representation ρ:

```
ρt = (1 0; 0 1);  ρy = (0 0; -ϵ 0);
ρa = ((1+1/ϵ)/2  0; 0  -(1-1/ϵ)/2);  ρx = (0 1; 0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy,
  ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}

{True, True, True}
```

It is enough to verify the desired identity in ρ:

```
ME = MatrixExp;
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].
    ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /.
  {τ0 → -Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2, η0 → η1 + e^-α1 η2/(1-ϵ η2 ξ1),
   α0 → 2 Log[1 - ϵ η2 ξ1] + α1 + α2, ξ0 → e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2}]

True
```

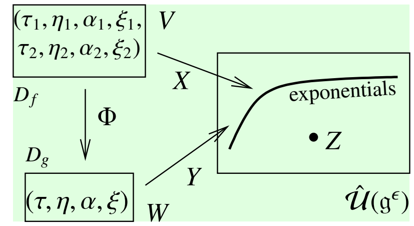
Solution, Part 2. But now, with $D_f = f(z \mapsto \partial_\zeta) = \partial_{\eta_1}^3\partial_{\alpha_1}^2\partial_{\xi_1}^2\partial_{\eta_2}^2\partial_{\alpha_2}^2\partial_{\xi_2}$,

$$Z = D_f X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2}\big|_{vs=0} = D_f Y_{\tau,\eta,\alpha,\xi}\big|_{vs=0} = \mathbb{O}\left(D_f e^{\tau t}e^{\eta y}e^{\alpha a}e^{\xi x}\big|_{vs=0} : yax\right) = \mathbb{O}(g\colon yax):$$

```
Expand[∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} Exp[
    (-Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2) t + (η1 + e^-α1 η2/(1-ϵ η2 ξ1)) y +
     (2 Log[1 - ϵ η2 ξ1] + α1 + α2) a + (e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2) x
   ] /. (τ | η | α | ξ)1|2 → 0]
```

$2a^4t^2xy^3 + 4tx^2y^4 - 16atx^2y^4 + 24a^2tx^2y^4 - 16a^3tx^2y^4 + 4a^4tx^2y^4 + 16x^3y^5 - 32ax^3y^5 + 24a^2x^3y^5 - 8a^3x^3y^5 + a^4x^3y^5 + 2a^4txy^3\epsilon - 8a^5txy^3\epsilon + 8x^2y^4\epsilon - 40ax^2y^4\epsilon + 80a^2x^2y^4\epsilon - 80a^3x^2y^4\epsilon + 40a^4x^2y^4\epsilon - 8a^5x^2y^4\epsilon - 4a^5xy^3\epsilon^2 + 8a^6xy^3\epsilon^2$

Note 2. Replacing $f \to D_f$ (and likewise $g \to D_g$), we find that $D_g = \Phi_* D_f$.



Note 3. The two great evils of mathematics are non-commutativity and non-linearity. We traded one for the other.

Note 4. We could have done similarly with $e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x} = e^{\tau t+\eta y+\alpha a+\xi x}$, and with $S(e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x})$, $\Delta(e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x})$, $\prod_{i=1}^5 e^{\tau_i t}e^{\eta_i y}e^{\alpha_i a}e^{\xi_i x}$.

Fact. $R_{12} \to \exp\left(\partial_{\tau_1}\partial_{\alpha_2}+\partial_{y_1}\partial_{x_2}\right)(1+\sum_{d\geq 1}\epsilon^d p_d)$, where the p_d are computable polynomials of a-priori bounded degrees.

Moral. We need to understand the pushforwards via maps like Φ of (formally ∞-order) "differential operators at 0", that in themselves are perturbed Gaussians. This turns out to be the same problem as "0-dimensional QFT" (except no integration is ever needed), and if $\epsilon^{k+1} = 0$, it is explicitly soluble.

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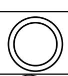
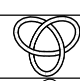
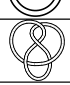
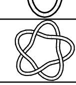
dog·ma (dôg′mə, dŏg′-) — The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.

2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral	diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral
	0_1^a 1 0	0 / ✔ 0 / ✔		3_1^a $t-1$ t	1 / ✘ 1 / ✘
	4_1^a $3-t$ 0	1 / ✘ 1 / ✔		5_1^a t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘

. . . the full Rolfsen table is at ωεβ/ld17