# A weird group

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[This could make a nice exercise in a group theory / Lie groups class]

**Claim.** Let $V$ be a vector space, $\varphi \in V^*$. Then the following defines an associative product on $V$:

$$x \times y := x + y + \varphi(x) \cdot y$$

**Proof.**

$$(x \times y) \times z = (x + y + \varphi(x) y) \times z = x + y + z + \varphi(x) y +$$
$$+ \left(\varphi(x) + \varphi(y) + \varphi(x)\varphi(y)\right) z$$

& similarly for $x \times (y \times z)$.

**Claim.** This is almost a group law.

**Proof.** Trivially, $0$ is the unit. Given $x$,

$$x^{-1} = -\frac{x}{1+\varphi(x)} \quad [\text{defined when } \varphi(x) \neq -1]$$

$$x \times x^{-1} = x - \frac{x}{1+\varphi(x)} - \frac{\varphi(x)}{1+\varphi(x)} x = 0$$

$$x^{-1} \times x = -\frac{x}{1+\varphi(x)} + x - \varphi\left(\frac{x}{1+\varphi(x)}\right) x = 0$$

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$$\varphi(x \times y) = \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) \quad \text{So}$$

$$1 + \varphi(x \times y) = (1 + \varphi(x))(1 + \varphi(y))$$

So if $\varphi(x) \neq -1$ & $\varphi(y) \neq -1$, then also $\varphi(x \times y) \neq -1$. Thus

**Claim.** $[\varphi \neq -1] \subset V$ is a group.

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WLOG, $\varphi(x) = x_1$ ; rename $x_{rest} \to x$, $x_1 \to \alpha$, & get

$$(\alpha, x) \times (\beta, y) = (\alpha + \beta + \alpha\beta, x + (1+\alpha) y)$$

Now with $a = 1+\alpha$, $b = 1+\beta$, get

$$(a, x) \times (b, y) = (ab, x + ay)$$

So our group is the $(n-1)$-dimensional "translation & rescaling group".