Claim. Let $V$ be a vector space, $\Psi \in V^*$. Then the following defines an associative product on $V$:

$$x \ast y := x + y + \Psi(x) \cdot y$$

Proof.

$$(x \ast y) \ast z = (x + y + \Psi(x) \cdot y) \ast z = x + y + z + \Psi(x) \cdot y + \Psi(z) \cdot y$$

More generally, for $x \ast (y \ast z)$.

Claim. This is almost a group law.

Proof. Trivially, $0$ is the unit. Given $x$,

$$x^{-1} = -\frac{x}{1 + \Psi(x)} \quad \text{[defined when } \Psi(x) \neq -1\] }$$

$$x \ast x^{-1} = x - \frac{x}{1 + \Psi(x)} - \frac{\Psi(x)}{1 + \Psi(x)} x = 0$$

$$x^{-1} \ast x = -\frac{x}{1 + \Psi(x)} + x - \frac{\Psi(x)}{1 + \Psi(x)} x = 0$$

$\Psi(x \ast y) = \Psi(x) + \Psi(y) + \Psi(x) \cdot \Psi(y)$, so

$$1 + \Psi(x \ast y) = (1 + \Psi(x)) (1 + \Psi(y))$$

So if $\Psi(x) \neq -1$ & $\Psi(y) \neq -1$, then also

$\Psi(x \ast y) \neq -1$. Thus

Claim. $[\Psi \neq -1] \subset V$ is a group.
WLOG, \( \psi(x) = x \), rename \( x_{\text{ast}} \rightarrow x, x_{i-} \rightarrow x_i \),

we get

\[(\alpha, x) \times (\beta, y) = (\alpha + \beta + k \beta, x + (1 + k)y)\]

Now with \( \alpha = 1 + \alpha \), \( \beta = 1 + \beta \), we get

\[(a, x) \times (b, y) = (ab, x + ay)\]

so our group is the \((n - 1)\)-dimensional

"translation & rescaling group".