

Pensieve Header: The U(I2D) program, extending U(I2D)-PolyTime.nb (“GSeries” version, aborted).

```

I2DRules = {
  u[lft___, 1, rgt___] :=> u[lft, rgt],
  u[lft___, sn·, sm·, rgt___] :=> u[lft, sn+m, rgt],
  u[lft___, yn·, xm·, rgt___] :=>  $\sum_{k=0}^m (-n)^k \text{Binomial}[m, k] u[\text{lft}, x^{-k+m}, y^n, \text{rgt}]$ ,
  u[lft___, pyn·, pxm·, rgt___] :=> u[lft, pxm, pyn, rgt],
  u[lft___, xn·, pxm·, rgt___] :=> u[lft, pxm, xn, rgt],
  u[lft___, xn·, pym·, rgt___] :=>  $\sum_{k=0}^n (-m)^k \text{Binomial}[n, k] u[\text{lft}, py^m, x^{-k+n}, \text{rgt}]$ ,
  u[lft___, yn·, pxm·, rgt___] :=> u[lft, pxm, yn, rgt],
  u[lft___, yn·, pym·, rgt___] :=>
     $\sum_{k=0}^{\text{Min}[m,n]} \text{Binomial}[n, k] \text{Binomial}[m, k] k! u[\text{lft}, px^k, py^{m-k}, y^{n-k}, \text{rgt}]$ 
};

Outer[u[#1, #2] - u[#2, #1] &, {px, py, x, y}, {px, py, x, y}] //. I2DRules // Expand //
  MatrixForm

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & u[py] & -u[px] \\ 0 & -u[py] & 0 & u[y] \\ 0 & u[px] & -u[y] & 0 \end{pmatrix}$$

u[y^3, x, x, py^2, px^4] //. I2DRules // Expand
54 u[px^6, y] + 25 u[px^4, py^2, y^3] + 96 u[px^5, py, y^2] - 36 u[px^6, x, y] + 6 u[px^6, x^2, y] -
  10 u[px^4, py^2, x, y^3] + u[px^4, py^2, x^2, y^3] - 48 u[px^5, py, x, y^2] + 6 u[px^5, py, x^2, y^2]
Unprotect[NonCommutativeMultiply];
0 ** _ = 0;
_ ** 0 = 0;
(c_?NumberQ * a_) ** b_ := Expand[c * (a ** b)];
a_ ** (c_?NumberQ * b_) := Expand[c * (a ** b)];
a_Plus ** b_ := (# ** b) & /@ a;
a_ ** b_Plus := (a ** #) & /@ b;
T[{m1_, m2_, m3_, m4_}] ** T[{n1_, n2_, n3_, n4_}] := Expand[
  u[px^m1, py^m2, x^m3, y^m4, px^n1, py^n2, x^n3, y^n4] //. I2DRules
] /. ut_u => T[Exponent[Times @@ ut, #] & /@ {px, py, x, y}];
T[t1_] ** T[t2_] := Module[
  {cc, tt},
  Distribute[
    OT @@ MapThread[T[#1] ** T[#2] &, {{t1}, {t2}}]
  ] /. ot_OT => (
    cc = 1; tt = T[];
    Replace[ot, c_. * t_T => (cc *= c; AppendTo[tt, t];), {1}];
    cc * Flatten[tt]
  )
]

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```

T[{2, 1, 3, 4}] ** T[{1, 1, 0, 1}]
-T[{3, 2, 0, 5}] + 3 T[{3, 2, 1, 5}] - 3 T[{3, 2, 2, 5}] + T[{3, 2, 3, 5}] + 4 T[{4, 1, 3, 4}]
T[{2, 1, 3, 4}, {0, 0, 0, 1}] ** T[{1, 1, 0, 1}, {0, 0, 1, 0}]
T[{3, 2, 0, 5}, {0, 0, 0, 1}] - T[{3, 2, 0, 5}, {0, 0, 1, 1}] - 3 T[{3, 2, 1, 5}, {0, 0, 0, 1}] +
3 T[{3, 2, 1, 5}, {0, 0, 1, 1}] + 3 T[{3, 2, 2, 5}, {0, 0, 0, 1}] -
3 T[{3, 2, 2, 5}, {0, 0, 1, 1}] - T[{3, 2, 3, 5}, {0, 0, 0, 1}] + T[{3, 2, 3, 5}, {0, 0, 1, 1}] -
4 T[{4, 1, 3, 4}, {0, 0, 0, 1}] + 4 T[{4, 1, 3, 4}, {0, 0, 1, 1}]
MakeGSeries[m_Integer, t_T] := Module[
  {a, d},
  a = GSeries @@ Table[0, {m+1}];
  d = Plus @@ Take[Plus@@t, 2];
  If[d ≤ m, a[[d+1]] = t];
  a
];
MakeGSeries[m_Integer, a_] := a /. t_T => MakeGSeries[m, t];
a_GSeries + b_GSeries ^=
  GSeries @@ Table[a[[m]] + b[[m]], {m, 1, Min[Length[a], Length[b]]}];
a_GSeries ** b_GSeries := GSeries @@ Table[
  Sum[a[[k+1]] ** b[[m-k+1]], {k, 0, m}],
  {m, 0, Min[Length[a], Length[b]] - 1}
];
MakeGSeries[7, T[{3, 2, 2, 5}, {0, 0, 0, 1}]]
GSeries[0, 0, 0, 0, 0, T[{3, 2, 2, 5}, {0, 0, 0, 1}], 0, 0]
MakeGSeries[7, T[{2, 1, 3, 4}] ** T[{1, 1, 0, 1}]]
-GSeries[0, 0, 0, 0, 0, T[{3, 2, 0, 5}], 0, 0] +
3 GSeries[0, 0, 0, 0, 0, T[{3, 2, 1, 5}], 0, 0] -
3 GSeries[0, 0, 0, 0, 0, T[{3, 2, 2, 5}], 0, 0] +
GSeries[0, 0, 0, 0, 0, T[{3, 2, 3, 5}], 0, 0] + 4 GSeries[0, 0, 0, 0, 0, T[{4, 1, 3, 4}], 0, 0]
MakeGSeries[7, T[{1, 0, 0, 0}]]
GSeries[0, T[{1, 0, 0, 0}], 0, 0, 0, 0, 0, 0]
MakeGSeries[7, T[{2, 1, 3, 4}] ** T[{1, 1, 0, 1}]] ** MakeGSeries[7, T[{1, 0, 0, 0}]]
-GSeries[0, 0, 0, 0, 0, 0, T[{4, 2, 0, 5}], 0] +
3 GSeries[0, 0, 0, 0, 0, 0, T[{4, 2, 1, 5}], 0] -
3 GSeries[0, 0, 0, 0, 0, 0, T[{4, 2, 2, 5}], 0] +
GSeries[0, 0, 0, 0, 0, 0, T[{4, 2, 3, 5}], 0] + 4 GSeries[0, 0, 0, 0, 0, 0, T[{5, 1, 3, 4}], 0]
GExp[n_Integer, d_Integer, expr_] := Module[
  {total, term, k},
  term = MakeGSeries[d, T@@Table[{0, 0, 0, 0}, {n}]];
  term + Sum[
    term = Expand[term ** expr / k],
    {k, d}
  ]
];

```