Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⎴, ⁀}.
```

Initialization

```
FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

Initialization

```
AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm,
  hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.
```

Initialization

```
AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x = LW@"x"; y = LW@"y";
{α, β, γ} = LS /@ {x + b[x, y], y - b[x, b[x, y]], x + y - 2 b[x, y]}
```

alphabetagamma

{LS[$\overline{x}$, $\overline{x\,y}$, 0, 0, ...], LS[$\overline{y}$, 0, $-\overline{x\,\overline{x\,y}}$, 0, ...], LS[$\overline{x}+\overline{y}$, $-2\,\overline{x\,y}$, 0, 0, ...]}

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

{LS[0, $\overline{x\,y}$, $\overline{\overline{x\,y}\,y}$, $-\overline{x\,\overline{x\,\overline{x\,y}}}$, ...], LS[0, 0, 0, 0, ...]}

bch

```
bch = BCH[x, y]
```

bch

$$\mathrm{LS}\left[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\,\overline{x\overline{xy}}+\frac{1}{12}\,\overline{\overline{xy}y},\ \frac{1}{24}\,\overline{x\overline{\overline{xy}y}},\ \ldots\right]$$

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]]}
```

bch16

$$\left\{39.5313,\ \left\{2181,\ -\frac{17\,\overline{x\,x\,\overline{x\overline{xy}}\,\overline{xy}\,\overline{x\overline{xy}}\,x\,x\,\overline{\overline{xy}y}\,y}}{179\,625\,600}+\frac{389\,\overline{x\,x\,\overline{x\overline{xy}}\,\overline{xy}\,\overline{x\overline{xy}}\,\overline{x\overline{xy}}\,\overline{x\overline{xy}}}}{1\,320\,883\,200}+\frac{53\,\overline{x\,x\,\overline{x\overline{xy}}\,\overline{xy}\,\overline{x\overline{xy}}\,x\overline{xy}\,\overline{\overline{xy}y}}}{1\,089\,728\,640}\right\}\right\}$$

omegas

```
{ω1, ω2} = CWS /@ {cw[x] - 3 cw[y, x, x], cw[y] + cw[y, y]}
```

omegas

$$\{\mathrm{CWS}[\overset{\frown}{x},\ 0,\ -3\,\overset{\frown}{xxy},\ 0,\ \ldots],\ \mathrm{CWS}[\overset{\frown}{y},\ \overset{\frown}{yy},\ 0,\ 0,\ \ldots]\}$$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$$\left\{\mathrm{CWS}\left[h\,\overset{\frown}{x},\ 0,\ -3\,h^3\,\overset{\frown}{xxy},\ 0,\ \ldots\right],\ \mathrm{CWS}\left[h\,\overset{\frown}{y},\ h^2\,\overset{\frown}{yy},\ 0,\ 0,\ \ldots\right]\right\}$$

TangentialDerivative

```
{λ = ⟨x → α, y → β⟩, γ // Dλ}
```

TangentialDerivative

$$\left\{\left\langle\overline{x}\to\mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots],\ \overline{y}\to\mathrm{LS}\left[\overline{y},\ 0,\ -\overline{x\overline{xy}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ \ldots\right]\right\}$$

tb

```
λ1 = λ; λ2 = ⟨x → β, y → γ⟩; tb[λ1, λ2]
```

tb

$$\left\langle\overline{x}\to\mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ \ldots\right],\ \overline{y}\to\mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ \ldots\right]\right\rangle$$

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

$$\{\mathrm{CWS}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 18\,\overset{\frown}{xxxxyxyy}-18\,\overset{\frown}{xxxxyyxy}-36\,\overset{\frown}{xxxyxxyy}+36\,\overset{\frown}{xxxyyxxy},\ \ldots],$$
$$\mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$$

TestingGammaODE

lhs = $\partial_t\Gamma_t[\lambda]$; rhs = λ // $e^{-t\,D_\lambda}$ // adSeries$\left[\frac{ad}{e^{ad}-1},\ \Gamma_t[\lambda]\right]$;

{$\Gamma_0[\lambda]$, lhs, (lhs ≡ rhs) @ {6}}

TestingGammaODE

$\{\langle \overline{x}\to LS[0, 0, 0, 0, \ldots], \overline{y}\to LS[0, 0, 0, 0, \ldots]\rangle,$

$\left\langle \overline{x}\to LS\left[\overline{x}, \overline{xy}, -t\,\overline{x\overline{xy}}, \frac{1}{4}t^2\,\overline{x\overline{x\overline{xy}}} - t\,\overline{x\overline{\overline{xy}y}}, \ldots\right],\right.$

$\left.\overline{y}\to LS\left[\overline{y}, 0, -\overline{x\overline{xy}}, -t\,\overline{x\overline{\overline{xy}y}}, \ldots\right]\right\rangle, BS[7\ True, \ldots]\}$

TestingGamma

{γ // $e^{-t\,D_\lambda}$, γ // CC[$\Gamma_t[\lambda]$]}

TestingGamma

$\left\{LS\left[\overline{x}+\overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots\right], LS\left[\overline{x}+\overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots\right]\right\}$

TestingLambdaODE

lhs = $\partial_t\Lambda_t[\lambda]$; rhs = λ // $e^{D_{\Lambda_t[\lambda]}}$ // adSeries$\left[\frac{ad}{e^{ad}-1},\ \Lambda_t[\lambda],\ tb\right]$;

{$\Lambda_0[\lambda]$, lhs, (lhs ≡ rhs) @ {6}}

TestingLambdaODE

$\{\langle \overline{x}\to LS[0, 0, 0, 0, \ldots], \overline{y}\to LS[0, 0, 0, 0, \ldots]\rangle,$

$\left\langle \overline{x}\to LS\left[\overline{x}, \overline{xy}, t\,\overline{x\overline{xy}}, \frac{1}{2}t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots\right],\right.$

$\left.\overline{y}\to LS\left[\overline{y}, 0, -\overline{x\overline{xy}}, t\,\overline{x\overline{\overline{xy}y}}, \ldots\right]\right\rangle, BS[7\ True, \ldots]\}$

TestingLambda

{γ // CC[t λ], γ // $e^{-D_{\Lambda_t[\lambda]}}$}

TestingLambda

$\left\{LS\left[\overline{x}+\overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, -\frac{1}{2}t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots\right],\right.$

$\left.LS\left[\overline{x}+\overline{y}, -2\,\overline{xy}, -t\,\overline{x\overline{xy}}, -\frac{1}{2}t^2\,\overline{x\overline{x\overline{xy}}} + t\,\overline{x\overline{\overline{xy}y}}, \ldots\right]\right\}$

Unclassified aside: an alternative formulation of Λ (on March 1, 2015, this took 61 Seconds to degree 8):

λ3 = ⟨x → RandomLieSeries[{x, y}], y → RandomLieSeries[{x, y}]⟩;

{lhs = λ3 // EulerE // adSeries$\left[\frac{e^{ad}-1}{ad},\ \lambda 3\right]$ // RC[−λ3],

rhs = Λ[λ3] // EulerE // adSeries$\left[\frac{e^{ad}-1}{ad},\ \Lambda[\lambda 3],\ tb\right]$; (lhs ≡ rhs) @ {7}} // Timing

$\{13.1406,$

$\left\{\left\langle \overline{x}\to LS\left[-2\,\overline{x}-2\,\overline{y}, 6\,\overline{xy}, -\frac{17}{2}\,\overline{x\overline{xy}} - \frac{23}{2}\,\overline{\overline{xy}y}, \frac{23}{6}\,\overline{x\overline{x\overline{xy}}} + \frac{89}{2}\,\overline{x\overline{\overline{xy}y}} + \frac{73}{6}\,\overline{\overline{\overline{xy}y}y}, \ldots\right],\right.\right.$

$\left.\overline{y}\to LS\left[-2\,\overline{y}, -2\,\overline{xy}, \frac{9}{2}\,\overline{x\overline{xy}} + 2\,\overline{\overline{xy}y}, -\frac{47}{6}\,\overline{x\overline{x\overline{xy}}} - 3\,\overline{x\overline{\overline{xy}y}} + \frac{5}{6}\,\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\rangle, BS[$

$8\ True, \ldots]\}\}$

CCAndRC

```
{α // CCx[-γ], α // CCx[-γ] // RCx[γ], α // CCx[-γ] // CCx[γ]}
```

CCAndRC

$$\left\{\mathrm{LS}\left[\overline{x},\ 2\,\overline{xy},\ -\frac{5}{2}\,\overline{x\,\overline{xy}}+\frac{3}{2}\,\overline{\overline{xy}\,y},\ \frac{7}{6}\,\overline{x\,\overline{x\,\overline{xy}}}-\frac{23}{6}\,\overline{x\,\overline{\overline{xy}\,y}}+\frac{2}{3}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\right],\right.$$
$$\left.\mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots],\ \mathrm{LS}\left[\overline{x},\ \overline{xy},\ -\overline{x\,\overline{xy}},\ 2\,\overline{x\,\overline{x\,\overline{xy}}}+\overline{x\,\overline{\overline{xy}\,y}},\ \ldots\right]\right\}$$

tru

```
u = LW@"u"; v = LW@"v";
With[{γ = b[b[v, u], u]}, tru[γ]]
```

tru

$-\overset{\frown}{uv}$

divu

```
With[{γ = u + b[b[v, u], u]}, divu[γ]]
```

divu

$\overset{\frown}{u} - \overset{\frown}{uuv}$

Ju

```
Jx[γ]
```

Ju

$$\mathrm{CWS}\left[\overset{\frown}{x},\ \frac{5\,\overset{\frown}{xy}}{2},\ -\frac{7\,\overset{\frown}{xxy}}{6}+\frac{7\,\overset{\frown}{xyy}}{6},\ \frac{3\,\overset{\frown}{xxxy}}{8}-\frac{11\,\overset{\frown}{xxyy}}{4}-\frac{3\,\overset{\frown}{xyxy}}{4}+\frac{3\,\overset{\frown}{xyyy}}{8},\ \ldots\right]$$

j

```
{div[λ]@{5}, j[λ]@{5}}
```

j

$$\{\mathrm{CWS}[\overset{\frown}{x}+\overset{\frown}{y},\ -\overset{\frown}{xy},\ -\overset{\frown}{xxy},\ 0,\ 0,\ \ldots],$$
$$\mathrm{CWS}[\overset{\frown}{x}+\overset{\frown}{y},\ -\overset{\frown}{xy},\ -\overset{\frown}{xxy},\ -\overset{\frown}{xxyy}+\overset{\frown}{xyxy},\ -\overset{\frown}{xxxyy}+\overset{\frown}{xxyxy},\ \ldots]\}$$

cocycle4j

```
lhs = j[BCHtb[λ1, λ2]]; rhs = j[λ1] + e^(Dλ1)[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

cocycle4j

$$\{\mathrm{CWS}[\overset{\frown}{x}+2\,\overset{\frown}{y},\ -3\,\overset{\frown}{xy},\ 0,\ -9\,\overset{\frown}{xxyy}+9\,\overset{\frown}{xyxy},\ \ldots],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$$\{\mathrm{CWS}[\epsilon\,\overset{\frown}{x}+\epsilon\,\overset{\frown}{y},\ -\epsilon\,\overset{\frown}{xy},\ -\epsilon\,\overset{\frown}{xxy},\ 0,\ \ldots],\ \mathrm{BS}[5\ \mathrm{True},\ \ldots]\}$$

Section 2.3 - The [AT]-inspired presentation El of $A^{w}{}_{exp}$

ElSetup

```
x = LW@"x"; y = LW@"y";
{ζa = El[⟨x → LS[x + b[x, y]], y → LS[y - b[x, b[x, y]]]⟩, CWS[cw[x] - 3 cw[x, y, x]]],
 ζb = El[⟨x → LS[y - b[x, y]], y → LS[x + y + b[y, b[x, y]]]⟩, CWS[cw[y] - 2 cw[x, y]]],
 ζc = El[⟨x → LS[x - b[b[x, y], b[x, y]]], y → LS[y + 3 b[x, b[x, y]]]⟩,
   CWS[cw[x] - 2 cw[x, y] + cw[x, y, x]]]}
```

ElSetup

```
{El[⟨x → LS[x, xy, 0, 0, ...], y → LS[y, 0, -xxy, 0, ...]⟩,
  CWS[x, 0, -3 xxy, 0, ...]],
 El[⟨x → LS[y, -xy, 0, 0, ...], y → LS[x + y, 0, -xyy, 0, ...]⟩,
  CWS[y, -2 xy, 0, 0, ...]], El[
  ⟨x → LS[x, 0, 0, 0, ...], y → LS[y, 0, 3 xxy, 0, ...]⟩, CWS[x, -2 xy, xxy, 0, ...]]}
```

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

```
{El[⟨x → LS[2 x + y, 0, 1/2 xxy, ...], y → LS[x + 3 y, 0, 5/2 xxy - xyy, ...]⟩,
  CWS[2 x + y, -4 xy, -2 xxy, ...]], BS[9 True, ...]}
```

detaExample

```
{ζa // dη^x, ζa // dη^y}
```

detaExample

```
{El[⟨y → LS[y, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨x → LS[x, 0, 0, 0, ...]⟩, CWS[x, 0, 0, 0, ...]]}
```

dA1

```
{ζd = El[λ, CWS[0]], ζd // dA}
```

dA1

```
{El[⟨x → LS[x, xy, 0, 0, ...], y → LS[y, 0, -xxy, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨x → LS[-x, -xy, 0, 0, ...], y → LS[-y, 0, xxy, 0, ...]⟩,
  CWS[-x - y, xy, xxy, xxyy - xyxy, ...]]}
```

dA2

```
(ζd ≡ (ζd // dA // dA))@{8}
```

dA2

```
BS[9 True, ...]
```

dA3

```
lhs = (ζa ** ζb) // dA; rhs = (ζb // dA) ** (ζa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dA3

```
{El[⟨x → LS[-x - y, 0, -1/2 xxy, ...], y → LS[-x - 2 y, 0, 1/2 xxy + xyy, ...]⟩,
  CWS[-y, -2 xy, -2 xxy - xyy, ...]], BS[9 True, ...]}
```

dS

```
ξd // dS
```

dS

```
El[⟨x̄ → LS[x̄, -x̄ȳ, 0, 0, ...], ȳ → LS[ȳ, 0, -x̄x̄ȳ, 0, ...]⟩,
 CWS[x̂ + ŷ, x̂ŷ, -x̂x̂ŷ, x̂x̂ŷŷ - x̂ŷx̂ŷ, ...]]
```

dD1

```
{ξa, ξa // dΔ[y, y, z]}
```

dD1

```
{El[⟨x̄ → LS[x̄, x̄ȳ, 0, 0, ...], ȳ → LS[ȳ, 0, -x̄x̄ȳ, 0, ...]⟩,
  CWS[x̂, 0, -3 x̂x̂ŷ, 0, ...]],
 El[⟨z → LS[ȳ + z̄, 0, -x̄x̄ȳ - x̄x̄z̄, 0, ...], x̄ → LS[x̄, x̄ȳ + x̄z̄, 0, 0, ...],
   ȳ → LS[ȳ + z̄, 0, -x̄x̄ȳ - x̄x̄z̄, 0, ...]⟩, CWS[x̂, 0, -3 x̂x̂ŷ - 3 x̂x̂ẑ, 0, ...]]}
```

dD2

```
lhs = (ξa ** ξb) // dΔ[y, y, z]; rhs = (ξa // dΔ[y, y, z]) ** (ξb // dΔ[y, y, z]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dD2

```
{El[⟨z → LS[x̄ + 2 ȳ + 2 z̄, 0, -1/2 x̄x̄ȳ - 1/2 x̄x̄z̄ - x̄ȳz̄ - x̄ȳȳ - 2 x̄z̄ȳ - x̄z̄z̄, ...],
   x̄ → LS[x̄ + ȳ + z̄, 0, 1/2 x̄x̄ȳ + 1/2 x̄x̄z̄, ...],
   ȳ → LS[x̄ + 2 ȳ + 2 z̄, 0, -1/2 x̄x̄ȳ - 1/2 x̄x̄z̄ - x̄ȳz̄ - x̄ȳȳ - 2 x̄z̄ȳ - x̄z̄z̄, ...]⟩,
  CWS[x̂ + ŷ + ẑ, -2 x̂ŷ - 2 x̂ẑ, -3 x̂x̂ŷ - 3 x̂x̂ẑ, ...]], BS[9 True, ...]}
```

Section 2.4 - The factored presentation Ef of $A^w{}_{exp}$ and its stronger precursor Es

EsSetup1

```
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
   CWS[cw[u] - 3 cw[u, v, u]]]
```

EsSetup1

```
Es[⟨1 → LS[ū, ūv̄, 0, 0, ...], 2 → LS[v̄, 0, -ūūv̄, 0, ...], 3 → LS[ū, 0, 0, 0, ...]⟩,
 CWS[û, 0, -3 ûûv̂, 0, ...]]
```

EsSetup2

```
ξb = RandomEsSeries[0, {1, 2, 3, 4}];
ξb@{2}
```

EsSetup2

$$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[-\overline{1}-2\,\overline{2}+2\,\overline{3}-2\,\overline{4},\ 2\,\overline{1\,2}+\frac{\overline{1\,3}}{2}+\overline{1\,4}-\frac{\overline{2\,3}}{2}-\frac{\overline{2\,4}}{2}+2\,\overline{3\,4},\ \ldots\Big],$$

$$2\to \mathrm{LS}\Big[2\,\overline{1}-\overline{2}-2\,\overline{3}+\overline{4},\ 2\,\overline{1\,2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{1\,4}-\overline{2\,3}-\overline{2\,4}-\frac{\overline{3\,4}}{2},\ \ldots\Big],$$

$$3\to \mathrm{LS}\Big[-\overline{1}+\overline{2}+2\,\overline{4},\ -2\,\overline{1\,2}+2\,\overline{1\,3}-\overline{1\,4}-\frac{3\,\overline{2\,3}}{2}+2\,\overline{2\,4}-2\,\overline{3\,4},\ \ldots\Big],$$

$$4\to \mathrm{LS}\Big[-2\,\overline{1}+2\,\overline{2}+2\,\overline{3}+\overline{4},\ -\frac{\overline{1\,2}}{2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{2\,4}+\overline{3\,4},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[\overset{\frown}{3}-\overset{\frown}{4},\ \frac{3\,\overset{\frown}{11}}{2}+\frac{3\,\overset{\frown}{12}}{2}-2\,\overset{\frown}{13}+\overset{\frown}{14}+\overset{\frown}{22}+2\,\overset{\frown}{23}-\frac{\overset{\frown}{24}}{2}-2\,\overset{\frown}{33}-\overset{\frown}{34}+\overset{\frown}{44},\ \ldots\Big]\Big]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$$\Big\{\mathrm{Es}\Big[\Big\langle 3\to \mathrm{LS}\Big[\overline{u},\ -\overline{u\,v},\ -\overline{u\,\overline{u\,v}}+\frac{1}{2}\,\overline{\overline{u\,v}\,v},\ \frac{3}{2}\,\overline{u\,\overline{u\,\overline{u\,v}}}+\overline{u\,\overline{\overline{u\,v}\,v}}-\frac{1}{6}\,\overline{\overline{\overline{u\,v}\,v}\,v},\ \ldots\Big],$$

$$4\to \mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{u\,v}}{2},\ -\frac{23}{12}\,\overline{u\,\overline{u\,v}}-\frac{5}{12}\,\overline{\overline{u\,v}\,v},\ \overline{u\,\overline{u\,\overline{u\,v}}}+\frac{13}{24}\,\overline{u\,\overline{\overline{u\,v}\,v}}+\frac{1}{12}\,\overline{\overline{\overline{u\,v}\,v}\,v},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[2\,\overset{\frown}{u},\ -\overset{\frown}{uv},\ -\frac{3\,\overset{\frown}{uuv}}{2},\ -\frac{\overset{\frown}{uuuv}}{6}+\overset{\frown}{uuvv}-\overset{\frown}{uvuv},\ \ldots\Big]\Big],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\Big\}$$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$$\Big\{\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[-2\,\overline{1}+\overline{4},\ -\frac{3\,\overline{1\,4}}{2},\ 20\,\overline{1\,\overline{1\,4}}-\frac{19}{3}\,\overline{\overline{1\,4}\,4},\ \ldots\Big],$$

$$4\to \mathrm{LS}\Big[2\,\overline{1}+\overline{4},\ \overline{1\,4},\ -\frac{31}{2}\,\overline{1\,\overline{1\,4}}-\frac{13}{6}\,\overline{\overline{1\,4}\,4},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[3\,\overset{\frown}{1}-\overset{\frown}{4},\ -3\,\overset{\frown}{11}+\frac{\overset{\frown}{14}}{2}+\overset{\frown}{44},\ \frac{71\,\overset{\frown}{111}}{4}+\frac{19\,\overset{\frown}{114}}{4}-\frac{7\,\overset{\frown}{144}}{6}-\frac{2\,\overset{\frown}{444}}{3},\ \ldots\Big]\Big],\ \mathrm{BS}[6\ \mathrm{True},\ \ldots]\Big\}$$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$$\{\mathrm{El}[\langle 1\to \mathrm{LS}[0,0,0,\ldots],\ 2\to\mathrm{LS}[\overline{1},0,0,\ldots],\ 3\to\mathrm{LS}[\overline{1}+\overline{2},0,0,\ldots]\rangle,\ \mathrm{CWS}[0,0,0,\ldots]],\ \mathrm{BS}[6\,\mathrm{True},\ldots]\}$$

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

$$\mathrm{Es}\left[\langle 1\to\mathrm{LS}[0,0,0,0,0,0,\ldots]\rangle,\ \mathrm{CWS}\left[0,-\widehat{11},0,-\frac{31\,\widehat{1111}}{12},0,-\frac{1351\,\widehat{111111}}{360},\ldots\right]\right]$$

Borromean

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

$$\mathrm{Es}\Big[\Big\langle b\to\mathrm{LS}\Big[0,\ \overline{gr},\ \frac{1}{2}\overline{g\overline{gr}}+\overline{\overline{br}g}+\frac{1}{2}\overline{\overline{gr}r},$$
$$-\frac{1}{2}\overline{b\overline{\overline{br}g}}+\frac{1}{6}\overline{g\overline{g\overline{gr}}}+\frac{1}{4}\overline{g\overline{\overline{gr}r}}-\frac{1}{2}\overline{\overline{bg}\,\overline{br}}-\frac{1}{2}\overline{\overline{\overline{br}g}g}-\frac{1}{2}\overline{\overline{\overline{br}r}g}+\frac{1}{6}\overline{\overline{\overline{gr}r}r},\ \ldots\Big],\ g\to$$
$$\mathrm{LS}\Big[0,\ -\overline{br},\ \frac{1}{2}\overline{b\overline{br}}-\overline{b\overline{gr}}-\overline{\overline{br}g}+\frac{1}{2}\overline{\overline{br}r},\ -\frac{1}{6}\overline{b\overline{b\overline{br}}}-\frac{1}{2}\overline{b\overline{b\overline{gr}}}-\frac{1}{2}\overline{b\overline{g\overline{gr}}}-\frac{1}{2}\overline{b\overline{\overline{br}g}}-$$
$$\frac{1}{4}\overline{b\overline{\overline{br}r}}+\frac{1}{2}\overline{b\overline{\overline{gr}r}}+\frac{1}{2}\overline{\overline{bg}\,\overline{br}}+\overline{\overline{br}\,\overline{gr}}-\overline{\overline{b\overline{gr}}g}-\frac{1}{2}\overline{\overline{\overline{br}g}g}+\frac{1}{2}\overline{\overline{\overline{br}r}g}-\frac{1}{6}\overline{\overline{\overline{br}r}r},\ \ldots\Big],$$
$$r\to\mathrm{LS}\Big[0,\ \overline{bg},\ \frac{1}{2}\overline{b\overline{bg}}+\overline{b\overline{gr}}+\frac{1}{2}\overline{\overline{bg}g},\ \frac{1}{6}\overline{b\overline{b\overline{bg}}}+\frac{1}{2}\overline{b\overline{b\overline{gr}}}+$$
$$\frac{1}{2}\overline{b\overline{g\overline{gr}}}+\frac{1}{4}\overline{b\overline{\overline{bg}g}}+\frac{1}{2}\overline{b\overline{\overline{gr}r}}+\frac{1}{6}\overline{\overline{\overline{bg}g}g},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ 0,\ 2\,\widehat{bgr},\ \widehat{bbgr}-\widehat{bgbr}+\widehat{bggr}-\widehat{bgrg}+\widehat{bgrr}-\widehat{brgr},\ \ldots\Big]\Big]$$

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
x = LW["x"]; y = LW["y"]; z = LW["z"];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

CapSetup

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

VCapEqns

```
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn = (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

VCapSolution

```
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ}, (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V0@{4}, κ@{6}}
```

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

VCapSolution

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

VCapSolution

$$\Big\{\mathrm{Es}\Big[\Big\langle \overline{x} \to \mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ \ldots\Big],$$
$$\overline{y} \to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880} + \frac{\overline{xxyy}}{2880} + \frac{\overline{xyxy}}{5760} + \frac{\overline{xyyy}}{2880},\ \ldots\Big]\Big],$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11\,520},\ 0,\ -\frac{\overline{xxxxxx}}{725\,760},\ \ldots\Big]\Big\}$$

Sinh

```
Series[1/4 Log[(ℏ/2)/Sinh[ℏ/2]], {ℏ, 0, 12}]
```

Sinh

$$-\frac{\hbar^2}{96} + \frac{\hbar^4}{11\,520} - \frac{\hbar^6}{725\,760} + \frac{\hbar^8}{38\,707\,200} - \frac{\hbar^{10}}{1\,916\,006\,400} + \frac{691\,\hbar^{12}}{62\,768\,369\,664\,000} + O[\hbar]^{13}$$

LambdaV

```
Λ[V0]
```

LambdaV

$$\mathrm{El}\Big[\Big\langle \bar{x} \to \mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ \frac{1}{96}\overline{x\overline{xy}},\ \frac{\overline{x\overline{x\overline{xy}}}}{2880} - \frac{1}{480}\overline{x\overline{\overline{xy}y}} + \frac{\overline{\overline{\overline{xy}y}y}}{1440},\ \ldots\Big],$$
$$\bar{y} \to \mathrm{LS}\Big[\frac{\bar{x}}{2},\ -\frac{\overline{xy}}{12},\ \frac{1}{96}\overline{x\overline{xy}},\ \frac{1}{960}\overline{x\overline{x\overline{xy}}} - \frac{1}{320}\overline{x\overline{\overline{xy}y}} + \frac{1}{720}\overline{\overline{\overline{xy}y}y},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880} + \frac{\overline{xxyy}}{2880} + \frac{\overline{xyxy}}{5760} + \frac{\overline{xyyy}}{2880},\ \ldots\Big]\Big]$$

logF

```
logF = Λ[V0][[1]] // dσ[{x, y} → {y, x}]
```

logF

$$\Big\langle \bar{x} \to \mathrm{LS}\Big[\frac{\bar{y}}{2},\ \frac{\overline{xy}}{12},\ \frac{1}{96}\overline{\overline{xy}y},\ -\frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{320}\overline{x\overline{\overline{xy}y}} - \frac{1}{960}\overline{\overline{\overline{xy}y}y},\ \ldots\Big],$$
$$\bar{y} \to \mathrm{LS}\Big[0,\ \frac{\overline{xy}}{24},\ \frac{1}{96}\overline{\overline{xy}y},\ -\frac{\overline{x\overline{x\overline{xy}}}}{1440} + \frac{1}{480}\overline{x\overline{\overline{xy}y}} - \frac{\overline{\overline{\overline{xy}y}y}}{2880},\ \ldots\Big]\Big\rangle$$

atkv

$$\texttt{atkv = logF // EulerE // adSeries}\Big[\frac{e^{ad}-1}{ad},\ \texttt{logF, tb}\Big];$$

```
{f = atkv_x, g = atkv_y}
```

atkv

$$\Big\{\mathrm{LS}\Big[\frac{\bar{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y},\ \ldots\Big],$$
$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y},\ \ldots\Big]\Big\}$$

On March 1, 2015, the following took 379 seconds in degree 8:

KVTest

$$\Big(\hbar^{-1}\,(\mathrm{LS}[x+y] - \mathrm{BCH}[y,x] \equiv f - g - \mathrm{Ad}[-x][f] + \mathrm{Ad}[y][g]) \bigwedge$$
$$\mathrm{div}_x[f] + \mathrm{div}_y[g] \equiv \frac{1}{2}\,\mathrm{tr}_u\Big[\mathrm{adSeries}\Big[\frac{ad}{e^{ad}-1},\ x\Big][u] + \mathrm{adSeries}\Big[\frac{ad}{e^{ad}-1},\ y\Big][u] -$$
$$\mathrm{adSeries}\Big[\frac{ad}{e^{ad}-1},\ \mathrm{BCH}[x,y]\Big][u]\Big]\Big) @\{6\}\ //\ \mathrm{Timing}$$

KVTest

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

KVTest

```
{13.8281, BS[7 True, ...]}
```

KVDirect

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡ 1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] +
       adSeries[ad/(e^ad - 1), y][u] - adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F,
 G}
```

KVDirect

$\{LS[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots],$
$LS[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots]\}$

Section 3.3 - The involution τ and the twist equation

Theta

```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

Theta

$\{El[\langle \overline{x} \to LS[\overline{y}, 0, 0, 0, \ldots], \overline{y} \to LS[\overline{x}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],$
$Es[\langle \overline{x} \to LS[\overline{y}, \frac{\overline{xy}}{2}, \frac{1}{6}\overline{x\overline{xy}} - \frac{1}{12}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{x\overline{xy}}} - \frac{1}{24}\overline{x\overline{\overline{xy}y}}, \ldots], \overline{y} \to$
$LS[\overline{x}, -\frac{\overline{xy}}{2}, -\frac{1}{12}\overline{x\overline{xy}} + \frac{1}{6}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{\overline{xy}y}} - \frac{1}{24}\overline{\overline{\overline{xy}y}y}, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]]\}$

Vtau

```
τV = Rs[x, y] ** (V₀ // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1/2];
(V₀ ≡ τV) @ {6}
```

Vtau

BS[7 True, ...]

Linearized

```
{A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
msgs = SeriesSolve[{A, B},
   ℏ^-1 (b[x, A] + b[y, B] ≡ LS[0]) ⋀ (div_x[A] + div_y[B] ≡ CWS[0])];
{A, B}
```

Linearized

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {As[y] → 0}.

Linearized

{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}

msgs

```
Read[msgs]
```

msgs

```
{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}
```

dims

```
A@12; Length[Last[#]] & /@ Read[msgs]
```

dims

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y, y] → 0}.

dims

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

dims1

```
{A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
msgs1 = SeriesSolve[{A1, B1},
   ℏ^-1 (b[x, A1] + b[y, B1] ≡ LS[0]) ⋀
    (div_x[A1] + div_y[B1] ≡ CWS[0]) ⋀ (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]
```

dims1

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {A1s[y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {A1s[x, x, x, x, y, x, y, y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y, y] → 0}.

dims1

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims1

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

Section 3.4 - Drinfel’d Associators

4T

```
{b[t[1, 3], t[4, 2]], b[t[1, 2] + t[1, 3], t[2, 3]]}
```

4T

```
{0, 0}
```

DKExample

```
b[t[1, 3], t[1, 2]]
```

DKExample

$\mathrm{DK}\left[3, -\overline{1\,2}\right]$

DKSExample

```
b[t[1, 3], t[1, 2]] // DKS
```

DKSExample

$\text{DKS}\left[0,\ -\overline{t_{13}\,t_{23}},\ 0,\ 0,\ \ldots\right]$

sigmaExample

```
{t[2, 3]^σ[{{2,4},{1,5},{3,7,8},{9}] // DKS, t[2, 3]^σ[24,15,378,9] // DKS}
```

sigmaExample

$\left\{\text{DKS}\left[\overline{t_{13}}+\overline{t_{17}}+\overline{t_{18}}+\overline{t_{35}}+\overline{t_{57}}+\overline{t_{58}},\ 0,\ 0,\ 0,\ \ldots\right],\ \text{DKS}\left[\overline{t_{13}}+\overline{t_{17}}+\overline{t_{18}}+\overline{t_{35}}+\overline{t_{57}}+\overline{t_{58}},\ 0,\ 0,\ 0,\ \ldots\right]\right\}$

BCH4DK

```
R = DKS[t[1, 2] / 2];
{R ** R^σ[2,3], R ** R^σ[12,3]}
```

BCH4DK

$\left\{\text{DKS}\left[\frac{\overline{t_{12}}}{2}+\frac{\overline{t_{23}}}{2},\ -\frac{1}{8}\overline{t_{13}\,t_{23}},\ -\frac{1}{48}\overline{\overline{t_{13}\,t_{23}}\,t_{23}}+\frac{1}{96}\overline{t_{13}\,\overline{t_{13}\,t_{23}}},\ -\frac{1}{384}\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}+\frac{1}{384}\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}},\ \ldots\right],\ \text{DKS}\left[\frac{\overline{t_{12}}}{2}+\frac{\overline{t_{13}}}{2}+\frac{\overline{t_{23}}}{2},\ 0,\ 0,\ 0,\ \ldots\right]\right\}$

Phi

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1 / 24; Φ0 = DKS[3, Φs];
SeriesSolve[Φ0,
  (Φ0^σ[3,2,1] ≡ -Φ0) ⋀ (Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34])];
Φ0@
 {6}
```

Phi

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

Phi

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

Phi

$\text{DKS}\Big[0,\ \frac{1}{24}\overline{t_{13}\,t_{23}},\ 0,\ -\frac{7\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}{5760}+\frac{7\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}{5760}-\frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}{1440},$

$0,\ \frac{31\,\overline{\overline{\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}\,t_{23}}\,t_{23}}}{967\,680}-\frac{157\,\overline{t_{13}\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,\overline{t_{13}\,t_{23}}}}}{1\,935\,360}-\frac{31\,\overline{\overline{t_{13}\,t_{23}}\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}}{387\,072}-$

$\frac{31\,\overline{t_{13}\,\overline{\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}\,t_{23}}}}{483\,840}+\frac{11\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,\overline{t_{13}\,t_{23}}}}}}{290\,304}+\frac{31\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}}{725\,760}+$

$\frac{83\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}}}{967\,680}-\frac{13\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}}}{241\,920}+\frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}}}{60\,480},\ \ldots\Big]$

Hexagons

```
R = DKS[t[1, 2] / 2];
(R^σ[12,3] ≡ Φ0 ** R^σ[2,3] ** (-Φ0)^σ[1,3,2] ** R^σ[1,3] ** Φ0^σ[3,1,2] ⋀
  (-R)^σ[12,3] ≡ Φ0 ** (-R)^σ[2,3] ** (-Φ0)^σ[1,3,2] ** (-R)^σ[1,3] ** Φ0^σ[3,1,2]) @{6}
```

Hexagons

BS[7 True, ...]

Section 3.5 - Associators in $\mathcal{A}^w$

PhiV

```
V12 = V0 // dσ[{x, y} → {1, 2}];
ΦV = (V12 // dA)^σ[12,3] ** (V12 // dA)^σ[1,2] ** V12^σ[2,3] ** V12^σ[1,23]
```

PhiV

Es[

$$\Big\langle 1 \to LS\Big[0,\ \frac{\overline{2\,3}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1440}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{5760}-\frac{7\,\overline{2\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{2\,\overline{\overline{2\,3}\,3}}}{5760}+\frac{1}{480}\overline{\overline{1\,2}\,\overline{1\,3}}-\frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1920}+\frac{1}{640}\overline{\overline{1\,\overline{2\,3}}\,2}-\frac{\overline{\overline{\overline{1\,3}\,2}\,2}}{1152}-\frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1152}-\frac{\overline{\overline{\overline{2\,3}\,3}\,3}}{1440},\ \ldots\Big],$$

$$2 \to LS\Big[0,\ -\frac{\overline{1\,3}}{24},\ 0,\ \frac{\overline{1\,\overline{1\,\overline{1\,3}}}}{1440}-\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1152}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{1920}-\frac{1}{480}\overline{1\,\overline{\overline{1\,3}\,2}}-\frac{\overline{1\,\overline{\overline{1\,3}\,3}}}{5760}+\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1152}+\frac{7\,\overline{\overline{1\,2}\,\overline{1\,3}}}{5760}+\frac{19\,\overline{\overline{1\,3}\,\overline{2\,3}}}{5760}+\frac{7\,\overline{\overline{1\,\overline{2\,3}}\,2}}{1920}+\frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760}+\frac{7\,\overline{\overline{\overline{1\,3}\,3}\,2}}{5760}+\frac{\overline{\overline{\overline{1\,3}\,3}\,3}}{1440},\ \ldots\Big],$$

$$3 \to LS\Big[0,\ \frac{\overline{1\,2}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{1440}+\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{5760}-\frac{\overline{1\,\overline{\overline{1\,3}\,2}}}{1440}-\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1440}+\frac{\overline{\overline{1\,2}\,\overline{1\,3}}}{5760}+\frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1440}-\frac{\overline{\overline{1\,\overline{2\,3}}\,2}}{1152}-\frac{7\,\overline{\overline{\overline{1\,2}\,2}\,2}}{5760}-\frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760}-\frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1440},\ \ldots\Big]\Big\rangle,\ CWS[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

PentPhiV

```
ΦV ** ΦV^σ[1,23,4] ** ΦV^σ[2,3,4] ≡ ΦV^σ[12,3,4] ** ΦV^σ[1,2,34]
```

PentPhiV

```
BS[5 True, ...]
```

Phi_is_sder

```
ϕ = (ΦV // Λ)[[1]];
(b[LW@1, ϕ1] + b[LW@2, ϕ2] + b[LW@3, ϕ3])@{6}
```

Phi_is_sder

```
LS[0, 0, 0, 0, 0, 0, ...]
```

DK2Es

```
DK2Es[s___][ζ_] := El[ζ // αMap[s], CWS[0]] // Γ;
DK2Es[1, 2, 3][Φ0]
```

DK2Es

```
Es[
 ⟨1 → LS[0, (2 3)/24, 0, -(1 1 2 3)/1440 + (7 1 2 2 3)/5760 + (1 2 3 3)/5760 - (7 2 2 2 3)/5760 + (7 2 2 3 3)/5760 + 1/480 (1 2 1 3) - (1 3 2 3)/1920 +
    1/640 (1 2 3 2) - (1 3 2 2)/1152 - (1 3 3 2)/1152 - (2 3 3 3)/1440, ...],
  2 → LS[0, -(1 3)/24, 0, (1 1 1 3)/1440 - (1 1 2 3)/1152 + (7 1 2 2 3)/1920 - 1/480 (1 1 3 2) - (1 1 3 3)/5760 + (1 2 3 3)/1152 +
    (7 1 2 1 3)/5760 + (19 1 3 2 3)/5760 + (7 1 2 3 2)/1920 + (7 1 3 2 2)/5760 + (7 1 3 3 2)/5760 + (1 3 3 3)/1440, ...],
  3 → LS[0, (1 2)/24, 0, -(1 1 1 2)/1440 + (1 1 2 3)/5760 + (7 1 2 2 3)/5760 + (7 1 1 2 2)/5760 - (1 1 3 2)/1440 - (1 2 3 3)/1440 +
    (1 2 1 3)/5760 + (1 3 2 3)/1440 - (1 2 3 2)/1152 - (7 1 2 2 2)/5760 - (7 1 3 2 2)/5760 - (1 3 3 2)/1440, ...]⟩, CWS[0, 0, 0, 0, ...]]
```

The computation below takes a a couple of hours and yields “BS[8 True,False,...]”:

```
TrueQ[DK2Es[1, 2, 3][Φ0] ≡ Φv]@{8}
```

```
BS[8 True, False, ...]
```

Section 3.6 - Solving the Kashiwara-Vergne Equations Using a Drinfel'd Associator

ZB

```
R = DKS[t[1, 2] / 2];
ZB = (-Φ0)^σ[13,2,4] ** Φ0^σ[1,3,2] ** R^σ[2,3] ** (-Φ0)^σ[1,2,3] ** Φ0^σ[12,3,4]
```

ZB

$$\mathrm{DKS}\Big[\frac{\overline{t_{23}}}{2},\ -\frac{1}{12}\overline{t_{13}t_{23}}-\frac{1}{24}\overline{t_{14}t_{24}}+\frac{1}{24}\overline{t_{14}t_{34}}+\frac{1}{12}\overline{t_{24}t_{34}},\ 0,\ \frac{\overline{\overline{\overline{t_{13}t_{23}}t_{23}}t_{23}}}{5760}+\frac{7\,\overline{\overline{\overline{t_{14}t_{24}}t_{24}}t_{24}}}{5760}+\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{24}}t_{24}}}{1920}-\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{24}}}{1920}-\frac{7\,\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{34}}}{5760}-$$
$$\frac{\overline{\overline{\overline{t_{24}t_{34}}t_{34}}t_{34}}}{5760}+\frac{\overline{\overline{t_{14}\overline{t_{24}t_{34}}}t_{24}}}{1920}+\frac{\overline{\overline{t_{14}t_{24}}\,\overline{t_{14}t_{34}}}}{1920}-\frac{\overline{\overline{t_{14}t_{34}}\,\overline{t_{24}t_{34}}}}{1920}-\frac{1}{720}\overline{t_{13}\overline{\overline{t_{13}t_{23}}t_{23}}}+$$
$$\frac{1}{720}\overline{t_{13}\overline{t_{13}\overline{t_{13}t_{23}}}}-\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{24}}t_{24}}}}{5760}+\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{34}}t_{34}}}}{5760}-\frac{\overline{t_{14}\overline{\overline{t_{24}t_{34}}t_{34}}}}{5760}+\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{24}}}}}{1440}-$$
$$\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{34}}}}}{1440}-\frac{1}{960}\overline{t_{14}\overline{t_{14}\overline{t_{24}t_{34}}}}+\frac{\overline{t_{14}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760}-\frac{1}{960}\overline{t_{24}\overline{\overline{t_{24}t_{34}}t_{34}}}-\frac{\overline{t_{24}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760},\ \ldots\Big]$$

VfromPhi

```
ZB // DK2Es[1, 2, 3, 4] // tη^1 // tη^3
```

VfromPhi

$$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[0,\ -\frac{\overline{24}}{24},\ 0,\ \frac{\overline{72\overline{2\overline{24}}}}{5760}-\frac{\overline{72\overline{\overline{24}4}}}{5760}+\frac{\overline{\overline{\overline{24}4}4}}{1440},\ \ldots\Big],$$
$$2\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots],\ 3\to \mathrm{LS}\Big[\frac{\overline{2}}{2},\ -\frac{\overline{24}}{12},\ 0,\ \frac{\overline{2\overline{2\overline{24}}}}{5760}-\frac{1}{720}\overline{2\overline{\overline{24}4}}+\frac{1}{720}\overline{\overline{\overline{24}4}4},\ \ldots\Big],$$
$$4\to \mathrm{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big\rangle,\ \mathrm{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

The computation below takes a few hours and yields "BS[8 True,False,...]":

```
VB = ZB // DK2Es[1, 2, 3, 4] // tη^1 // tη^3 // hη^2 // hη^4 // hσ[{1, 3} → {x, y}] //
  tσ[{2, 4} → {x, y}]; TrueQ[VB[[1]] ≡ V0[[1]]] @ {8}
```

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

```
BS[8 True, False, ...]
```

nu

```
νinv = Φ0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

nu

$\text{Es}\left[\langle \overline{x} \to \text{LS}[0, 0, 0, 0, \ldots]\rangle, \text{CWS}\left[0, \frac{\widehat{xx}}{24}, 0, -\frac{\widehat{xxxx}}{2880}, \ldots\right]\right]$

nucap4

```
(νinv ** Cap ** Cap ** Cap ** Cap) @ {6}
```

nucap4

$\text{Es}[\langle \overline{x} \to \text{LS}[0, 0, 0, 0, 0, 0, \ldots]\rangle, \text{CWS}[0, 0, 0, 0, 0, 0, \ldots]]$

Phi1

```
Φ1 = (Φ0)3 // LieMorphism[LW@1 → -x - y, LW@2 → y]
```

Phi1

$\text{LS}\left[0, -\frac{\overline{x\,y}}{24}, 0, \frac{\overline{x\,\overline{x\,\overline{x\,y}}}}{1440} - \frac{\overline{x\,\overline{\overline{x\,y}\,y}}}{5760} + \frac{\overline{\overline{\overline{x\,y}\,y}\,y}}{1440}, \ldots\right]$

F

```
F = ⟨x → LieMorphism[y → -x - y][-Φ1],
   y → LS[(x + y) / 2] ~BCH~ LieMorphism[x → y, y → -x - y][-Φ1] ~BCH~ LS[-y / 2]⟩
```

F

$\left\langle \overline{x} \to \text{LS}\left[0, -\frac{\overline{x\,y}}{24}, 0, \frac{7\,\overline{x\,\overline{x\,\overline{x\,y}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{x\,y}\,y}}}{5760} + \frac{\overline{\overline{\overline{x\,y}\,y}\,y}}{1440}, \ldots\right],\right.$

$\left. \overline{y} \to \text{LS}\left[\frac{\overline{x}}{2}, -\frac{\overline{x\,y}}{12}, 0, \frac{\overline{x\,\overline{x\,\overline{x\,y}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{x\,y}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{x\,y}\,y}\,y}, \ldots\right]\right\rangle$

FV

```
(F ≡ V0⟦1⟧) @ {7}
```

FV

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

FV

```
BS[8 True, ...]
```

Section 3.7 - A Potential S_4 Action on Solutions of KV

rho2

```
ρ2[V_] := V // (-1)^deg;
V1 = Es[⟨x → LS[0], y → LS[-x / 2]⟩, CWS[0]] ** V0;
{(V1 ≡ ρ2[V1]) @ {8}, (V0 ≡ Rs[x, y] ** ρ2[V0]) @ {8}}
```

rho2

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

rho2

```
{BS[9 True, ...], BS[9 True, ...]}
```

rho3

```
ρ3[ξ_Es] := ξ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
ξc = RandomEsSeries[1, {x, y}];
ξc ≡ (ξc // ρ3 // ρ3 // ρ3)
```

rho3

```
BS[5 True, ...]
```

V2

```
V2 = V0 ** Θs[x, y, -1/4] **
    Es[⟨x → LS@0, y → LS@0⟩, CWS[cw[x]/12 - cw[y]/12] - (2 Cap〚2〛 // tΔ[x, x, y])];
(V2 ≡ ρ3[V2])@{6}
```

V2

```
BS[7 True, ...]
```