

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS IV: SOME COMPUTATIONS

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ABSTRACT. In the previous three papers in this series, [WK01]–[WK03], Z. Dancso and I studied a certain theory of “homomorphic expansions” of “w-knotted objects”, a certain class of knotted objects in 4-dimensional space. When all layers of interpretation are stripped off, what remains is a study of a certain number of equations written in a family of spaces \mathcal{A}^w , closely related to degree-completed free Lie algebras and to degree-completed spaces of cyclic words.

The purpose of this paper is to introduce mathematical and computational tools that enable explicit computations (up to a certain degree) in these \mathcal{A}^w spaces and to use these tools to solve the said equations and verify some properties of their solutions, and as a consequence, to carry out the computation (up to a certain degree) of certain knot-theoretic invariants discussed in [WK01]–[WK03] and in my related paper [BN4].

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1. INTRODUCTION

Within the previous three papers in this series [WKO1]–[WKO3]¹ a number of intricate equations written in various graded spaces related to free Lie algebras and to spaces of cyclic words were examined in detail, for good reasons that were explained there and elsewhere. The purpose of this paper is to introduce mathematical tools (on the upper parts of pages) and computational tools (on the lower parts of pages, below the bold dividing lines^{C1}) that allow for the explicit solution of these equations, at least up to a certain degree.

The equations we have in mind arise in other papers and appear throughout this paper. Yet to help our impatient readers orient themselves, Figure 1.1 contains a “flash summary” of the most important equations and their topological and algebraic significance.

Why bother? What do limited explicit computations add, given that these intricate equations are known to be soluble, and given that the conceptual framework within which these equations make sense is reasonably well understood [WKO1]–[WKO3]? My answers are three:

¹Also within my [BN4], and within papers by Alekseev, Enriquez, and Torossian [AT, AET], and within Kashiwara’s and Vergne’s [KV], and also within many older papers about Drinfel’d associators (e.g. Drinfel’d’s [Dr1, Dr2] and my [BN2]).

^{C1}If you are not interested in the actual computations, it is safe to ignore the parts of pages below the bold dividing lines and restrict to “strict” mathematics, which is always above these lines. **Alert.** If you are interested in the computations, note that the computational footnotes are sometimes long and crawl across page boundaries. This footnote is the first example.

The programs described in this paper were written in Mathematica [Wo] and are available at [WKO4]. Before starting with any computations, download the packages `FreeLie.m` and `AwCalculus.m` and type within Mathematica: (the interactive Mathematica session demonstrated in this paper is available as [WKO4]/WKO4Session.nb)

```

<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;

```

```

FreeLie` implements / extends
{*, +, **, $SeriesShowDegree, <>, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Λ, σ, ℏ, ←, ↗}.

```

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150909.

The last input (“human”) line above declares that by default we wish the computer to print series within graded spaces (such as free Lie algebras) to degree 4. Note that we highlight in pink input lines that affect later computations.

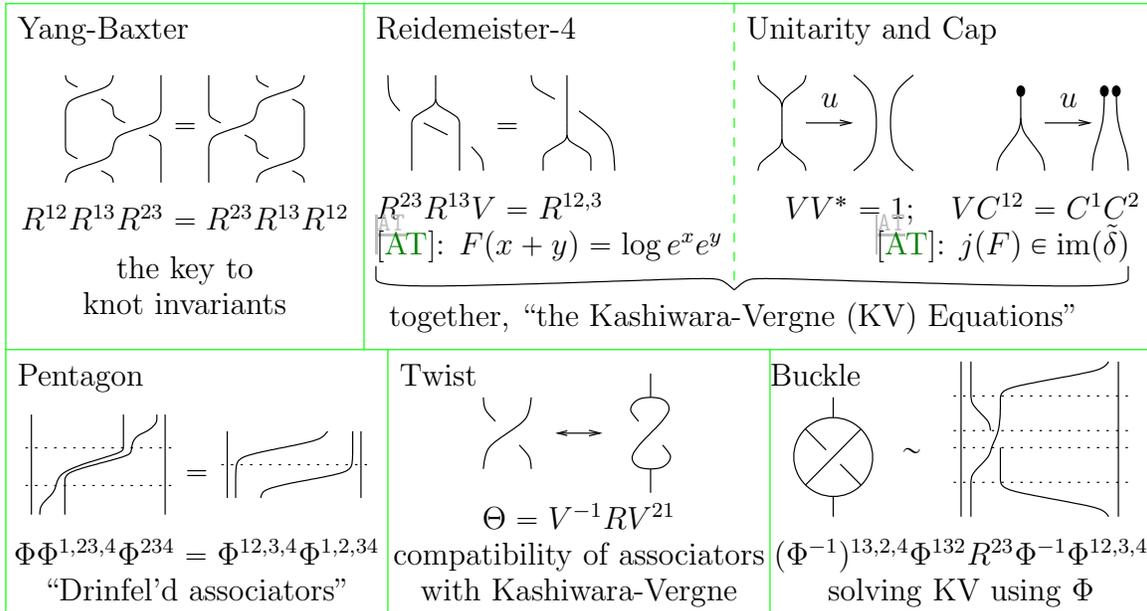


Figure 1.1. The most important equations.

fig:flash

- (1) Personally, my belief in what I can’t compute decays quite rapidly as a function of the complexity involved. Even if the overall picture is clear, the details will surely go wrong, and sooner or later, something bigger than a detail will go wrong. Even a limited computation may serve as a wonderful sanity check. In situations such as ours, where many signs and conventions need to be decided and may well go wrong, even a low-degree computation increases my personal confidence level by a great degree. Given computations that work to degree 6 (say), it is hard to imagine that a detail was missed or that conventions were established in an inconsistent manner. In fact, if the computer programs are clear enough and are shown to work, these programs become the authoritative declarations of the details and conventions.
- (2) The computational tools introduced here may well be useful in other contexts where free Lie algebras and/or cyclic words arise.
- (3) The papers [WKO1, WKO2] (and likewise [BN4]) are about equations, but even more so, about the construction of certain knot and tangle invariants. With the tools presented here, the invariants of arbitrary knotted objects of the types studied in [WKO1, WKO2, BN4] may be computed.

The equations of [WKO1]–[WKO3] always involve group-like, or “exponential” elements, and are written in some spaces of “arrow diagrams” that go under the umbrella name \mathcal{A}^w . Hence a crucial first step is to find convenient presentations for the group-like elements $\mathcal{A}_{\text{exp}}^w$ in \mathcal{A}^w -spaces. It turns out that there are (at least) two such presentations, each with its own advantages and disadvantages. Hence in Section 2 we recall \mathcal{A}^w briefly (2.1), then discuss some free-Lie-algebra preliminaries (2.2), then describe the Aleksey-Torossian-[AT]-inspired “lower-interlaced” presentation E_l of $\mathcal{A}_{\text{exp}}^w$ (2.3), then describe the [BN4]-inspired “factored” presentation E_f of $\mathcal{A}_{\text{exp}}^w$ and its stronger precursor “split” presentation E_s (2.4), and then describe how to convert between the two primary presentations (2.5).

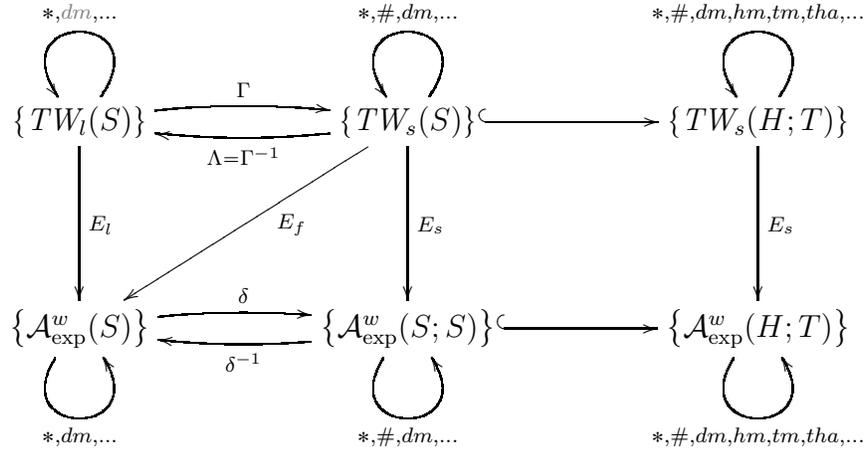


Figure 1.2. The main spaces and maps appearing in this paper.

fig:diagram

We then present our computations in Section 3: Some knot and tangle invariants are computed in Section 3.1 and solutions of the Kashiwara-Vergne (KV) equations in Section 3.2. In Section 3.3 we discuss the “Twist Equation” and compute dimensions of spaces of solutions of the linearized KV equations, with and without the Twist Equation. In Section 3.4 we compute a Drinfel’d associator, in Section 3.5 we compute associators in \mathcal{A}^w starting from a solution of the KV equations, and in Section 3.6 we show how to compute a solution of KV from a Drinfel’d associator. The last computational result is in Section 3.7, where we give computational support to the existence of an action of the symmetric group S_4 on the set of solutions of the Kashiwara-Vergne Equations.

We conclude this introduction with a description of the commutative diagram in Figure 1.2 which displays the main spaces and maps appearing in this paper, as described in detail in Section 2. The bottom row of this diagram consists of spaces of “group-like” elements inside spaces \mathcal{A}^w of “arrow diagrams”; these are the spaces that have direct knot-theoretic significance. The top row are spaces of “trees and wheels”, or more precisely, various elements of free Lie algebras and various cyclic words. They are the spaces of “primitives” corresponding to the group-like elements at the bottom, via various “exponentiation” maps E_l , E_f , and E_s . In this paper we study^{C2} the spaces on the bottom row by means of their presentations by elements in the top row.

The collection $\{\mathcal{A}_{\text{exp}}^w(S)\}$ of spaces we primarily wish to study (and in which most of the equations of Figure 1.1 are written) appears on the bottom left. There are many binary and unary operations acting on the spaces within $\{\mathcal{A}_{\text{exp}}^w(S)\}$ as indicated by the circular self-arrow appearing there, which is labelled with the most important of these operations, the binary $*$ and the unary dm . On the top left of the diagram are the spaces $\{TW_l(S)\}$ of trees and wheels which represent $\{\mathcal{A}_{\text{exp}}^w(S)\}$ via the E_l presentation. The same collection of operations acts here too, though notice that the operation dm is grayed-out, because we have no direct implementation for it in TW_l language.

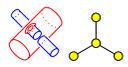
On the bottom right is a bigger collection of spaces, $\{\mathcal{A}_{\text{exp}}^w(H; T)\}$, which contains as a subset the collection $\{\mathcal{A}_{\text{exp}}^w(S; S)\}$ (bottom middle), which is isomorphic in a non-trivial

^{C2}Or “implement”, in computer-speak.

manner (via δ and δ^{-1}) to $\{\mathcal{A}_{\text{exp}}^w(S)\}$. A richer collection of operations act on $\{\mathcal{A}_{\text{exp}}^w(H;T)\}$, and the most important of those are $*$, $\#$, dm , hm , tm , and tha .

On the top right is the collection $\{TW_s(H;T)\}$ of spaces of trees and wheels which represent $\{\mathcal{A}_{\text{exp}}^w(H;T)\}$ via the E_s presentation. When restricted to $H = T = S$, this is the collection $\{TW_s(S)\}$ representing $\{\mathcal{A}_{\text{exp}}^w(S;S)\}$, and representing our primary interest $\{\mathcal{A}_{\text{exp}}^w(S)\}$ via E_f , the composition of E_s with δ^{-1} .

Note that TW_l and TW_s are set-theoretically the same spaces of trees and wheels. Yet the operations $*$, dm , etc. act on them in a different manner, and hence they deserve to have different names². Note also that TW_l and TW_s are in fact isomorphic via structure-preserving isomorphisms (denoted Γ and $\Lambda = \Gamma^{-1}$). These isomorphisms are compositions of the relatively simple-minded δ and δ^{-1} with the more complex “exponentiations” E_l and E_s and their inverses. Thus the isomorphisms Γ and Λ are non-linear and quite complicated.



We will occasionally comment on the relationship between the constructs appearing in this papers and three related topics: “topology”, or more precisely certain aspects of the theory of 2-knots, “Lie theory”, or more precisely certain classes of formulas that make sense in arbitrary finite-dimensional Lie algebras, and “Alekseev-Torossian”, or more precisely, issues related to the paper [AT]. These comments will in general be incomplete and should be regarded as “hints for the already initiated” — people familiar with the papers [WKO1, WKO2, WKO3, BN4, AT] will hopefully find that these comments help to put the current paper in context. These comments will always be labelled by one (or more) of the three logos at the head of this paragraph, which correspond, in order, to “topology”, “Lie theory”, and “Alekseev-Torossian”.



Within the study of simply-knotted (ribbon) 2-knots, or more precisely w-knotted-objects as they appear in [WKO1, WKO2, BN4], the rows of Figure I.2 correspond to the extra row

$$\begin{array}{ccccc}
 \{\mathcal{K}^w(S)\} & \xrightleftharpoons[\delta^{-1}]{\delta} & \{\mathcal{K}^w(S;S)\} & \hookrightarrow & \{\mathcal{K}^w(H;T)\}, \\
 \text{⤵} & & \text{⤵} & & \text{⤵} \\
 *,dm,\dots & & *,\#,dm,\dots & & *,\#,dm,hm,tm,tha,\dots
 \end{array}$$

via the “associated graded” procedure described in [WKO2]. Here $\mathcal{K}^w(S)$ is the set of S -labelled w-tangles [WKO2], $\mathcal{K}^w(H;T)$ is the set of w-knotted H -labelled hoops and T -labelled balloons [BN4], $\mathcal{K}^w(S;S)$ is the same but with $H = T = S$, and δ is the same as in [BN4]. This correspondence is further recalled throughout the rest of this paper.



The corresponding Lie-theoretic spaces (compare [WKO1, Section 3.5]) are

$$\begin{array}{ccccc}
 \{\mathcal{U}(I\mathfrak{g})^{\otimes S}\} & \xrightleftharpoons[\delta^{-1}]{\delta} & \{\mathcal{U}(\mathfrak{g})^{\otimes S} \otimes \mathcal{S}(\mathfrak{g}^*)^{\otimes S}\} & \hookrightarrow & \{\mathcal{U}(\mathfrak{g})^{\otimes H} \otimes \mathcal{S}(\mathfrak{g}^*)^{\otimes T}\}. \\
 \text{⤵} & & \text{⤵} & & \text{⤵} \\
 *,dm,\dots & & *,\#,dm,\dots & & *,\#,dm,hm,tm,tha,\dots
 \end{array}$$

This correspondence is further recalled throughout the rest of this paper.

²Much as in group theory, a direct product $N \times H$ is set-theoretically the same as a semi-direct product $N \rtimes H$, yet it is wrong to refer to them by the same name.

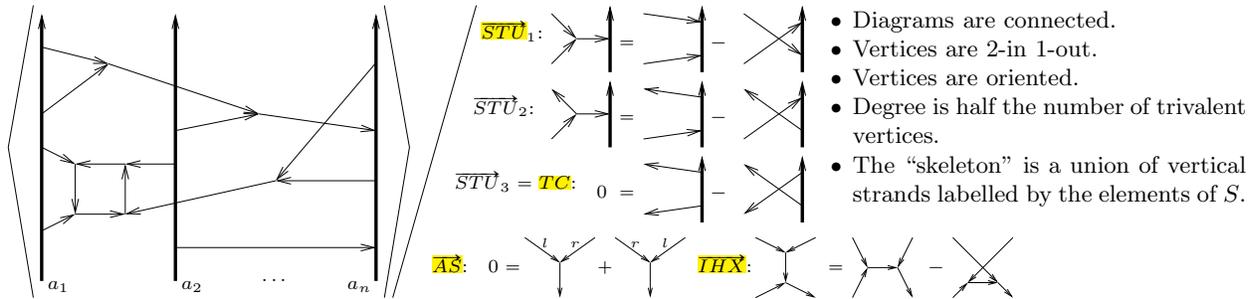
[AT] In [AT] there is no good counterparts for last two columns of our diagram. The counterpart of the first (and primary) column is a mixture $\hat{\mathcal{U}}((\mathbf{a}_n \oplus \mathbf{t} \mathbf{d} \mathbf{e} \mathbf{t}_n) \times \mathbf{t} \mathbf{r}_n)$ containing the most important spaces occurring in [AT]. More in the next section.

1.1. **Acknowledgement.** This paper was written almost entirely with Z. Dancso in the room (physically or virtually via Skype), working on various parts of our joint series [WKO1]–[WKO3]. Hence her indirect contribution to it, in a huge number of routine consultations, should be acknowledged in capitals: THANKS, ZSUZSI. I would like to further thank A. Alekseev and S. Morgan for their comments and suggestions.

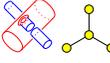
subsec:Aw

2. GROUP-LIKE ELEMENTS IN \mathcal{A}^w

2.1. **A brief review of \mathcal{A}^w .** Let $S \doteq \{a_1, a_2, \dots\}^3$ be a finite set of “strand labels”. The space $\mathcal{A}^w(S)$ is the completed graded vector space⁴ of diagrams made of (vertical) “strands” labelled by the elements of S , and “arrows” as summarized by the following picture:



When $S = \{1, 2, \dots, n\}$ we abbreviate $\mathcal{A}^w(\uparrow_n) \doteq \mathcal{A}^w(S)$.

 **[AT]** In topology, elements of $\mathcal{A}^w(S)$ are closely related to (finite type invariants of) simply knotted 2-dimensional tubes in \mathbb{R}^4 ([WKO1]–[WKO3], [BN4]). In Lie theory, they represent “universal” \mathfrak{g} -invariant tensors in $\mathcal{U}(I\mathfrak{g})^{\otimes S}$, where $I\mathfrak{g} \doteq \mathfrak{g} \times \mathfrak{g}^*$ ⁵ and \mathfrak{g} is some finite dimensional Lie algebra ([WKO1]–[WKO3]). Readers of Alekseev and Torossian [AT] may care about \mathcal{A}^w because using notation from [AT], $\mathcal{A}^w(\uparrow_n)$ is the completed universal enveloping algebra of $(\mathfrak{a}_n \oplus \mathfrak{tder}_n) \times \mathfrak{tr}_n$ (see [WKO2]), and hence much of the [AT] story can be told within \mathcal{A}^w . Several significant Lie theoretic problems (e.g., the Kashiwara-Vergne problem, [KV, AT, WKO2]) can be interpreted as problems about $\mathcal{A}^w(\uparrow_n)$.

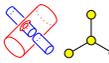
SortedForm

Comment 2.1. Using the \overrightarrow{STU}_2 relation one may sort the skeleton vertices in every $D \in \mathcal{A}^w(S)$ so that along every skeleton component all arrow heads appear ahead of all arrow tails, and by a diagrammatic analogue of the PBW theorem (compare [BN1, Theorem 8]), this sorted form is unique modulo \overrightarrow{STU}_1 , TC , \overrightarrow{AS} and \overrightarrow{IHX} relations.

Operations

Definition 2.2. A number of operations are defined on elements of the $\mathcal{A}^w(S)$ spaces:

- (1) If S_1 and S_2 are disjoint, then given $D_1 \in \mathcal{A}^w(S_1)$ and $D_2 \in \mathcal{A}^w(S_2)$, their union $D_1 D_2 = D_1 \sqcup D_2 \in \mathcal{A}^w(S)$, where $S = S_1 \sqcup S_2$, is obtained by placing them side by side as illustrated on the right.

 In topology, \sqcup corresponds to the disjoint union of 2-tangles⁶. In Lie theory, it corresponds to the map $\mathcal{U}(I\mathfrak{g})^{\otimes S_1} \otimes \mathcal{U}(I\mathfrak{g})^{\otimes S_2} \rightarrow \mathcal{U}(I\mathfrak{g})^{\otimes (S_1 \sqcup S_2)}$.

³Yellow highlighting corresponds to the [glossary](#), Section 4.

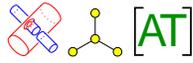
⁴For simplicity we always work over \mathbb{Q} .

⁵In earlier papers we have used the order $I\mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g}$.

⁶To be clear, the “2” in “2-tangles” refers to the dimension of the things being knotted, and not to the number of components.

- (2) Given $D_1 \in \mathcal{A}^w(S)$ and $D_2 \in \mathcal{A}^w(S)$, their product $D_1 \boxed{*} D_2 \in \mathcal{A}^w(S)$ is obtained by “stacking D_2 on top of D_1 ”:

$$(D_1, D_2) = \left(\begin{array}{|c|} \hline \text{---} \\ \hline \boxed{D_1} \\ \hline \text{---} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{---} \\ \hline \boxed{D_2} \\ \hline \text{---} \\ \hline \end{array} \right) \mapsto \begin{array}{|c|} \hline \text{---} \\ \hline \boxed{D_2} \\ \hline \text{---} \\ \hline \boxed{D_1} \\ \hline \text{---} \\ \hline \end{array} = D_1 * D_2. \quad (1) \quad \text{eq: TubePr}$$

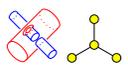


In topology, the stacking product corresponds to the concatenation operation on knotted tubes, akin to the standard stacking product of tangles. In Lie theory, it comes from the algebra structure of $\mathcal{U}(\mathfrak{I}\mathfrak{g})^{\otimes S}$. In [AT], it is the product of the completed universal enveloping algebra $\hat{\mathcal{U}}((\mathfrak{a}_n \oplus \mathfrak{t}\mathfrak{d}\mathfrak{e}\mathfrak{r}_n) \times \mathfrak{t}\mathfrak{r}_n)$.

Note that below and throughout this paper we use \boxed{f} for postfix operator application and for “composition done right”. Meaning that $x \boxed{f}$ is equivalent to $f(x)$ and $f \boxed{g}$ is $g \circ f$ is “do f then do g ”.

- (3) Given $D \in \mathcal{A}^w(S)$ and $a \in S$, $D \boxed{d\eta^a}$ is the result of deleting strand a from D and mapping it to 0 if any arrow connects to a , as illustrated on the right.

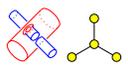
$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \xrightarrow{\boxed{d\eta^a}} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$$



In topology, $d\eta^a$ is the removal of one component from a 2-tangle. In Lie theory it corresponds to the co-unit $\eta: \mathcal{U}(\mathfrak{I}\mathfrak{g}) \rightarrow \mathbb{Q}$.

- (4) Given $D \in \mathcal{A}^w(S)$ and $a \in S$, $D \boxed{dA^a}$ is the result of “flipping over stand a and multiplying by a $(-)$ sign for each arrow whose head connects to a ”, as illustrated above. We denote by dA the operation of likewise flipping (with signs) *all* strands: $dA = dA^S := \prod_{a \in S} dA^a$.

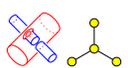
$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \xrightarrow{\boxed{dA^a}} \left(\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right)$$



In topology, dA^a is the reversal of the 1D orientation of a knotted tube [WKO2]. In Lie theory, it is the antipode of $\mathcal{U}(\mathfrak{I}\mathfrak{g})$ combined with the sign reversal $\varphi \rightarrow -\varphi$ acting on the \mathfrak{g}^* factor of $\mathfrak{I}\mathfrak{g}$. When elements of $\mathcal{U}(\mathfrak{I}\mathfrak{g})^{\otimes S}$ are interpreted as differential operators acting on functions on \mathfrak{g}^S , dA corresponds to the L^2 adjoint.

- (5) Similarly, $D \boxed{dS^a}$ is the result of “flipping over stand a and multiplying by a $(-)$ sign for each arrow head or tail that connects to a ”, as illustrated above⁷.

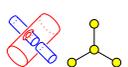
$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \xrightarrow{\boxed{dS^a}} \left(\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}, \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right)$$



In topology, dS^a is the reversal of both the 1D and the 2D orientation of a knotted tube [WKO2]. In Lie theory, it is the antipode of $\mathcal{U}(\mathfrak{I}\mathfrak{g})$.

- (6) Given $D \in \mathcal{A}^w(S)$, given $a, b \in S$, and given $c \notin S \setminus \{a, b\}$, $D \boxed{dm_c^{ab}}$ is the result of “stitching strands a and b and calling the resulting strand c ”, as illustrated on the right.

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \xrightarrow{\boxed{dm_c^{ab}}} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$$

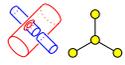
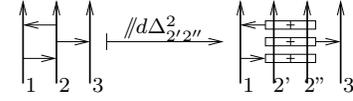


In topology, dm_c^{ab} is the “internal stitching” of two tubes within a single 2-link, akin to the “stitching” operation that combines two strands of an ordinary

⁷The letter S is used here for both “a set of strands” and “an operation similar to an antipode”. Hopefully no confusion will arise.

tangle into a single “longer” one. In Lie theory, it is an “internal product” $\mathcal{U}(\mathfrak{g})^{\otimes n} \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes(n-1)}$ which “merges” two factors within $\mathcal{U}(\mathfrak{g})^{\otimes n}$.

- (7) Given $D \in \mathcal{A}^w(S)$, given $a \in S$, and given $b, c \notin S \setminus a$, $D // d\Delta_{bc}^a$ is the result of “doubling” strand a , calling the resulting “daughter strands” b and c , and summing over all ways of lifting the arrows that were connected to a to either b or c (so if there are k arrows connected to a , $D // d\Delta_{bc}^a$ is a sum of 2^k diagrams).



In topology, $d\Delta$ is the operation of “doubling” one component in a 2-link. In Lie theory, it is the co-product $\Delta: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes 2}$ acting on the a factor in $\mathcal{U}(\mathfrak{g})^{\otimes S}$, extended by the identity acting on all other factors. In [AT], it is the coface maps of [AT, Example 3.14].

- (8) Finally, the operation $d\sigma_b^a: \mathcal{A}(S) \rightarrow \mathcal{A}(S \setminus \{a\} \sqcup \{b\})$ does nothing but renaming the strand a to b (assuming $a \in S$ and $b \notin S \setminus \{a\}$). 2.2

We note that the product operation $(D_1, D_2) \mapsto D_1 * D_2$ can be implemented using the union operation \sqcup , the stitching operation dm , and some renaming — namely, if $\bar{S} = \{\bar{a} : a \in S\}$ is some set of “temporary” labels disjoint from S but in a bijection with S , then

$$D_1 * D_2 = \left(D_1 \sqcup \left(D_2 // \prod_a d\sigma_a^{\bar{a}} \right) \right) // \prod_a dm_a^{\bar{a}}. \tag{2}$$

Therefore below we will sometimes omit the implementation of $(D_1, D_2) \mapsto D_1 D_2$ provided all other operations are implemented.

We note that $\mathcal{A}^w(S)$ is a co-algebra, with the co-product $\square(D)_x$ for a diagram D representing an element of $\mathcal{A}^w(S)$, being the sum of all ways of dividing D between a “left co-factor” and a “right co-factor” so that connected components of $D \setminus (\uparrow \times S)$ (D with its skeleton removed) are kept intact (compare with [BN1, Definition 3.7]).

Definition 2.3. An element Z of $\mathcal{A}^w(S)$ is “group-like” if $\square(Z) = Z \otimes Z$. We denote the set of group-like elements in $\mathcal{A}^w(S)$ by $\mathcal{A}_{\text{exp}}^w(S)$.

We leave it for the reader to verify that all the operations defined above restrict to operations $\mathcal{A}_{\text{exp}}^w \rightarrow \mathcal{A}_{\text{exp}}^w$.

In topology, \square is the operation of “cloning” an entire 2-link. It is not to be confused with $d\Delta$; one dimension down and with just one component, the pictures are:



In [AT], \square is the co-product of $\hat{U}((\mathfrak{a} \oplus \mathfrak{tder}) \times \mathfrak{tr})$ and moding out by wheels, $\mathcal{A}_{\text{exp}}^w$ is TAut.

In Lie theory, \square is *not* the co-product $\Delta: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes 2}$. Rather, given two finite dimensional Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , \square corresponds to the map

$$\square: \mathcal{U}(I(\mathfrak{g}_1 \oplus \mathfrak{g}_2))^{\otimes S} \rightarrow \mathcal{U}(I\mathfrak{g}_1)^{\otimes S} \otimes \mathcal{U}(I\mathfrak{g}_2)^{\otimes S}.$$

Discussion 2.4. We seek to have efficient descriptions of the elements of $\mathcal{A}_{\text{exp}}^w(S)$ and efficient means of computing the above operations on such elements.

Let $\mathcal{A}_{\text{prim}}^w(S)$ ⁸ denote the set of primitives of $\mathcal{A}^w(S)$: these are the elements $\zeta \in \mathcal{A}^w(S)$ satisfying $\square(\zeta) = \zeta \otimes 1 + 1 \otimes \zeta$. Let $FL(S)$ denote the degree-completed free Lie algebra with generators S , and let $CW(S)$ denote the degree-completed vector space spanned by non-empty cyclic words on the alphabet S . In [WKO2, Proposition 3.19] we have shown that there is a short exact sequence of vector spaces

$$0 \rightarrow CW(S) \rightarrow \mathcal{A}_{\text{prim}}^w(S) \rightarrow FL(S)^S \rightarrow 0, \quad (3)$$

where $FL(S)^S$ denotes the set of all functions $S \rightarrow FL(S)$. Hence $\mathcal{A}_{\text{prim}}^w(S) \simeq FL(S)^S \oplus CW(S)$ (not canonically!). Often in bi-algebras there is a bijection given by $\zeta \mapsto e^\zeta$ between primitive elements ζ and group-like elements e^ζ . Hence we may expect to be able to present elements of $\mathcal{A}_{\text{exp}}^w(S)$ as formal exponentials of combinations of “trees” (elements of $FL(S)^S$) and “wheels” (elements of $CW(S)$)⁹:

$$\mathcal{A}_{\text{exp}}^w(S) \sim TW(S) := FL(S)^S \times CW(S) = \left\{ (\lambda; \omega) : \begin{array}{l} \lambda = \{a \rightarrow \lambda_a\}_{a \in S}, \lambda_a \in FL(S) \\ \omega \in CW(S) \end{array} \right\}. \quad (4)$$

We implement Equation (4) in a more-or-less straightforward way in Section 2.3 and in a less straightforward but somewhat stronger way in Section 2.4.

Discussion 2.5. Why are there two presentations for elements of $\mathcal{A}_{\text{exp}}^w$?

Because as we shall see, \mathcal{A}^w is a bi-algebra in two ways, using two different products, yet with the same co-product \square . In \mathcal{A}^w , the notions “primitive” and “group-like”, whose definition involves only \square , are canonical. Yet the bijection between primitive and group-like elements, $\zeta \leftrightarrow e^\zeta$, depends also on the product used within the power-series interpretation of e^ζ . Thus there are two different ways to describe the group-like elements $\mathcal{A}_{\text{exp}}^w$ of \mathcal{A}^w in terms of its primitives TW .

The first product on \mathcal{A}^w is the stacking product of Equation (1). The second will be introduced later, in Equations (18) and (42).

Very roughly speaking, \mathcal{A}^w is a combinatorial model of “ $\pi_1 \times \pi_2$ ” (with homotopies replaced by isotopies; see [BN4]). The other product on \mathcal{A}^w is the one coming from the direct product “ $\pi_1 \times \pi_2$ ”.

Very roughly speaking, \mathcal{A}^w is a combinatorial model of (tensor powers of a completion of) $\mathcal{U}(I\mathfrak{g})$. By PBW, $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$ as co-algebras but not as algebras. The other product on \mathcal{A}^w is the one corresponding to the natural product on $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$. The reality is a bit more delicate, though. \mathcal{A}^w is only a model of (a small part of) the \mathfrak{g} -invariant part of $\mathcal{U}(I\mathfrak{g})$, and the co-product \square of \mathcal{A}^w does not correspond to the co-product Δ of $\mathcal{U}(I\mathfrak{g})$.

⁸ $\mathcal{A}_{\text{prim}}^w$ is elsewhere denoted \mathcal{P}^w .

⁹ We use the set-theoretic notation “ \times ” rather than the linear-algebraic “ \oplus ” in Equation (4) to emphasize that the two sides of that equation are only expected to be set-theoretically isomorphic. The left-hand-side, in fact, is not even a linear space in a natural way.

subsec:FL

2.2. **Some preliminaries about free Lie algebras and cyclic words.** It should be clear from Discussion 2.4 that free Lie algebras and cyclic words play a prominent role in this paper. For the convenience of our readers we collect in this section some preliminaries about about these topics. Almost everything in this section comes either from Alekseev-Torossian's [AT], or from [WKO2, BN4], and the detailed proofs of the assertions made here can be found in these papers.

 Note that Lie algebras appear in two distinct roles in this paper. *Free* Lie algebras *FL* appear along with cyclic words *CW* as the primitives of \mathcal{A}^w (Equation (3)). *Finite dimensional* Lie algebras \mathfrak{g} appear only as motivational comments, always marked with a  symbol. As already indicated, elements in \mathcal{A}^w , and hence elements of *FL* and of *CW* can represent “universal” formulas that make sense in any finite dimensional Lie algebra \mathfrak{g} . Hence part of our discussion of *FL* and *CW* is a discussion of things that make sense universally for all finite dimensional Lie algebras.

Recall that $FL(S)$ denotes the graded completion of the free Lie algebra over a set of generators S , all considered to have degree 1. In the case when $S = \{x_1, \dots, x_n\}$, Alekseev and Torossian [AT] denote this space \mathfrak{lie}_n^{C3}

A noteworthy element of $FL(x, y)$ is the Baker-Campbell-Hausdorff series, ^{C4}

$$\mathbf{BCH}(x, y) := \log(e^x e^y) = x + y + \frac{[x, y]}{2} + \frac{[x, [x, y]] + [[x, y], y]}{12} + \dots$$

Recall also that $CW(S)$ (\mathfrak{lc}_n in [AT]) denotes the graded completion of the vector space spanned by non-empty cyclic words in the alphabet S . Our convention is to crown cyclic

^{C3}In computer talk, generators of $FL(S)$ are always single-character “Lyndon words” (e.g. [Re]); in our case we set x and y to be the single-character words “ x ” and “ y ”, and then α, β , and γ to be the Lie series $x + [x, y]$, $y - [x, [x, y]]$, and $x + y - 2[x, y]$ (elements of *FL* are infinite series, in general, but these examples are finite):

 $\mathbf{x} = \mathbf{LW@} \text{"x"}; \mathbf{y} = \mathbf{LW@} \text{"y"};$
 $\{\alpha, \beta, \gamma\} = \mathbf{LS} / @ \{\mathbf{x} + \mathbf{b}[x, y], \mathbf{y} - \mathbf{b}[x, \mathbf{b}[x, y]], \mathbf{x} + \mathbf{y} - 2 \mathbf{b}[x, y]\}$

 $\{\mathbf{LS}[\overline{x}, \overline{xy}, 0, 0, \dots], \mathbf{LS}[\overline{y}, 0, -\overline{xxy}, 0, \dots], \mathbf{LS}[\overline{x+y}, -2\overline{xy}, 0, 0, \dots]\}$

Note that as we requested earlier, our example series are printed to degree 4. Note also that they are printed using “top bracket” $\overline{xy} := [x, y]$ notation, which is easier to read when many brackets are nested.

We then compute $[\alpha, \beta]$ and verify the Jacobi identity for α, β , and γ :

 $\{\mathbf{b}[\alpha, \beta], \mathbf{b}[\alpha, \mathbf{b}[\beta, \gamma]] + \mathbf{b}[\beta, \mathbf{b}[\gamma, \alpha]] + \mathbf{b}[\gamma, \mathbf{b}[\alpha, \beta]]\}$

 $\{\mathbf{LS}[0, \overline{xy}, \overline{xyy}, -\overline{xxxy}, \dots], \mathbf{LS}[0, 0, 0, 0, \dots]\}$

^{C4}In computer talk:

 $\mathbf{bch} = \mathbf{BCH}[\mathbf{x}, \mathbf{y}]$

 $\mathbf{LS}[\overline{x+y}, \frac{\overline{xy}}{2}, \frac{1}{12} \overline{xxxy} + \frac{1}{12} \overline{xyy}, \frac{1}{24} \overline{xxxyy}, \dots]$

Just to show that we can, here are the lexicographically middle three of the 2,181 terms of the BCH series in degree 16, along with the time in seconds it took my humble laptop to compute it:

C:bch

C:alphabet

C:BracketE

C:bch16

Fuller output:
[\[WKO4\]/bch.nb](http://www.wko4.net/bch.nb)

There is a unique Lie bracket $[\cdot, \cdot]_{tb}$ (the “tangential bracket”) on $FL(S)^S$ which makes (5) a split exact sequence of Lie algebras, and hence $(FL(S)^S, [\cdot, \cdot]_{tb}) \simeq A_S \oplus \text{tder}_S$ as Lie algebras. With $[\cdot, \cdot]$ denoting the ordinary direct-sum bracket on $FL(S)^S$ and with the action of ∂_λ extended to $\partial_\lambda: FL(S)^S \rightarrow FL(S)^S$ in the obvious manner, we have^{C8}

$$[\lambda_1, \lambda_2]_{tb} = [\lambda_1, \lambda_2] + \partial_{\lambda_1} \lambda_2 - \partial_{\lambda_2} \lambda_1.$$

The $\lambda \mapsto \partial_\lambda$ action of $(FL(S)^S, [\cdot, \cdot]_{tb})$ on $FL(S)$ extends to an action on the universal enveloping algebra of $FL(S)$, the free associative algebra $FA(S)$ on S generators, and then descends to the vector-space quotient of $FA(S)$ by commutators, namely to cyclic words. Leaving aside the empty word, we find that $(FL(S)^S, [\cdot, \cdot]_{tb})$ acts on $CW(S)$, and hence also on $TW(S)$.^{C9}

There are two ways to assign an automorphism of the free Lie algebra $FL(S)$ to an element $\lambda \in FL(S)^S$:

- (1) One may exponentiate the derivation ∂_λ to get $e^{\partial_\lambda}: FL(S) \rightarrow FL(S)$.
- (2) One may define an automorphism $C^\lambda: FL(S) \rightarrow FL(S)$ by setting its values on the generators by $C^\lambda(a) := e^{\lambda a} a e^{-\lambda a} = e^{\text{ad}^{\lambda a}} a$. We denote the inverse of C^λ by $RC^{-\lambda}$ and note that it is *not* $C^{-\lambda}$.

[AT] In [AT], (1) corresponds to the presentation of elements of the automorphism group TAut_n as exponentials of elements of its Lie algebra tder_n , while (2) corresponds to its presentation in terms of “basis conjugating automorphisms” $x_i \mapsto g_i^{-1} x_i g_i$ where $g_i = e^{-\lambda_i}$. Compare with [AT, Section 5.1].

The following pair of propositions, which we could not find elsewhere, relates these two automorphisms:

¹⁰Using the notation of [BN4], $\partial_{\lambda_i} = -\sum_{a \in S} \text{ad}_a^{\lambda_i} = -\sum_{a \in S} \text{ad}_a \{\lambda_i\}$. I apologize for the minus sign which stems from a bad choice made in [BN4].

computations below

C:tb ^{C8}For example:

$$\odot \odot \lambda_1 = \lambda; \lambda_2 = \langle \mathbf{x} \rightarrow \beta, \mathbf{y} \rightarrow \gamma \rangle; \text{tb}[\lambda_1, \lambda_2]$$

$$\left(\overline{\mathbf{x}} \rightarrow \text{LS} [0, 0, \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y}}}}, -\overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y} \overline{\mathbf{y}}}}, \dots], \overline{\mathbf{y}} \rightarrow \text{LS} [0, 0, \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y}}}}, -\overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y} \overline{\mathbf{y}}}}, \dots] \right)$$

^{C9}We check that up to degree 8, $\partial_{[\lambda_1, \lambda_2]_{tb}}(\omega_1) = [\partial_{\lambda_1}, \partial_{\lambda_2}](\omega_1)$ (for our choice of λ_1, λ_2 , and ω_1 , both sides vanish below degree 8):

C:tb2

$$\odot \odot \text{lhs} = \mathbf{D}_{\text{tb}[\lambda_1, \lambda_2]}[\omega_1]; \text{rhs} = \mathbf{b}[\mathbf{D}_{\lambda_1}, \mathbf{D}_{\lambda_2}][\omega_1];$$

$$\{\text{lhs}@\{8\}, (\text{lhs} \equiv \text{rhs})@\{8\}\}$$

$$\left\{ \text{CWS}[0, 0, 0, 0, 0, 0, 0, 0, 18 \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y} \overline{\mathbf{y} \overline{\mathbf{y}}}}}}, -18 \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y} \overline{\mathbf{y} \overline{\mathbf{y}}}}}}, -36 \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y} \overline{\mathbf{y} \overline{\mathbf{y} \overline{\mathbf{y}}}}}}, +36 \overline{\mathbf{x} \overline{\mathbf{x} \overline{\mathbf{y} \overline{\mathbf{y} \overline{\mathbf{y} \overline{\mathbf{y}}}}}}, \dots], \right.$$

$$\left. \text{BS}[9 \text{ True}, \dots] \right\}$$

Note that the comparison operator \equiv returns a “Boolean Sequence” (BS) rather than a single True/False value, as the computer has no way of knowing whether two series are equal without computing them up to a given degree. In our case, we’ve asked for the comparison of **lhs** with **rhs** up to degree 8, and the output, including degree 0, is a sequence of 9 affirmations, summarized as “9 True”.

prop:Gamma

Proposition 2.6. Given $\lambda \in FL(S)^S$, let t be a scalar-valued formal variable and let $\Gamma_t(\lambda) \in \text{at}$ $FL(S)^S$ be the (unique) solution of the ordinary differential equation

$$\Gamma_0(\lambda) = 0 \quad \text{and} \quad \frac{d\Gamma_t(\lambda)}{dt} = \lambda // e^{-t\partial_\lambda} // \frac{\text{ad } \Gamma_t(\lambda)}{e^{\text{ad } \Gamma_t(\lambda)} - 1}. \quad (6)$$

Then
$$e^{-t\partial_\lambda} = C^{\Gamma_t(\lambda)}. \quad (7)$$

Proof. The two sides L_t and R_t of Equation (7) are power-series perturbations of the identity automorphism of $FL(S)$. More fully, L_t can be written $L_t = \sum_{d \geq 0} t^d L(d)$ where $L(d): FL(S) \rightarrow FL(S)$ raises degrees by at least d (and so the sum converges), and where $L(0)$ is the identity. R_t can be written in a similar way. We claim that it is enough to prove that

$$A_t := \left(\frac{dL_t}{dt} \right) // L_t^{-1} = \left(\frac{dR_t}{dt} \right) // R_t^{-1} =: B_t. \quad (8)$$

Indeed, if otherwise $L_t \neq R_t$, consider the minimal d for which $L(d) \neq R(d)$. Then $d > 0$ and the least-degree term in $A_t - B_t$ is the degree $d - 1$ term, which equals $dt^{d-1}L(d) // L_t^{-1} - dt^{d-1}R(d) // R_t^{-1} = dt^{d-1}(L(d) - R(d)) // L_t^{-1} \neq 0$ (the last equality is because $L_t^{-1} = R_t^{-1}$ to degree d), contradicting Equation (8). Note that in fact we have shown that if $A_t = B_t$ to degree d in t , then Equation (7) holds to degree $d + 1$.

To compute B_t we need the differential of C^μ (at $\mu = \Gamma_t(\lambda)$) and the chain rule. The differential of C^μ is quite difficult; fortunately, we have computed it in the case where $\mu = (u \rightarrow \gamma)$ is supported on just one $u \in S$, in [BN4, Lemma 10.7]. Both the result and its proof generalize simply, and so we have

$$\delta C^\mu = -\partial \left\{ \delta \mu // \frac{e^{\text{ad } \mu} - 1}{\text{ad } \mu} // RC^{-\mu} \right\} // C^\mu,$$

where we have written $\partial\{\text{mess}\}$ instead of ∂_{mess} because mess is too big to fit as a subscript. Hence by the chain rule and then by Equation (6),

$$B_t = -\partial \left\{ \frac{d\Gamma_t(\lambda)}{dt} // \frac{e^{\text{ad } \mu} - 1}{\text{ad } \mu} // RC^{-\mu} \right\} \Big|_{\mu=\Gamma_t(\lambda)} = -\partial \left\{ \lambda // e^{-t\partial_\lambda} // RC^{-\Gamma_t(\lambda)} \right\} = -\partial_{\lambda // e^{-t\partial_\lambda} // RC^{-\Gamma_t(\lambda)}}.$$

^{C10} We verify that the computer-calculated $\Gamma_t(\lambda)$ satisfies the ODE in (6) and then that the operator equality (7) holds, at least when evaluated on “our” γ :

```

(oo) lhs = ∂t Γt[λ]; rhs = λ // e-t∂λ // adSeries[ $\frac{\text{ad}}{e^{\text{ad}} - 1}$ , Γt[λ]];
      {Γ0[λ], lhs, (lhs == rhs) @ {6}}

```

```

[ ] { < x̄ → LS[0, 0, 0, 0, ...], ȳ → LS[0, 0, 0, 0, ...] >,
      { x̄ → LS[ x̄, x̄ȳ, -t x̄x̄ȳ,  $\frac{1}{4} t^2 \overline{x x x y}$  - t x̄x̄ȳȳ, ... ], ȳ → LS[ ȳ, 0, -x̄x̄ȳ, -t x̄x̄ȳȳ, ... ] },
      BS[7 True, ...] }

```

```

(oo) {γ // e-t∂λ, γ // cc[Γt[λ]]}

```

```

[ ] { LS[ x̄ + ȳ, -2 x̄ȳ, -t x̄x̄ȳ,  $\overline{x x x y}$ , ... ], LS[ x̄ + ȳ, -2 x̄ȳ, -t x̄x̄ȳ,  $\overline{x x x y}$ , ... ] }

```

On the other hand, computing A_t is a simple differentiation, and we get that $A_t = -\partial_\lambda$. Comparing with the line above, we find that if Equation (7) holds to degree d , then Equation (8) also holds to degree d . But then as we noted, (7) holds to degree $d + 1$. As Equation (7) clearly holds at $t = 0$, we find that it holds to all orders. \square

Comment 2.7. It is easier (though insufficient) to assume that there is a solution $\Gamma_t(\lambda)$ to Equation (7) and deduce that it must satisfy the differential equation (6): simply differentiate (7) with respect to t and simplify as much as you can allowing yourself to use (7) as needed within the simplification process. The result is (6), and the steps follow the computational steps of the above proof rather closely. The actual proof is a bit harder because if we cannot assume (7) while deriving it, so we have to resort to an inductive process.

Proposition 2.8. *As in the previous proposition, let $\Lambda_t(\lambda)$ be the (unique) solution of*

$$\Lambda_0(\lambda) = 0 \quad \text{and} \quad \frac{d\Lambda_t(\lambda)}{dt} = \lambda // e^{\partial_{\Lambda_t(\lambda)}} // \frac{\text{ad}_{\text{tb}} \Lambda_t(\lambda)}{e^{\text{ad}_{\text{tb}} \Lambda_t(\lambda)} - 1}. \quad (9)$$

Then $C^{t\lambda} = e^{-\partial_{\Lambda_t(\lambda)}}. \quad (10)$

The proof of this proposition is very similar and not even a tiny bit nicer than the proof of the previous one. So we skip it and instead include a computer verification.^{C11}

As special cases, we denote $\Gamma_1(\lambda)$ by $\Gamma(\lambda)$ and $\Lambda_1(\lambda)$ by $\Lambda(\lambda)$.

One special case of C^λ deserves to be named:

Definition 2.9. (Compare [BN4, Section 4.2]) Given $u \in S$ and $\gamma \in FL(S)$ let C_u^γ denote the automorphism of $FL(S)$ defined by mapping the generator u to its “conjugate” $e^\gamma u e^{-\gamma} = e^{-\text{ad} \gamma}(u)$ (this is simply C^λ , where λ is the length 1 sequence $(u \rightarrow \gamma)$). Let $RC_u^{-\gamma}$ be the inverse of C_u^γ (which is *not* $C_u^{-\gamma}$).^{C12}

^{C11}We verify that the computer-calculated $\Lambda_t(\lambda)$ satisfies the ODE in (9) and then that the operator equality (10) holds, at least when evaluated on “our” γ :

```

(oo) lhs = ∂t Λt [λ]; rhs = λ // eDΛt[λ] // adSeries [  $\frac{\text{ad}}{e^{\text{ad}-1}}$ , Λt [λ], tb ];
      {Λ0 [λ], lhs, (lhs ≡ rhs) @ {6}}

```

```

(oo) { < x̄ → LS [0, 0, 0, 0, ...], ȳ → LS [0, 0, 0, 0, ...] >,
      { x̄ → LS [ x̄, x̄ȳ, t x̄ȳȳ,  $\frac{1}{2} t^2 \overline{x x̄ȳ} + t \overline{x x̄ȳȳ}$ , ... ], ȳ → LS [ ȳ, 0, -x̄ȳȳ, t x̄ȳȳȳ, ... ] },
      BS [7 True, ...] }

```

```

(oo) { γ // CC [t λ], γ // e-DΛt[λ] }

```

```

(oo) { LS [ x̄ + ȳ, -2 x̄ȳ, -t x̄ȳȳ, - $\frac{1}{2} t^2 \overline{x x̄ȳ} + t \overline{x x̄ȳȳ}$ , ... ],
      LS [ x̄ + ȳ, -2 x̄ȳ, -t x̄ȳȳ, - $\frac{1}{2} t^2 \overline{x x̄ȳ} + t \overline{x x̄ȳȳ}$ , ... ] }

```

^{C12}Just testing:

```

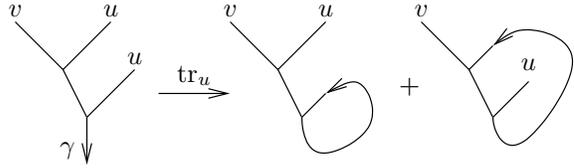
(oo) { α // CCx [-γ], α // CCx [-γ] // RCx [γ], α // CCx [-γ] // CCx [γ] }

```

Last we define/recall a number of functionals $FL(S) \rightarrow CW(S)$:

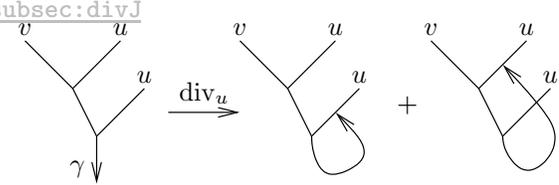
def:tru

Definition 2.10. For $u \in S$ we let $\text{tr}_u: FL(S) \rightarrow CW(S)$ be the sum of all ways of connecting the head of γ to any of its u -labelled tails and regarding the result as an element of $CW(FL(S)) \rightarrow CW(S)$. The example on the right corresponds to the specific computation $\text{tr}_u[[v, u], u] = \overline{[v, u]} + \overline{v(-u)} = -\overline{uv}$ ^{C13}



def:J

Definition 2.11. (Compare ^{KBH}[BN4, Section 5.1]) For $u \in S$ we let $\text{div}_u: FL(S) \rightarrow CW(S)$ be the functional defined schematically by the picture on the right, which corresponds to the specific computation $\text{div}_u[[v, u], u] = \overline{u[v, u]} + \overline{uv(-u)} = -\overline{uuv}$ ^{C14} (more details in ^{KBH}[BN4]). Given also $\gamma \in FL(S)$, set



$$J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma // RC_u^{s\gamma}) // C_u^{-s\gamma}. \text{C15}$$

def:j

Definition 2.12. Let $\text{div}: FL(S) \rightarrow CW(S)$ be the Alekseev-Torossian “divergence” functional, as in ^{AT}[AT, Section 5.1], but extended by 0 on A_S . In our language, $\text{div} \lambda = \sum_{u \in S} \text{div}_u \lambda$.

computations below

$$\left\{ \text{LS} \left[\overline{x}, 2 \overline{xy}, -\frac{5}{2} \overline{xxxy} + \frac{3}{2} \overline{xyxy}, \frac{7}{6} \overline{xxxxy} - \frac{23}{6} \overline{xyxyy} + \frac{2}{3} \overline{xyyy}, \dots \right], \right. \\ \left. \text{LS} \left[\overline{x}, \overline{xy}, 0, 0, \dots \right], \text{LS} \left[\overline{x}, \overline{xy}, -\overline{xxxy}, 2 \overline{xxxxy} + \overline{xyxyy}, \dots \right] \right\}$$

C:tru

^{C13}In computer talk, and using a temporary value for γ , so as not to interfere with its existing value:

```
oo u = LW@"u"; v = LW@"v";
With[{γ = b[b[v, u], u]}, tr_u[γ]]
```

$$\overline{-uv}$$

C:divu

^{C14}In computer talk:

```
oo With[{γ = u + b[b[v, u], u]}, div_u[γ]]
```

$$\overline{u - uuv}$$

C:JDef

^{C15}We quote the implementation of J in `FreeLie.m` (FL) and, reverting to the “old” γ , compute $J_1(\gamma)$:

$$\text{FL } J_u[\gamma] := J_u[\gamma] = \text{Module} \left[\{s\}, \int_0^1 (\gamma // RC_u[s\gamma] // \text{div}_u // CC_u[-s\gamma]) ds \right];$$

C:Ju

```
oo J_x[γ]
```

$$\text{CWS} \left[\overline{x}, \frac{5}{2} \overline{xy}, -\frac{7}{6} \overline{xxxy} + \frac{7}{6} \overline{xyxy}, \frac{3}{8} \overline{xxxxy} - \frac{11}{4} \overline{xyxyy} - \frac{3}{4} \overline{xyxyy} + \frac{3}{8} \overline{xyyyy}, \dots \right]$$

Let $j: FL(S) \rightarrow CW(S)$ is the Alekseev-Torossian “logarithm of the Jacobian”: $j(\lambda) = \frac{e^{\partial_\lambda} - 1}{\partial_\lambda}(\text{div } \lambda)$.^{C16}

Alekseev and Torossian prove in [AT] that j is the unique functional $j: FL(S) \rightarrow CW(S)$ satisfying the “cocycle condition” $j(\text{BCH}_{tb}(\lambda_1, \lambda_2)) = j(\lambda_1) + e^{\partial_{\lambda_1}} j(\lambda_2)$, where BCH_{tb} stands for the BCH formula using the tangential bracket $[\cdot, \cdot]_{tb}$ on $FL(S)^S$:

$$\text{BCH}_{tb}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 + \frac{1}{2}[\lambda_1, \lambda_2]_{tb} + \dots,$$

and the “initial condition” $\frac{\partial}{\partial \epsilon} j(\epsilon \lambda) = \text{div } \lambda$.^{C17}

computations below

^{C16} A quote of the computer-definition, and then $\text{div } \lambda$ and $j(\lambda)$, computed to degree 5:

```

FL div [λ_AngleBracket] := Sum[div_a[λ_a], {a, Support[λ]}];
j[λ_AngleBracket] := div[λ] // DerivationSeries[e^der - 1, D_λ];

```

```

○○ {div[λ]@{5}, j[λ]@{5}}

```

```

☒ {CWS[x + y, -xy, -xxy, 0, 0, ...], CWS[x + y, -xy, -xxy, -xxyy + xyxy, -xxyyy + xxyxy, ...]}

```

^{C17} We verify the cocycle condition and the initial condition. For the latter, we first declare ϵ to be “an infinitesimal” by declaring that $\epsilon^2 = 0$, and then we verify that $j(\epsilon \lambda) = \epsilon \text{div } \lambda$:

```

○○ lhs = j[BCH_tb[λ1, λ2]]; rhs = j[λ1] + e^D_λ1[j[λ2]];
{lhs, (lhs == rhs)@{8}}

```

```

☒ {CWS[x + 2 y, -3 xy, 0, -9 xxyy + 9 xyxy, ...], BS[9 True, ...]}

```

```

○○ ε /: ε^2 = 0;
{j[ε λ], j[ε λ] == ε div[λ]}

```

```

☒ {CWS[ε x + ε y, -ε xy, -ε xxy, 0, ...], BS[5 True, ...]}

```


subsec:AT

2.3. The lower-interlaced presentation E_l of $\mathcal{A}_{\text{exp}}^w$. For a finite set S let $TW_l(S)$ be set-theoretically the same as $TW(S) = FL(S)^S \times CW(S)$ — we only add the “ l ” subscript to emphasize that TW_l carries an algebraic structure, and that it is different from the algebraic structure on TW_s , which we will study later. Elements of $TW_l(S)$ are ordered pairs $(\lambda; \omega)_l$ where $\lambda \in FL(S)^S$, $\omega \in CW(S)$, and the subscript l is there only to remind us of the context.

Set $\mathbf{g}: \text{ElDef}$

$$E_l(\lambda; \omega)_l := \exp(\mathbf{l}\lambda) * \exp(\omega) \in \mathcal{A}_{\text{exp}}^w(S), \quad \left(\begin{array}{l} \text{“}E_l\text{” for “}E\text{xpone} \\ \text{after using } \underline{l}\text{”} \end{array} \right)$$

where $l: FL(S)^S = A_S \oplus \text{tder}_S \rightarrow \mathcal{A}^w(S)$ is the “lower” Lie embedding¹¹ of trees into $\mathcal{A}^w(S)$ (see [WKO2, Section 3.2]), where ι is the obvious inclusion of wheels ($= CW(S) = \text{tt}_S$) into $\mathcal{A}^w(S)$, and where exponentiation is taken using the stacking product (1) of $\mathcal{A}^w(S)$. A pictorial representation of $E_l(\lambda; \omega)_l$ appears on the right: Reading from the bottom up, we see “exponentially many” copies of λ (meaning, a sum over n of n copies with coefficient $1/n!$). Each λ is a linear combination of trees with one head and many tails, which are attached to the strands in T with the head below the tails. Each copy of λ appears on the right as a gray “wizard’s cap” whose tip corresponds to the head of λ , and is therefore tipped downward. Above $\exp(\mathbf{l}\lambda)$ is our symbolic representation of $\exp(\omega)$.

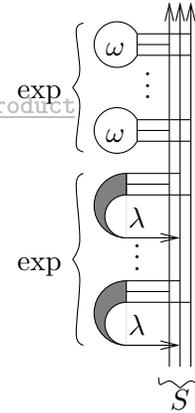


Figure 2.13. $E_l(\lambda; \omega)_l$.

Figure 2.13 also explains the name “interlaced” for this presentation, for in it heads and tails are interlaced along the strands of S (contrast with E_s in Figure 2.19 and with E_f in Figure 2.28).

It follows from the results of [WKO2, Section 3.2] that the map $E_l: TW_l(S) \rightarrow \mathcal{A}_{\text{exp}}^w(S)$ is a set-theoretic bijection. Hence the operations of Definition 2.2 induce corresponding operations on $TW_l(S)$. We list these within the (long!) definition-proposition below.

Definition-Proposition 2.14. *The bijection E_l intertwines the operations defined below with the operations in Definition 2.2:*

¹¹We could have equally well used the “upper” Lie embedding u , setting $E_u(\lambda; \omega)_u := \exp(\omega) \exp(\mathbf{u}\lambda)$ with only minor modifications to the formulas that follow.

^{C18}We cannot verify Definition-Proposition 2.14 *per se* on the computer, as we have no direct computer implementation of \mathcal{A}^w . Indeed, the whole point of this paper is to provide an implementation of \mathcal{A}^w by means of E_l (and later, E_s and E_f). Instead, we verify below that many properties of operations on \mathcal{A}^w (the associativity of the stacking product, etc.) indeed hold for their E_l implementations. We start by setting the values of some “sample” elements on which we will run our tests (note that on the computer we represent $(\lambda; \omega)_l$ as $\text{El}[\lambda, \omega]$):

```

x = LW@ "x"; y = LW@ "y";
{L_a = El [<x -> LS[x + b[x, y]], y -> LS[y - b[x, b[x, y]]], CWS[cw[x] - 3 cw[x, y, x]]],
  L_b = El [<x -> LS[y - b[x, y]], y -> LS[x + y + b[y, b[x, y]]], CWS[cw[y] - 2 cw[x, y]]],
  L_c = El [<x -> LS[x - b[b[x, y], b[x, y]], y -> LS[y + 3 b[x, b[x, y]]],
  CWS[cw[x] - 2 cw[x, y] + cw[x, y, x]]]

```

C:ElSetup

(1) If $S_1 \cap S_2 = \emptyset$ and $(\lambda_i; \omega_i)_l \in TW_l(S_i)$, `g:lsqcup`

$$(\lambda_1; \omega_1)_l (\lambda_2; \omega_2)_l = (\lambda_1; \omega_1)_l \sqcup (\lambda_2; \omega_2)_l := (\lambda_1 \sqcup \lambda_2; \omega_1 + \omega_2)_l, \quad (11) \quad \text{eq:ElCup}$$

where $\sqcup: FL(S_1)^{S_1} \times FL(S_2)^{S_2} \rightarrow FL(S_1 \sqcup S_2)^{S_1 \sqcup S_2}$ is the union operation of functions (or, in computer speak, the concatenation of associative arrays) followed by the inclusions $FL(S_i) \rightarrow FL(S_1 \sqcup S_2)$, and $\omega_1 + \omega_2$ is defined using the inclusions $CW(S_i) \rightarrow CW(S_1 \sqcup S_2)$.

(2) If $(\lambda_i; \omega_i)_l \in TW_l(S)$, `g:last`

$$(\lambda_1; \omega_1)_l \ast (\lambda_2; \omega_2)_l := (\text{BCH}_{tb}(\lambda_1, \lambda_2); e^{-\partial_{\lambda_2}}(\omega_1) + \omega_2)_l. \quad (12) \quad \text{eq:ElProdu}$$

(3) If $(\lambda; \omega)_l \in TW_l(S)$ and $a \in S$, `g:ldeta`

$$(\lambda; \omega)_l // d\eta^a := ((\lambda \setminus a) // (a \rightarrow 0); \omega // (a \rightarrow 0))_l, \quad (13) \quad \text{eq:ElEta}$$

where $\lambda \setminus a$ denotes the function λ with the element a removed from its domain (in computer talk, “remove the key a ”), and $(a \rightarrow 0)$ denotes the substitution $a = 0$, which is defined on both FL and CW and maps $FL(S) \rightarrow FL(S \setminus a)$ and $CW(S) \rightarrow CW(S \setminus a)$.^{C20}

(4) For a single $a \in S$, I don't know a simple description of the operation dA^a in E_l language¹². Yet the composition $dA^a := \prod_{a \in S} dA^a$ is manageable: (j is defined in Definition 2.12)

$$(\lambda; \omega)_l // dA^S := (-\lambda; e^{\partial_{\lambda}}(\omega) - j(\lambda))_l. \quad (14) \quad \text{eq:ElA}$$

¹² A not-so-simple description would be to use the language of the factored presentation of Section 2.4, converting back and forth using the results of Section 2.5. `subsec:Ef`

computations below

```

{El [⟨x̄ → LS [x̄, x̄ȳ, 0, 0, ...], ȳ → LS [ȳ, 0, -x̄x̄ȳ, 0, ...]⟩, CWS [x̄, 0, -3 x̄x̄ȳ, 0, ...]],
  El [⟨x̄ → LS [ȳ, -x̄ȳ, 0, 0, ...], ȳ → LS [x̄ + ȳ, 0, -x̄ȳȳ, 0, ...]⟩, CWS [ȳ, -2 x̄ȳ, 0, 0, ...]],
  El [⟨x̄ → LS [x̄, 0, 0, 0, ...], ȳ → LS [ȳ, 0, 3 x̄x̄ȳ, 0, ...]⟩, CWS [x̄, -2 x̄ȳ, x̄x̄ȳ, 0, ...]]}

```

^{C19}We quote the E_l implementation of the stacking product from `AwCalculus.m` (AC) and verify that it is associative, at least to degree 8:

```

AC El /: El [λ1_, ω1_] ** El [λ2_, ω2_] /: Support [λ1] == Support [λ2] :=
  El [BCHtb [λ1, λ2], e-Dλ2 [ω1] + ω2];

```

```

lhs = ℒa ** (ℒb ** ℒc); rhs = (ℒa ** ℒb) ** ℒc;
{lhs@{3}, (lhs == rhs)@{8}}

```

```

{El [⟨x̄ → LS [2 x̄ + ȳ, 0, 1/2 x̄x̄ȳ, ...], ȳ → LS [x̄ + 3 ȳ, 0, 5/2 x̄x̄ȳ - x̄ȳȳ, ...]⟩,
  CWS [2 x̄ + ȳ, -4 x̄ȳ, -2 x̄x̄ȳ, ...]], BS [9 True, ...]}

```

^{C20}Example:

```

{ℒa // dηx, ℒa // dηy}

```

```

{El [⟨ȳ → LS [ȳ, 0, 0, 0, ...]⟩, CWS [0, 0, 0, 0, ...]],
  El [⟨x̄ → LS [x̄, 0, 0, 0, ...]⟩, CWS [x̄, 0, 0, 0, ...]]}

```

(5) For a single $a \in S$, I don't know a simple description of the operation dS^a in E_l language¹². Yet the composition $dS := \prod_{a \in S} dS^a$ is manageable:

$$(\lambda; \omega)_l // dS^S := (-\lambda // (-1)^{\deg}; (e^{\partial\lambda}(\omega) - j(\lambda)) // (-1)^{\deg})_l. \quad \text{C22} \quad \text{eq:ElS}$$

(6) I don't know a simple description of the operation dm_c^{ab} in E_l language¹². Yet note that Equation (2) implies that “applying dm to all strands” is manageable, being the stacking product described in (12).

(7) We have

$$(\lambda; \omega)_l // d\Delta_{bc}^a := ((\lambda \setminus a) \sqcup (b \rightarrow \lambda_a, c \rightarrow \lambda_a) // (a \rightarrow b + c); \omega // (a \rightarrow b + c))_l, \quad \text{C23} \quad \text{eq:ElDelta}$$

where $(a \rightarrow b + c)$ denotes the obvious replacement of the generator a with the sum $b + c$. It represents morphisms $FL(S) \rightarrow FL((S \setminus a) \sqcup \{b, c\})$, $FL(S)^H \rightarrow FL((S \setminus a) \sqcup \{b, c\})^H$ (for any set H), and $CW(S) \rightarrow CW((S \setminus a) \sqcup \{b, c\})$.

computations below

^{C21}We quote the computer-definition of dA , compute an example, verify that dA is an involution, and then that it is an anti-homomorphism relative to the stacking product:

 `El[\lambda_, \omega_] // dA := El[-\lambda, e^{\partial\lambda}[\omega] - j[\lambda]];`

 `{\xi_d = El[\lambda, CWS[0]], \xi_d // dA}`

 `{El[{\overline{x} \to LS[\overline{x}, \overline{x}\overline{y}, 0, 0, \dots], \overline{y} \to LS[\overline{y}, 0, -\overline{x}\overline{y}, 0, \dots]}], CWS[0, 0, 0, 0, \dots]}, El[{\overline{x} \to LS[-\overline{x}, -\overline{x}\overline{y}, 0, 0, \dots], \overline{y} \to LS[-\overline{y}, 0, \overline{x}\overline{y}, 0, \dots]}], CWS[-\overline{x} - \overline{y}, \overline{x}\overline{y}, \overline{x}\overline{y}, \overline{x}\overline{y}\overline{y} - \overline{x}\overline{y}\overline{y}, \dots]}]}`

 `(\xi_d = (\xi_d // dA // dA)) @ {8}`

 `BS[9 True, \dots]`

 `lhs = (\xi_a ** \xi_b) // dA; rhs = (\xi_b // dA) ** (\xi_a // dA); {lhs@{3}, (lhs = rhs)@{8}}`

 `{El[{\overline{x} \to LS[-\overline{x} - \overline{y}, 0, -\frac{1}{2} \overline{x}\overline{y}, \dots], \overline{y} \to LS[-\overline{x} - 2\overline{y}, 0, \frac{1}{2} \overline{x}\overline{y} + \overline{x}\overline{y}\overline{y}, \dots]}], CWS[-\overline{y}, -2\overline{x}\overline{y}, -2\overline{x}\overline{y} - \overline{x}\overline{y}\overline{y}, \dots]}], BS[9 True, \dots]}`

C:dS ^{C22}An example:

 `\xi_d // dS`

 `El[{\overline{x} \to LS[\overline{x}, -\overline{x}\overline{y}, 0, 0, \dots], \overline{y} \to LS[\overline{y}, 0, -\overline{x}\overline{y}, 0, \dots]}], CWS[\overline{x} + \overline{y}, \overline{x}\overline{y}, -\overline{x}\overline{y}, \overline{x}\overline{y}\overline{y} - \overline{x}\overline{y}\overline{y}, \dots]}`

^{C23}The computer-definition, an example, and then a verification that $d\Delta$ is homomorphism relative to the stacking product:

C:EldDelta

(8) We have

$$(\lambda; \omega)_l // d\sigma_b^a := (((\lambda \setminus a) \sqcup (b \rightarrow \lambda_a)) // (a \rightarrow b); \omega // (a \rightarrow b))_l, \quad (17)$$

where $(a \rightarrow b)$ denotes the obvious “generator renaming” morphisms $FL(S) \rightarrow FL((S \setminus a) \sqcup b)$, $FL(S)^H \rightarrow FL((S \setminus a) \sqcup b)^H$ (for any set H), and $CW(S) \rightarrow CW((S \setminus a) \sqcup b)$.

Proof. Equations (11), (13), (16), and (17) are trivial and were stated only to introduce notation. The tree-level part of Equation (12) follows from the fact that l is a morphism of Lie algebras (see within the proof of [WKO2, Proposition 3.19]). The wheels part of Equation (12) follows from [WKO2, Remark 3.24]. Equation (14) follows from the observation that dA^S is the adjoint map $*$ of [WKO2, Definition 3.26] and then from [WKO2, Proposition 3.27]. Equation (15) is the easily-established fact that on \mathcal{A}^w , $dS^S = (-1)^{\deg} dA^S$. \square

 Note that the absence of simple descriptions of dA^a , dS^a , and dm_c^{ab} in the E_l language is fatal for its applicability to knot theory, as these operations are needed within the computation of knot and tangle invariants. See Section 3.1.

[AT] *Comment 2.15.* Let $\pi_T: TW(S) \rightarrow FL(S)^S$ denote the projection onto the first factor (“trees”) of $TW(S) = FL(S)^S \times CW(S)$, and recall that up to a minor central factor, $(FL(S)^S, tb)$ is tder_S . Recall also that tder_S is the Lie algebra of TAut_S , and that elements of tder_S represent elements of TAut_S by exponentiation. With this in mind, the tree part of Equation (12) becomes the product of TAut_S . In other words, the diagram

$$\begin{array}{ccc} TW_l(S) \times TW_l(S) & \xrightarrow{*} & TW_l(S) \\ (\pi_T // \exp) \times (\pi_T // \exp) \downarrow & & \downarrow \pi_T // \exp \\ \text{TAut}_S \times \text{TAut}_S & \xrightarrow{\text{mult.}} & \text{TAut}_S \end{array}$$

computations below

 `AC El[λ_, ω_] // dΔ[a_, b_, c_] := El[
 (λ \ a) ∪ (b → λa, c → λa) // LieMorphism[LW@a → LW@b + LW@c],
 ω // LieMorphism[LW@a → LW@b + LW@c]]`

C:dD1

 `{ξa, ξa // dΔ[y, y, z]}`

 `{El[⟨x̄ → LS[x̄, x̄ȳ, 0, ...], ȳ → LS[ȳ, 0, -x̄xȳ, 0, ...]⟩, CWS[x̄, 0, -3x̄xȳ, 0, ...]],
 El[⟨z → LS[ȳ + z̄, 0, -x̄xȳ - x̄x̄z̄, 0, ...], x̄ → LS[x̄, x̄ȳ + x̄z̄, 0, 0, ...],
 ȳ → LS[ȳ + z̄, 0, -x̄xȳ - x̄x̄z̄, 0, ...]⟩, CWS[x̄, 0, -3x̄xȳ - 3x̄x̄z̄, 0, ...]]}`

C:dD2

 `lhs = (ξa ** ξb) // dΔ[y, y, z]; rhs = (ξa // dΔ[y, y, z]) ** (ξb // dΔ[y, y, z]);
 {lhs@{3}, (lhs ≡ rhs)@{8}}`

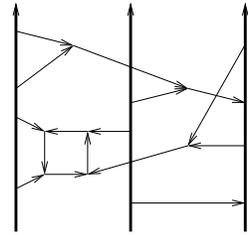
 `{El[⟨z → LS[x̄ + 2ȳ + 2z̄, 0, -1/2 x̄xȳ - 1/2 x̄x̄z̄ - x̄ȳz̄ - x̄ȳȳ - 2x̄z̄ȳ - x̄z̄z̄, ...],
 x̄ → LS[x̄ + ȳ + z̄, 0, 1/2 x̄xȳ + 1/2 x̄x̄z̄, ...],
 ȳ → LS[x̄ + 2ȳ + 2z̄, 0, -1/2 x̄xȳ - 1/2 x̄x̄z̄ - x̄ȳz̄ - x̄ȳȳ - 2x̄z̄ȳ - x̄z̄z̄, ...]⟩,
 CWS[x̄ + ȳ + z̄, -2x̄ȳ - 2x̄z̄, -3x̄xȳ - 3x̄x̄z̄, ...]], BS[9 True, ...]}`

is commutative. Hence the E_l presentation is valuable for $\overset{\text{AT}}{[\text{AT}]}$ as many of the $\overset{\text{AT}}{[\text{AT}]}$ equations involve the group structure of TAut_S .

subsec:Ef

2.4. The factored presentation E_f of $\mathcal{A}_{\text{exp}}^w$ and its stronger precursor E_s .

Following [BN4], in the “factored” presentation E_f of $\mathcal{A}_{\text{exp}}^w$ arrow heads are treated separately from arrow tails in diagrams such as the one on the right. This presentation of $\mathcal{A}_{\text{exp}}^w$ is more complicated than the previous one, yet it is also more powerful, and in some sense, it is made of simpler ingredients. We first enlarge the collection of spaces $\{\mathcal{A}^w(S)\}$ to a somewhat bigger collection $\{\mathcal{A}^w(H; T)\}$ on which a larger class of operations act. The new operations are more “atomic” than the old ones, in the sense that each of the operations of Definition 2.2 is a composition of 2-3 of the new operations. The advantage is that the new operations all have reasonably simple descriptions as operations on the group-like subsets $\{\mathcal{A}_{\text{exp}}^w(H; T)\}$ (the “split” presentation E_s below), and hence even the few operations whose description in the E_l presentation was omitted in Definition-Proposition 2.14 can be fully described and computed in the E_f presentation.

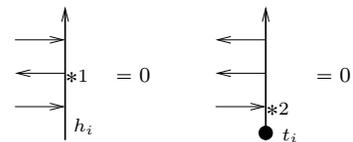


A sketch of our route is as follows: In Section 2.4.1, right below, we describe the spaces $\{\mathcal{A}^w(H; T)\}$. In Section 2.4.2 we describe the zoo of operations acting on $\{\mathcal{A}^w(H; T)\}$. Section 2.4.3 is the tofu of the matter — we describe the operations of the previous section in terms of spaces $\{TW_s(H; T)\}$ of trees and wheels, whose elements are in a bijection E_s with the group like elements of $\{\mathcal{A}^w(H; T)\}$. Finally in Section 2.4.4 we explain how the system of spaces $\{\mathcal{A}^w(S)\}$ includes into the system $\{\mathcal{A}^w(H; T)\}$ and how the operations of the former are expressed in terms of the latter, concluding the description of E_f .

sec:Family

2.4.1. *The family $\{\mathcal{A}^w(H; T)\}$.* Let $H \equiv \{h_1, h_2, \dots\}$ be some finite set of “head labels” and let $T \equiv \{t_1, t_2, \dots\}$ be some finite set of “tail labels” (these sets need not be of the same cardinality). Let $\mathcal{A}^w(H; T)$ be $\mathcal{A}^w(H \sqcup T)$ ¹³ moded out by the following further relations:

- If an arrow tail lands anywhere on a head strand (*1 on the right), the whole diagram is zero.
- The CP relation: If an arrow head is the lowest vertex on a tail strand (*2 on the right), the whole diagram is zero. (As on the right, we indicate the bottom ends of tail strands with bullets “•”).

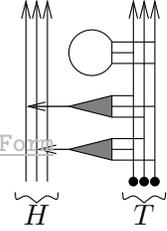


teDisjoint

¹³ We will often use sets of labels H and T that are *not* disjoint. The notation “ $H \sqcup T$ ” stands for the union of H and T , made disjoint by brute force; for example, by setting $H \sqcup T := (\{h\} \times H) \cup (\{t\} \times T)$, where h and t are two distinct labels chosen in advance to indicate “heads” and “tails”. In practise we will keep referring to the images of the elements of H within $H \sqcup T$ as h_i rather than (h, h_i) , and likewise for the t_i ’s. We will mostly avoid the confusion that may arise when $H \cap T \neq \emptyset$ by labelling operations as “head operations” which will always refer to labels in $H \hookrightarrow H \sqcup T$ or as “tail operations”, when referring to labels in $T \hookrightarrow H \sqcup T$.

m:PureForm

Comment 2.16. Using these two relations one may show that $\mathcal{A}^w(H;T)$ is isomorphic to the set of arrow diagrams in which only arrow heads land on the head strands (obvious, by the first relation) and in which only arrow tails meet the tail strands (use \overrightarrow{STU}_2 to slide any arrow head on a tail strand until it's near the bottom, then use the second relation; see also [Comment 2.1](#)), still modulo \overrightarrow{AS} , $\overrightarrow{IH\bar{X}}$, \overrightarrow{STU}_1 and TC . Thus a typical element of $\mathcal{A}^w(H;T)$ is shown on the right.



 In topology (see [\[BN4\]](#)), head strands correspond to “hoops”, or based knotted circles, and tail strands correspond to balloons, or based knotted spheres. The two relations and the isomorphism above are also meaningful [\[BN4\]](#).

 In Lie theory head strands represent $\mathcal{U}(\mathfrak{g})$ and tail strands represent the (right) Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}^*) \simeq \mathcal{S}(\mathfrak{g}^*)$. The evaluation $\mathfrak{g}^* \rightarrow 0$ induces a surjection of $\mathcal{U}(I\mathfrak{g})$ onto the first of these spaces whose kernel is “any word containing a letter in \mathfrak{g}^* ”, explaining the first relation above. The second relation is the definition of the Verma module.

Operations

2.4.2. Operations on $\{\mathcal{A}^w(H;T)\}$.

Operations

Definition 2.17. Just as in [Definition 2.2](#), there are several operations that are defined on $\mathcal{A}^w(H;T)$. In brief, these are:

- (1) A union operation $\sqcup: \mathcal{A}^w(H_1;T_1) \otimes \mathcal{A}^w(H_2;T_2) \rightarrow \mathcal{A}^w(H_1 \sqcup H_2;T_1 \sqcup T_2)$, defined when $H_1 \cap H_2 = T_1 \cap T_2 = \emptyset$, with obvious topological (compare with “ $*$ ” of [\[BN4, Figure 5\]](#)) and Lie theoretic meanings. (The symbol \sqcup is sometimes omitted: $D_1 D_2 := D_1 \sqcup D_2$).
- (2) A “stacking” product $\#$ can be defined on $\mathcal{A}^w(H;T)$ by stitching all pairs of equally-labelled head strands and then merging all pairs of equally-labelled tail strands in a pair of diagrams $D_1, D_2 \in \mathcal{A}^w(H;T)$. The “merging” of tail strands is described in more detail as the operation tm below. In fact, it may be better to define $\#$ using a formula similar to [Equation \(2\)](#) and the operations hm , tm , $h\sigma$, and $t\sigma$ defined below:

$$D_1 \# D_2 = \left(D_1 \sqcup \left(D_2 // \prod_{x \in H} h\sigma_x^x // \prod_{u \in T} t\sigma_u^u \right) \right) // \prod_{x \in H} hm_x^{x\bar{x}} // \prod_{u \in T} tm_u^{u\bar{u}}. \quad (18)$$

eq:AHTStack

  In topology, $\#$ is the stitching of hoops followed by the merging of balloons; this is not the same as the stitching of knotted tubes. In Lie theory, $\#$ corresponds to the componentwise product of $\mathcal{U}(\mathfrak{g})^{\otimes H} \otimes \mathcal{S}(\mathfrak{g}^*)^{\otimes T}$. Even when H and T are both singletons, this is not the same as the product of $\mathcal{U}(I\mathfrak{g})$, even though linearly $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$.

- (3) If $x \in H$ and $u \in T$, the operations $h\eta^x$ and $t\eta^u$ drop the head-strand x or the tail-strand u similarly to the operation $d\eta^a$ of [Definition 2.2](#).
- (4) hA^x reverses the head-strand x while multiplying by a (-1) factor for every arrow head on x . tA^u is the identity.
- (5) $hS^x = hA^x$ while tS^u multiplies by a factor of (-1) for every arrow tail on u (by TC , there's no need to reverse u).
- (6) The operation hm_z^{xy} is defined similarly to dm_c^{ab} of [Definition 2.2](#). Likewise for tm_w^{uv} , except in this case, the tail-strands u and v must first be cleared of all arrow-heads using the process of [Comment 2.16](#). Once u and v carry only arrow-tails, all these tail can be

put on a new tail-strand w in some arbitrary order (which doesn't matter, by TC). Note that $tm_w^{uv} = tm_w^{vu}$, so tm is “meta-commutative”.



In topology, tm_w^{uv} is the “merging of balloons” operation of [BN4, Section 3.1], which in itself is analogous to the (commutative) multiplication of π_2 .



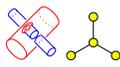
In Lie theory, tm_w^{uv} is the product of $\mathcal{S}(\mathfrak{g}^*)$. Note that tail strands more closely represent the Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g})$ whose isomorphism with $\mathcal{S}(\mathfrak{g}^*)$ involves “sliding all \mathfrak{g} -letters in a $\mathcal{U}(I\mathfrak{g})$ -word to the left and then cancelling them”. This is analogous to the process of cancelling arrow-heads which is a pre-requisite to the definition of tm_w^{uv} .

- (7) $h\Delta_{y_2}^x$ and $t\Delta_{vw}^u$ are defined similarly to $d\Delta_{bc}^a$.
- (8) $h\sigma_y^x$ and $t\sigma_v^u$ are defined similarly to $d\sigma_b^a$.
- (9) **New!** Given a tail $u \in T$, a “new” tail label $v \notin T \setminus u$ and a head $x \in H$ the operation $thm_y^{ux} : \mathcal{A}^w(H; T) \rightarrow \mathcal{A}^w(H \setminus x; (T \setminus u) \sqcup \{v\})$ is the obvious “tail-strand head-strand stitching” — similarly to dm_c^{ab} , stitch the strand u to the strand x putting u before x , and call the resulting “new” strand v . Note that for this to be well defined, v must be a tail strand.¹⁴

In practise, thm_y^{ux} is never used on its own, but the combination $h\Delta_{xx'}^x // thm_{v_u}^{ux'}$ (where x' is a temporary label) is very useful. Hence we set $tha^{ux} : \mathcal{A}^w(H; T) \rightarrow \mathcal{A}^w(H; T)$ (“tail by head action on u by x ”) to be that combination. In words, this is “double the strand x and put one of the copies on top of u ”.¹⁵



In topology, tha is the action of hoops on balloons as in [BN4, Section 3.1], which is similar to the action of π_1 on π_2 . In Lie theory, it is the right action of $\mathcal{U}(\mathfrak{g})$ on the Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g})$, or better, the action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{S}(\mathfrak{g}^*)$ induced from the co-adjoint action of \mathfrak{g} on \mathfrak{g}^* .



Exercise 2.18. In the cases when we did not state the topological or Lie theoretical meaning of an operation in Definition 2.17, find what it is.

2.4.3. *Group-like elements in $\{\mathcal{A}^w(H; T)\}$.* For any fixed finite sets H and T there is a co-product $\square : \mathcal{A}^w(H; T) \otimes \mathcal{A}^w(H; T) \rightarrow \mathcal{A}^w(H; T)$ defined just as in the case of $\mathcal{A}^w(S)$ (Definition 2.3), and along with the product $\#$ (and obvious units and co-units), $\mathcal{A}^w(H; T)$ is a graded connected co-commutative bi-algebra. Hence it makes sense to speak of the group-like elements $\mathcal{A}_{\text{exp}}^w(H; T)$ within $\mathcal{A}^w(H; T)$, and they are all $\#$ -exponentials of primitives in $\mathcal{A}^w(H; T)$. The primitives $\mathcal{A}_{\text{prim}}^w(H; T)$ in $\mathcal{A}^w(H; T)$ are connected diagrams and hence they are trees and wheels. As in Comment 2.16, the trees must have their roots on head strands and their leafs on tail strands, and the wheels must have all their “legs” on tail strands. As tails commute, we may think of the trees as abstract trees with leafs labelled by labels in T and roots in H , and the wheels are abstract cyclic words with letters in T . Hence canonically $\mathcal{A}_{\text{prim}}^w(H; T) \simeq FL(T)^H \oplus CW(T)$ and hence there is a bijection (called “the split presentation E_s ”)

$$E_s : \mathcal{A}_{\text{exp}}^w(H; T) := FL(T)^H \oplus CW(T) \xrightarrow{\sim} \mathcal{A}_{\text{exp}}^w(H; T) \quad (19)$$

¹⁴Note also that the analogous operation htm_v^{xu} “put x before u to get a tail v ” is 0 and hence we can safely ignore it, and that thm_y^{xu} and htm_v^{xu} , defined in the same way as thm_y^{ux} and htm_v^{xu} except to produce a head strand y , are not well defined because they do not respect the CP relation.

¹⁵Note that $thm_v^{ux} = tha^{ux} // h\eta^x // t\sigma_v^u$ so we lose no generality by considering tha^{ux} instead of thm_v^{ux} .

defined on an ordered pair $(\lambda; \omega)_s$ in $TW_s(H; T)$ by

$$(\lambda; \omega)_s \mapsto \exp_{\#}(e_s(\lambda; \omega)), \tag{20}$$

where $e_s(\lambda; \omega)_s$ is the sum over $x \in H$ of planting λ_x with its root on strand x and its leaves on the strands in T so that the labels match but at an arbitrary order on any T strand, plus the result of planting ω on just the T strands so that the labels match but at an arbitrary order on any T strand. A pictorial representation of $E_s(\lambda; \omega)_s$, using the same visual language as in Figure 2.13, appears on the right.

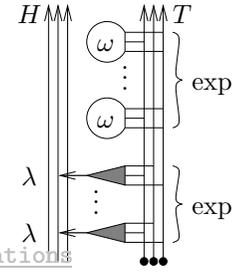


Figure 2.19. $E_s(\lambda; \omega)_s$.

It is easy to verify that the operations in Definition 2.17 intertwine \square and hence map group-like elements to group-like elements and hence they induce operations on $TW_s(H; T)$. These are summarized within the following definition-proposition.

Definition-Proposition 2.20. *The bijection E_s intertwines the operations defined below with the operations in Definition 2.17.*

¹⁶Here we no longer state conditions such as $H_1 \cap H_2 = \emptyset$, $u \in T$, $x \in H$. They are the same as in Definition 2.17, and more importantly, they are “what makes sense”.

^{C24}We quote from `AwCalculus.m` only the most interesting implementations — of \square (21), of hm (29), of tm (30), and of tha (35). Then we set the values of two “sample” elements in the E_s presentation (on the computer we represent $(\lambda; \omega)_s$ as $Es[\lambda, \omega]$):

```

AC Es /: Es[lambda_1, omega_1] Es[lambda_2, omega_2] /; Support[lambda_1] \cap Support[lambda_2] = {} := Es[lambda_1 \cup lambda_2, omega_1 + omega_2];
Es[lambda, omega] // hm[x, y, z] := Es[lambda // hm[x, y, z], omega];
Es[lambda, omega] // tm[u, v, w] := LieMorphism[LW@u -> LW@w, LW@v -> LW@w] /@ Es[lambda, omega];
Es[lambda, omega] // tha[u, x] := Es[lambda // RC_u[lambda_x], (omega + J_u[lambda_x]) // RC_u[lambda_x]];
    
```

```

Es_a = Es[<1 -> LS[u + b[u, v]], 2 -> LS[v - b[u, b[u, v]]], 3 -> LS[u - b[b[u, v], b[u, v]]],
CWS[cw[u] - 3 cw[u, v, u]]
    
```

```

Es[<1 -> LS[u, uv, 0, 0, ...], 2 -> LS[v, 0, -u uv, 0, ...], 3 -> LS[u, 0, 0, 0, ...],
CWS[u, 0, -3 uv, 0, ...]]
    
```

```

Es_b = RandomEsSeries[0, {1, 2, 3, 4}];
Es_b@{2}
    
```

```

Es[<1 -> LS[-1 - 2 2 + 2 3 - 2 4, 2 1 2 + 1 3 / 2 + 1 4 - 2 3 / 2 - 2 4 / 2 + 2 3 4, ...],
2 -> LS[2 1 - 2 - 2 3 + 4, 2 1 2 + 3 1 3 / 2 - 2 1 4 - 2 3 - 2 4 - 3 4 / 2, ...],
3 -> LS[-1 + 2 + 2 4, -2 1 2 + 2 1 3 - 1 4 - 3 2 3 / 2 + 2 2 4 - 2 3 4, ...],
4 -> LS[-2 1 + 2 2 + 2 3 + 4, -1 2 / 2 + 3 1 3 / 2 - 2 2 4 + 3 4, ...],
CWS[3 - 4, 3 1 1 / 2 + 3 1 2 / 2 - 2 1 3 + 1 4 + 2 2 + 2 2 3 - 2 4 / 2 - 2 3 3 - 3 4 + 4 4, ...]]
    
```

(Note that the second of sample elements was set to be a random series, with a seed of 0. It is printed only to degree 2, but it extends indefinitely as a random series.)

eq:EsCup
EsProduct
eq:EshEta
eq:EstEta
eq:EshA
eq:EstA
eq:EshS
eq:EstS
eq:Eshm
eq:Estm
q:EshDelta
q:EstDelta
q:EshSigma
q:EstSigma
eq:Estha

$$(1) (\lambda_1; \omega_1)_s (\lambda_2; \omega_2)_s = (\lambda_1; \omega_1)_s \square (\lambda_2; \omega_2)_s := (\lambda_1 \sqcup \lambda_2; \omega_1 + \omega_2)_s \quad (21)$$

$$(2) (\lambda_1; \omega_1)_s \# (\lambda_2; \omega_2)_s := ((x \rightarrow \text{BCH}(\lambda_{1x}, \lambda_{2x}))_{x \in H}; \omega_1 + \omega_2)_s \quad (22)$$

$$(3) (\lambda; \omega)_s // \text{h}\eta^x := (\lambda \setminus x; \omega)_s \quad (23)$$

$$(\lambda; \omega)_s // \text{t}\eta^u := (\lambda // (u \rightarrow 0); \omega // (u \rightarrow 0))_s \quad (24)$$

$$(4) (\lambda; \omega)_s // \text{h}A^x := ((\lambda \setminus x) \sqcup (x \rightarrow -\lambda_x); \omega)_s \quad (25)$$

$$\text{t}A^u := I \quad (26)$$

$$(5) \text{h}S^x := \text{h}A^x, \quad (27)$$

$$(\lambda; \omega)_s // \text{t}S^u := (\lambda // (u \rightarrow -u); \omega // (u \rightarrow -u))_s \quad (28)$$

$$(6) (\lambda; \omega)_s // \text{h}m_z^{xy} := ((\lambda \setminus \{x, y\}) \sqcup (z \rightarrow \text{BCH}(\lambda_x, \lambda_y))); \omega)_s \quad (29)$$

$$(\lambda; \omega)_s // \text{t}m_w^{uv} := (\lambda // (u, v \rightarrow w); \omega // (u, v \rightarrow w))_s \quad (30)$$

$$(7) (\lambda; \omega)_s // \text{h}\Delta_{yz}^x := ((\lambda \setminus x) \sqcup (y \rightarrow \lambda_x, z \rightarrow \lambda_x); \omega)_s \quad (31)$$

$$(\lambda; \omega)_s // \text{t}\Delta_{vw}^u := (\lambda // (u \rightarrow v + w); \omega // (u \rightarrow v + w))_s \quad (32)$$

$$(8) (\lambda; \omega)_s // \text{h}\sigma_y^x := ((\lambda \setminus x) \sqcup (y \rightarrow \lambda_x); \omega)_s \quad (33)$$

$$(\lambda; \omega)_s // \text{t}\sigma_v^u := (\lambda // (u \rightarrow v); \omega // (u \rightarrow v))_s \quad (34)$$

$$(9) (\lambda; \omega)_s // \text{t}ha^{ux} := (\lambda // RC_u^{\lambda_x}; (\omega + J_u(\lambda_x)) // RC_u^{\lambda_x})_s. \quad (35)$$

Proof. The first 8 assertions (14 operations) are very easy. The main challenge to the reader should be to gather her concentration for the 14-times repetitive task of unwrapping definitions. If you are ready to cut corners, only go over (21), (29), (30), (31), and (32). Let us turn to the proof of the last assertion, Equation (35). That proof is in fact in [BN4], or at least can be assembled from pieces already in [BN4]. Yet the assembly would be a bit delicate, and hence a proof is reproduced below which refers back to [BN4] only at one technical point.

By inspecting the definition of $\text{t}ha^{ux}$, it is clear that there is *some* assignment $\gamma \mapsto R_u^\gamma$ that assigns an operator $R_u^\gamma: FL(T) \rightarrow FL(T)$ to every $\gamma \in FL(T)$ and that there is *some* functional $K_u: FL(T) \rightarrow CW(T)$, for which a version of Equation (35) holds:

$$E_s(\lambda; \omega)_s // \text{t}ha^{ux} = E_s(\lambda // R_u^{\lambda_x}; (\omega + K_u(\lambda_x)) // R_u^{\lambda_x})_s \quad (36)$$

Indeed, $\text{t}ha^{ux}$ acts on $E_s(\lambda; \omega)_s$ by placing a copy of $\exp(\lambda_x)$ at the top of the tail strand u , and then re-writing the result without having any heads on strand u so as to invert E_s back again. The re-writing is done by sliding the heads of $\exp(\lambda_x)$ down to the bottom of strand u , where they cancel by *CP*. Every time a head slides past a tail we get a contribution from \overrightarrow{STU}_2 . Sometimes a head of a λ_x will slide against a tail of another λ_x , whose head will have to slide down too, leading to a rather complicated iterative process. Nevertheless, these contributions are the same for every tail on strand u , namely for every occurrence of the variable u in $FL(T)^H$ and/or in $CW(T)$. This explains the terms $\lambda // R_u^{\lambda_x}$ and $\omega // R_u^{\lambda_x}$ in Equation (36). We note that the degree 0 part of the operator $R_u^{\lambda_x}$ is the identity, and hence it is invertible.

But yet another type of term arises in the process — sometimes a head of some tree will slide against a tail of its own, and then the contribution arising from \overrightarrow{STU}_2 will be a wheel. Hence there is an additional contribution to the output, some $L_u(\lambda_x)$ which clearly can depend only on u and λ_x . Using the invertibility of $R_u^{\lambda_x}$ to write $L_u(\lambda_x) = K_u(\lambda_x) // R_u^{\lambda_x}$ we completely reproduce Equation (36).

We now need to show that R_u^γ and $K_u(\gamma)$ are RC_u^γ and $J_u(\gamma)$ of Definitions 2.9 and 2.11. Tracing again through the discussion in the previous two paragraphs, we see that at any fixed degree, R_u^γ and $K_u(\gamma)$ depend polynomially on the coefficients of γ , and hence it is legitimate

to study their variation with respect to γ . It is also easy to verify that $R_u^0 = RC_u^0 = I$ and that $K_u(0) = J_u(0) = 0$, and hence it is enough to show that, with an indeterminate scalar τ ,

$$\frac{d}{d\tau}R_u^{\tau\gamma} = \frac{d}{d\tau}RC_u^{\tau\gamma} \quad \text{and} \quad \frac{d}{d\tau}K_u(\tau\gamma) = \frac{d}{d\tau}J_u(\tau\gamma). \quad (37)$$

Let us compute the left-hand-sides of the above equations. If τ is an infinitesimal (so $\tau^2 = 0$), or more precisely, computing the above left-hand-sides at $\tau = 0$, we can re-trace the process described in the two paragraphs following Equation (36) keeping in mind that with $\lambda_x = \tau\gamma$ the \overrightarrow{STU}_2 relation can only be applied once (or else terms proportional to τ^2 will arise). The result is

$$\left. \frac{d}{d\tau}R_u^{\tau\gamma} \right|_{\tau=0} = \text{ad}_u^\gamma \quad \text{and} \quad \left. \frac{d}{d\tau}K_u(\tau\gamma) \right|_{\tau=0} = \text{div}_u(\gamma), \quad (38)$$

where $\text{ad}_u^\gamma: FL(T) \rightarrow FL(T)$ is the derivation which maps the generator u of $FL(T)$ to $[\gamma, u]$ and annihilates all other generators of $FL(T)$ (compare [BN4, Definition 10.5]) and where $\text{div}_u(\gamma)$ is the same as in Definition 2.11.

Moving on to general τ , we note that the operations hm and tha satisfy

$$hm_z^{xy} // tha^{uz} = tha^{ux} // tha^{uy} // hm_z^{xy} \quad (39)$$

(stitching strands x and y and then stitching a copy of the result to u is the same as stitching a copy of x to u , then a copy of y , and then stitching x to y ; compare [BN4, Equation (6)]). Applying the operators on the two sides of Equation (39) to $E_s(\lambda; \omega)$ (assuming H and T are such that it makes sense), then expanding using (29) and (36), and then ignoring the wheels in the resulting equality, we find that R_u satisfies

$$R_u^{\text{BCH}(\lambda_x, \lambda_y)} = R_u^{\lambda_x} // R_u^{\lambda_y} // R_u^{\lambda_x} \quad (40)$$

(compare [BN4, Equation (16)]). Similarly, looking only at the wheel part of (39) we get

$$K_u(\text{BCH}(\lambda_x, \lambda_y)) // R_u^{\text{BCH}(\lambda_x, \lambda_y)} = K_u(\lambda_x) // R_u^{\lambda_x} // R_u^{\lambda_y} // R_u^{\lambda_x} + K_u(\lambda_y // R_u^{\lambda_x}) // R_u^{\lambda_y} // R_u^{\lambda_x},$$

which, composing on the right with $R_u^{\text{BCH}(\lambda_x, \lambda_y)}$ and using (40), is equivalent to

$$K_u(\text{BCH}(\lambda_x, \lambda_y)) = K_u(\lambda_x) // R_u^{\lambda_x} + K_u(\lambda_y // R_u^{\lambda_x}) // C_u^{-\lambda_x} \quad (41)$$

(compare [BN4, Equation (19)]).

Equations (40) and (41) hold for any λ , and hence for any λ_x and λ_y . Specializing to $\lambda_x = \tau\gamma$ and $\lambda_y = \epsilon\gamma$, where ϵ is some new indeterminate scalar, and using the fact that

None should believe without a verification:

$$\text{lhs} = \xi_a // \text{hm}[1, 2, 4] // \text{tha}[u, 4]; \text{rhs} = \xi_a // \text{tha}[u, 1] // \text{tha}[u, 2] // \text{hm}[1, 2, 4]; \\ \{\text{lhs}, (\text{lhs} \equiv \text{rhs})@{8}\}$$

$$\left\{ \text{Es} \left[\left[3 \rightarrow \text{LS} \left[\overline{u}, -\overline{u\overline{v}}, -\overline{u\overline{u\overline{v}}} + \frac{1}{2} \overline{u\overline{u\overline{v}v}}, \frac{3}{2} \overline{u\overline{u\overline{u\overline{v}}}} + \overline{u\overline{u\overline{u\overline{v}v}}} - \frac{1}{6} \overline{u\overline{u\overline{v}v}v}, \dots \right], \right. \right. \\ \left. \left. 4 \rightarrow \text{LS} \left[\overline{u} + \overline{v}, \frac{\overline{u\overline{v}}}{2}, -\frac{23}{12} \overline{u\overline{u\overline{v}}} - \frac{5}{12} \overline{u\overline{u\overline{v}v}}, \overline{u\overline{u\overline{u\overline{v}}}} + \frac{13}{24} \overline{u\overline{u\overline{v}v}} + \frac{1}{12} \overline{u\overline{u\overline{v}v}v}, \dots \right] \right\}, \\ \text{cws} \left[2 \overline{u}, -\overline{u\overline{v}}, -\frac{3 \overline{u\overline{u\overline{v}}}}{2}, -\frac{\overline{u\overline{u\overline{u\overline{v}}}}}{6} + \overline{u\overline{u\overline{v}v}} - \overline{u\overline{v}u\overline{v}}, \dots \right], \text{BS}[9 \text{ True}, \dots] \left. \right\}$$

$\text{BCH}(\tau\gamma, \epsilon\gamma) = (\tau + \epsilon)\gamma$, Equations (40) and (41) become

$$R_u^{(\tau+\epsilon)\gamma} = R_u^{\tau\gamma} // R_u^{\epsilon\gamma} // R_u^{\tau\gamma} \quad \text{and} \quad K_u((\tau + \epsilon)\gamma) = K_u(\tau\gamma) // R_u^{\tau\gamma} + K_u(\epsilon\gamma // R_u^{\tau\gamma}) // C_u^{-\tau\gamma}.$$

Now differentiating with respect to ϵ at $\epsilon = 0$ and using Equation (38) with τ replaced with ϵ , we get

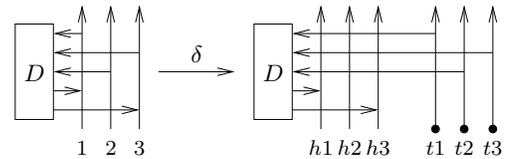
$$\frac{d}{d\tau} R_u^{\tau\gamma} = R_u^{\tau\gamma} // \text{ad}_u^{\gamma} // R_u^{\tau\gamma} \quad \text{and} \quad \frac{d}{d\tau} K_u(\tau\gamma) = \text{div}_u(\gamma // R_u^{\tau\gamma}) // C_u^{-\tau\gamma}.$$

The first of these equations is the same equation that is satisfied by RC_u (see [BN4, Lemma 10.7], with $\delta\gamma$ proportional to γ), and hence $R_u = RC_u$. By a simple change of variables, $J_u(\tau\gamma) = \int_0^\tau dt \text{div}_u(\gamma // RC_u^{t\gamma}) // C_u^{-t\gamma}$, and hence $\frac{d}{d\tau} J_u(\tau\gamma) = \text{div}_u(\gamma // RC_u^{\tau\gamma}) // C_u^{-\tau\gamma}$ (compare with the formula for the full differential of J , [BN4, Proposition 10.10]). Comparing with the above formula for the derivative of K_u , we find that $K_u = J_u$. \square

2.4.4. *The inclusion $\{\mathcal{A}^w(S)\} \leftrightarrow \{\mathcal{A}^w(H; T)\}$.* The following definition and proposition imply that there is no loss in studying the spaces $\mathcal{A}^w(H; T)$ rather than the spaces $\mathcal{A}^w(S)$.

Definition 2.21. Let $\delta: \mathcal{A}^w(S) \rightarrow \mathcal{A}^w(S; S)$ be the composition of the ‘‘double every strand’’ map $\prod_{a \in S} \Delta_{ha, ta}^a: \mathcal{A}^w(S) \rightarrow \mathcal{A}^w(hS \sqcup tS)$ with the projection $\mathcal{A}^w(hS \sqcup tS) \rightarrow \mathcal{A}^w(S; S)$ (as an exception to the rule of Footnote 13 we temporarily highlight the distinction between head and tail labels by affixing them with the prefixes h and t).

Comment 2.22. If $D \in \mathcal{A}^w(S)$ is sorted ‘‘heads below tails’’ as in Comment 2.1, then δD is D with its arrow heads placed on the head strands and its arrow tails placed on the tail strands, as shown on the right.



Proposition 2.23. δ is a (non-multiplicative) vector space isomorphism. The inverse of δ on $D \in \mathcal{A}^w(S; S)$ is given by the process

- (1) Write D with only arrow heads on the head strands and only arrow tails on the tail strands. By Comment 2.16 this produces a well-defined element D' of $\mathcal{A}^w(hS \sqcup tS)$.
- (2) Stitch all the head-tail pairs of strands in D' by putting each head ahead of its corresponding tail: $\delta^{-1}D = D' // \prod_a dm_a^{ha, ta}$.

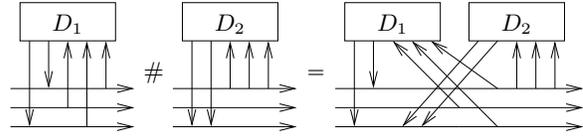
Proof. $\delta^{-1} // \delta = I$ by inspection, and $\delta // \delta^{-1}$ is clearly the identity on diagrams sorted to have heads ahead of tails as in Comment 2.1. \square

In topology, δ agrees with the δ of [BN4, Section 2.2]. In Lie theory, it agrees with the linear (non-multiplicative) isomorphism $\mathcal{U}(\mathfrak{I}\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$ and with similar isomorphisms considered by Etingof and Kazhdan within their work on the quantization of Lie bialgebras [EK] (albeit only when the Lie bialgebras in question are cocommutative).

Definition 2.24. The product $\#$ of $\mathcal{A}^w(S; S)$ induces a new product, also denoted $\#$, on $\mathcal{A}^w(S)$. If D_1 and D_2 are in $\mathcal{A}^w(S)$, set

$$D_1 \# D_2 := (\delta(D_1) \# \delta(D_2)) // \delta^{-1}. \tag{42}$$

Comment 2.25. With [Comment 2.22](#) in mind, we see that if D_1 and D_2 are sorted as in [Comment 2.1](#), then $D_1 \# D_2$ is “heads of D_1 , then of D_2 , then tails of D_1 , then of D_2 ” (with the last two parts interchangeable, by *TC*). The picture is nicer when rotated, as on the right.



See the comments following [Discussion 2.5](#).

The next proposition shows how the operations of defined on the $\mathcal{A}^w(S)$ -spaces in [Definition 2.2](#) can be written in terms of the “head and tail” operations of [Definition 2.17](#), thus completing the description of the E_s presentation.

Proposition 2.26. (1) If S_1 and S_2 are disjoint and $D_1 \in \mathcal{A}^w(S_1)$ and $D_2 \in \mathcal{A}^w(S_2)$, then $\delta(D_1 \sqcup D_2) = \delta(D_1) \sqcup \delta(D_2)$.

(2) Let $D_1, D_2 \in \mathcal{A}^w(S)$. Then $\delta(D_1 D_2)$ can be written in terms of $\delta(D_1)$ and $\delta(D_2)$ using its description in terms of \sqcup , $d\sigma$, and dm in [Equation \(2\)](#) and using the formulas for \sqcup , $d\sigma$, and dm that appear in parts (1), (8), and (6) of this proposition. ^{C26}

(3) $d\eta^a // \delta = \delta // h\eta^a // t\eta^a$.

[it:HTd\(6\)](#) $dm_c^{ab} // \delta = \delta // tha^{ab} // hm_c^{ab} // tm_c^{ab}$. ^{C26}

[it:HTd\(4\)](#) $dA^a // \delta = \delta // hA^a // tA^a // tha^{aa}$.

(7) $d\Delta_{bc}^a // \delta = \delta // h\Delta_{bc}^a // t\Delta_{bc}^a$.

[it:HTd\(5\)](#) $dS^a // \delta = \delta // hS^a // tS^a // tha^{aa}$.

[it:HTsigm\(8\)](#) $d\sigma_b^a // \delta = \delta // h\sigma_b^a // t\sigma_b^a$.

Proof. The only difficulty is with items (4)–(6). Item (4) is easier to understand in the form $\delta^{-1} // dA^a = hA^a // tA^a // tha^{aa} // \delta^{-1}$. Indeed, δ^{-1} plants heads ahead of tails on strand a . Applying dA^a reverses that strand (and adds some signs). This reversal can be achieved by reversing the head part (with signs), then the tail part (with signs), and then by swapping the two parts across each other. The first reversal is hA^a , the second is tA^a , and the swap

^{C26}As a sample for the whole proposition, we quote the implementation of dm and verify its meta-associativity $dm_a^{ab} // dm_a^{ac} = dm_b^{bc} // dm_a^{ab}$ (compare [BN4](#), [Equation \(32\)](#)). We then include our implementation of the stacking product (item (2) above) without further explanations:

```
AC   $\xi\_Es // dm[a\_ , b\_ , c\_ ] := \xi // tha[a, b] // tm[a, b, c] // hm[a, b, c];$ 
```

```
 lhs =  $\xi_b // dm[1, 2, 1] // dm[1, 3, 1];$  rhs =  $\xi_b // dm[2, 3, 2] // dm[1, 2, 1];$   
{lhs@{3}, (lhs = rhs)@{5}}
```

```
 {Es [ [ 1  $\rightarrow$  LS [ -2  $\overline{1}$  +  $\overline{4}$ , -  $\frac{3}{2} \overline{14}$ , 20  $\overline{114}$  -  $\frac{19}{3} \overline{144}$ , ... ],  
4  $\rightarrow$  LS [ 2  $\overline{1}$  +  $\overline{4}$ ,  $\overline{14}$ , -  $\frac{31}{2} \overline{114}$  -  $\frac{13}{6} \overline{144}$ , ... ] ],  
cws [ 3  $\overline{1}$  -  $\overline{4}$ , -3  $\overline{11}$  +  $\frac{14}{2}$  +  $\overline{44}$ ,  $\frac{71}{4} \overline{111}$  +  $\frac{19}{4} \overline{114}$  -  $\frac{7}{6} \overline{144}$  -  $\frac{2}{3} \overline{444}$ , ... ] ], BS[6 True, ... ] }
```

```
AC  Es /: Es[ $\lambda_1$ ,  $\omega_1$ ] ** Es[ $\lambda_2$ ,  $\omega_2$ ] /; Support[ $\lambda_1$ ] == Support[ $\lambda_2$ ] := Module[  
  {S = Support[ $\lambda_1$ ],  $\xi$ , a},  
   $\xi = Es[\lambda_1, \omega_1] * (Es[\lambda_2, \omega_2] // d\sigma[S \rightarrow (\text{bar} / @ S)]);$   
  Table[ $\xi = \xi // dm[a, \text{bar}[a], a]$ , {a, S}] // Last  
];
```

is tha^{aa} followed by δ^{-1} . Item (5) is proven in exactly the same way, and item (6) is proven in a similar way, where the right hand side traces the schematics $(ha\ ta\ hb\ tb) \xrightarrow{tha} (ha\ hb\ ta\ tb) \xrightarrow{hm/tm} ((ha\ hb)(ta\ tb))$. \square

Discussion 2.27. It is easy to verify that $\delta: \mathcal{A}^w(S) \rightarrow \mathcal{A}^w(S; S)$ is a co-algebra morphism, and hence it restricts to an isomorphism $\delta: \mathcal{A}_{\text{exp}}^w(S) \rightarrow \mathcal{A}_{\text{exp}}^w(S; S)$. Therefore $E_s//\delta^{-1}$ is a bijection between $TW_s(S) := TW_s(S; S)$ and $\mathcal{A}_{\text{exp}}^w(S)$. Proposition 2.26 now tells us how to write all the “ d ” operations of Definition 2.2 as compositions of “ h ” and “ t ” operations, and Definition-Proposition 2.20 tells us how to write these as operations on $TW_s(H; T)$ (the H and T label sets that occur here are always S with one or two labels added or removed). Hence overall $E_s//\delta^{-1}$, acting on $TW_s(S)$, is a complete presentation of $\mathcal{A}_{\text{exp}}^w(S)$.

Definition 2.29. The “factored” presentation E_f of $\mathcal{A}_{\text{exp}}^w$ is the composition $E_f := E_s//\delta^{-1}$. Namely, for a set S of strands, we define $E_f: TW_s(S) \xrightarrow{\sim} \mathcal{A}_{\text{exp}}^w(S)$ by $(\lambda; \omega)_s \mapsto E_s(\lambda; \omega)_s//\delta^{-1} = \text{exp}_{\#}(l\lambda + \omega)$. See the illustration on the right.

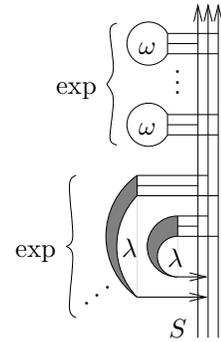


fig:Ef

Figure 2.28. $E_f(\lambda; \omega)_s$.

Conversion

2.5. Converting between the E_l and the E_f presentations.

We now have two presentations for elements of $\mathcal{A}_{\text{exp}}^w(S)$, and we wish to be able to convert between the two. This turns out to involve the maps Γ and Λ of Propositions 2.6 and 2.8.

Definition 2.30. Define a pair of inverse maps $\Gamma: TW_l(S) \rightarrow TW_s(S)$ and $\Lambda: TW_s(S) \rightarrow TW_l(S)$ by

$$\Gamma: (\lambda; \omega)_l \mapsto (\Gamma(\lambda); \omega)_s \quad \text{and} \quad \Lambda: (\lambda; \omega)_s \mapsto (\Lambda(\lambda); \omega)_l.$$

ammaLambda

Theorem 2.31. The left-most triangle in Figure 1.2 commutes. Namely,

$$E_l = \Gamma // E_f \quad \text{and} \quad E_f = \Lambda // E_l. \tag{43}$$

eq: convert

(All other parts of Figure 1.2 commute by definition).

Before we can prove this theorem we need a few preliminaries. For an element $D \in \mathcal{A}_{\text{exp}}^w(S)$, we can define three associated quantities:

- The projection of D to the degree 1 part of $\mathcal{A}^w(S)$, and especially, the projection $\pi_A(D)$ of the degree 1 part to its “framing” part A_S (consisting of self-arrows, that begin and end on the same strand and point, say, up).
- A conjugation automorphism C_D of $FL(S)$, defined as follows. First, embed $FL(S)$ into $\mathcal{A}^w(S \sqcup \{\infty\})$ by mapping any generator $a \in S$ to a degree 1 diagram in $\mathcal{A}^w(S \sqcup \{\infty\})$, the arrow whose tail is on strand a and whose head is on the new “ ∞ ” strand and extending in a bracket-preserving way, using the commutator of the stacking product as the bracket on $\mathcal{A}^w(S \sqcup \{\infty\})$. Then note that $FL(S) \subset \mathcal{A}^w(S \sqcup \{\infty\})$ is invariant under conjugation by D and let C_D denote this conjugation action.



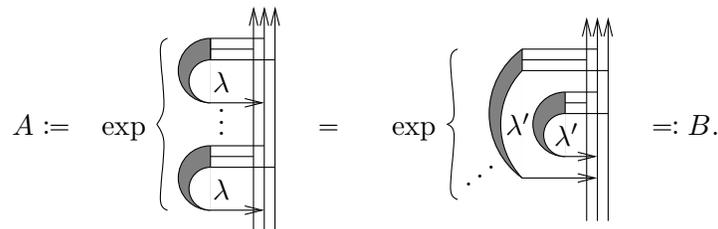
This is a direct analog of the Artin action of the pure braid groups PuB_n / PwB_n on the free group $FG(n)$.

- $\pi_{\bullet}(D)$ is the result of adding a bullet at the bottom of every strand of D , in the same sense as in Section 2.4.1. Equivalently, $\pi_{\bullet} = \delta // \prod_{a \in S} h\eta^a$ is the composition of δ with “delete all head strands”. The target space of π_{\bullet} is $\mathcal{A}^w(\emptyset; S)$, which is the symmetric algebra $\mathcal{S}(CW(S))$ generated by wheels.

Proposition 2.32. D is determined by the above three quantities $\pi_A(D)$, C_D , and $\pi_{\bullet}(D)$.

Proof. As in Section 2.3, every $D \in \mathcal{A}_{\text{exp}}^w(S)$ can be written uniquely in the form $D = e^{\lambda} e^{\omega}$, where $\lambda \in FL(S)^S$ and $\omega \in CW(S)$. One may easily verify that $\pi_{\bullet}(D)$ is ω , that C_D is the exponential of the derivation in tder_S corresponding to λ , and that $\pi_A(D)$ determines the part of λ lost by the projection $FL(S)^S \rightarrow \text{tder}_S$.

Proof of Theorem 2.31. For $\lambda \in FL(S)^S$ let $\lambda' = \Gamma(\lambda)$. Comparing Figures 2.13 and 2.28, we find that the ω parts drop out and we need to prove, schematically, that in $\mathcal{A}_{\text{exp}}^w(S)$,



A simple degree 1 calculation shows that $\pi_A(A) = \pi_A(B) = 0$. The CP relation of Section [2.4.1](#) shows that $\pi_1(A) = \pi_1(B) = 0$. Finally, it is easy to verify that $C_A = e^{-\partial\lambda}$ while $C_B = C^\lambda$, and hence $C_A = C_B$ follows from Proposition [2.6](#). \square

3. SOME COMPUTATIONS

3.1. Tangle Invariants.

3.1.1. *The General Framework.* Recall from [WKO2] that the assignment $Z^w \mapsto \exp(\uparrow) \otimes$ defined on S -component tangles and taking values in $\mathcal{A}_{\text{exp}}^w(S)$ (where \uparrow denotes an arrow connecting the upper strand to the lower strand and exponentiation is in a formal sense) defines an invariant of tangles with values in $\mathcal{A}_{\text{exp}}^w(S)$. We'd like to compute Z^w (more precisely, its logarithm), in as much as possible, using both the $TW_l(S)$ -valued [AT]-presentation E_l or using the $TW_s(S)$ -valued factored presentation E_f (recall Figure 1.2).

We let $R_l^+(a, b)$ and $R_s^+(a, b)$ denote the value $R(a, b) = Z^w \left(\begin{array}{c} \nearrow \\ a \quad b \end{array} \right)$ of the positive crossing in TW_l and TW_s , respectively, and similarly, let $R_l^-(a, b)$ and $R_s^-(a, b)$ denote the value $R^{-1}(a, b) = Z^w \left(\begin{array}{c} \searrow \\ b \quad a \end{array} \right)$ of the negative crossing in TW_l and TW_s , respectively (for both signs we label the upper strand a and the lower strand b). That is,

$$Z^w \left(\begin{array}{c} \nearrow \\ a \quad b \end{array} \right) = R_l^+(a, b) // E_l = R_s^+(a, b) // E_s \quad \text{and} \quad Z^w \left(\begin{array}{c} \searrow \\ b \quad a \end{array} \right) = R_l^-(a, b) // E_l = R_s^-(a, b) // E_s.$$

One may easily verify that $R_{l,s}^\pm(a, b) = (a \rightarrow 0, b \rightarrow \pm a; 0)_{l,s}$ ^{C27}, and it is a simple exercise to verify that R satisfies the Yang-Baxter / Reidemeister 3 relation $R_{l,s}^+(1, 2) * R_{l,s}^+(1, 3) * R_{l,s}^+(2, 3) = R_{l,s}^+(2, 3) * R_{l,s}^+(1, 3) * R_{l,s}^+(1, 2)$ ^{C28}.

3.1.2. *The Knot 8_{17} and the Borromean Tangle.* In this short section we evaluate Z^w on the knot 8_{17} and on the Borromean tangle, both shown in Figure 3.1. An expanded version of this section appears as [BN4, Sections 6.3 and 6.4].

For the 8-crossing knot 8_{17} we need to take 8 copies of R_s^\pm with strands labelled 1 through 16 as in Figure 3.1, and then stitch strands 1 to 2, 2 to 3, etc.^{C29}. This is done using dm operations, and hence we cannot use the E_l presentation.

Similarly for the 6-crossings Borromean tangle we need 6 copies of R_s^\pm followed by some stitching^{C30}. A colourful evaluation of the Borromean tangle appears in [BN4, Section 6.4].

computations below

^{C27}In computer talk, this is

```

⊙⊙ Rl[a_, b_] := E1[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
⊙⊙ iRl[a_, b_] := E1[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
⊙⊙ Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
⊙⊙ iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];

```

^{C28}Indeed, here's a computer verification in E_l , to degree 5:

```

⊙⊙ lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs == rhs)@{5}}

```

```

🖥️ {E1[⟨1 → LS[0, 0, 0, ...], 2 → LS[1̄, 0, 0, ...], 3 → LS[1̄+2̄, 0, 0, ...]⟩, CWS[0, 0, 0, ...]],
BS[6 True, ...]}

```

^{C29}Here it is, to degree 6:

```

⊙⊙ t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
⊙⊙ Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
⊙⊙ t1@{6}

```

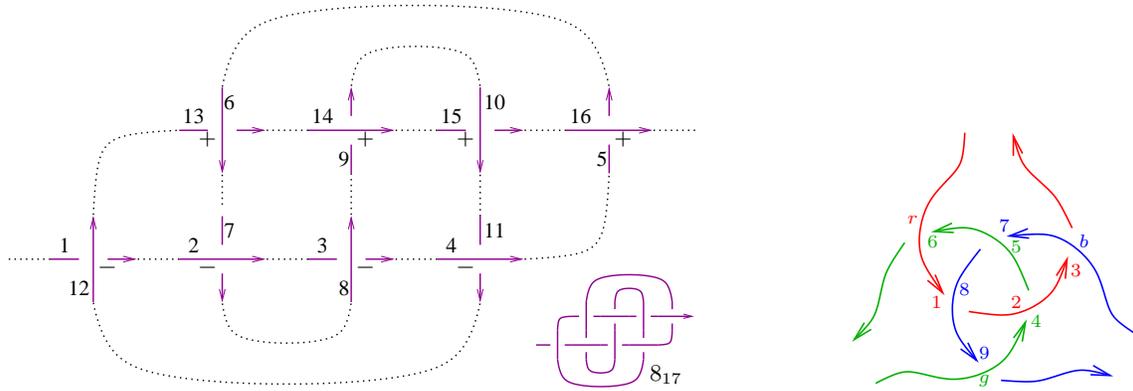


Figure 3.1. The knot 8_{17} and the Borromean tangle.

fig:817And

computations below

```
Es [⟨1 → LS[0, 0, 0, 0, 0, 0, ...]⟩,
  cws [0, -11, 0, -31 1111̄, 0, -1351 111111̄, ...]]
```

Fuller output:
[WKO4]/817.nb

:Borromean

C30 To degree 4, we get

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

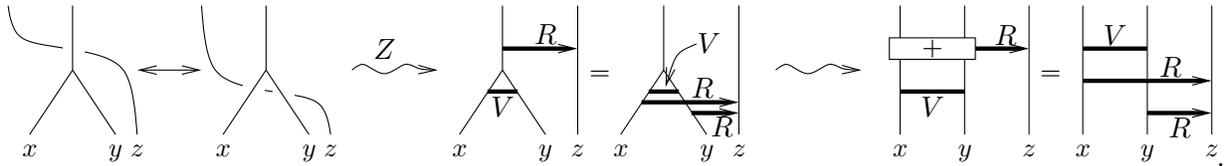
Fuller output:
[WKO4]/Borromean.nb

```
Es [⟨b → LS[0, gr, 1/2 ggr̄ + br̄g + 1/2 gr̄r,
  -1/2 bbr̄g + 1/6 ggr̄r̄ + 1/4 ggr̄r̄ - 1/2 bgr̄br̄ - 1/2 br̄ggr̄ - 1/2 br̄rgr̄ + 1/6 gr̄r̄r̄, ...],
  g → LS[0, -br, 1/2 bbr̄ - bgr̄ - br̄g + 1/2 br̄r̄, -1/6 bbr̄br̄ - 1/2 bbr̄gr̄ - 1/2 bgr̄gr̄ - 1/2 bbr̄gr̄ -
  1/4 bbr̄r̄ + 1/2 bgr̄r̄ + 1/2 bgr̄br̄ + br̄gr̄ - br̄gr̄g - 1/2 br̄ggr̄ + 1/2 br̄rgr̄ - 1/6 br̄r̄r̄, ...],
  r → LS[0, bg, 1/2 bbḡ + bgr̄ + 1/2 bgr̄g, 1/6 bbr̄bḡ + 1/2 bbr̄gr̄ + 1/2 bgr̄gr̄ +
  1/4 bbr̄gḡ + 1/2 bgr̄r̄r̄ + 1/6 bgr̄gḡ, ...]⟩,
  cws [0, 0, 2 bgr̄, bbr̄gr̄ - bgr̄br̄ + bgr̄gr̄ - bgr̄gr̄ + bgr̄r̄ - br̄gr̄, ...]]
```

bsec:SolKV

3.2. Solutions of the Kashiwara-Vergne Equations. In [WKO2, Section 4.1] we found that in order to construct a homomorphic expansion Z^w for the class wTF^o of orientable w-tangled foams, defined there, we need to find elements $V := Z^w(\text{cap}) \in \mathcal{A}_{\text{exp}}^w(x, y)^{C31}$ and $Cap := Z^w(\text{cup}) \in \mathcal{A}_{\text{exp}}^w(\text{cup})^{C32}$ that are required to satisfy the three equations in (44) and (45) below. Recall from [WKO2, Section 4.4] that these equations are equivalent to equations considered by Alekseev and Torossian in [AT] (see [WKO2, Equation 14] and [AT, Section 5.3]), and that the latter equations were shown in [AT, Section 5.2] to be equivalent to the Kashiwara-Vergne equations of [KV].

The purpose of this section is to trace through all that at the level of actual computations. Let us start by recalling from [WKO2] the equations for V and for Cap . The first of those is the R4 equation [WKO2, (11)], $V^{12}R^{12} = R^{23}R^{13}V^{12}$, coming from the picture



In the language of this paper, and denoting the three strands x , y , and z , this equation becomes

$$V * (R(x, z) // d\Delta_{xy}^x) = R(y, z) * R(x, z) * V^{C33} \tag{44}$$

The second and the third, “unitarity” and the “cap equation”, [WKO2, (12)] and [WKO2, (13)], are the equations

$$V * (V // dA) = 1 \quad \text{in } \mathcal{A}^w(x, y) \quad \text{and} \quad V * (Cap // d\Delta_{xy}^x) = Cap(Cap // d\sigma_y^x) \quad \text{in } \mathcal{A}^w(\text{cup}_{x,y}),^{C33} \tag{45}$$

¹⁷ Cap is called C in [WKO2] and we trust that the other minor notational differences with [WKO2] will cause no difficulty to the reader. Note that $\mathcal{A}^w(\text{cup}_S)$ is $\mathcal{A}^w(S)$ with CP relations imposed at the tops of the strands; compare with Section 2.4.1.

computations below

^{C31}For computations, we use the E_s presentation for V . As V is presented in $TW_s(\{x, y\})$, it is of the form $V = ((x \rightarrow \alpha, y \rightarrow \beta); \gamma)_s$, where $\alpha, \beta \in FL(x, y)$ and $\gamma \in CW(x, y)$, and where the coefficients of α , β , and γ , what we call the α s, the β s, and the γ s, will be determined later. The first line below sets α , β , and γ to be series with yet-unknown coefficients, and the second line sets V to be the appropriate combination of α , β , and γ :

```

⊙ x = LW["x"]; y = LW["y"]; z = LW["z"];
⊙ α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
⊙ V₀ = Es[⟨x → α, y → β⟩, γ];
    
```

(for a technical reason, in computations we use the symbol V_0 to denote V).

^{C32}Similarly, Cap is presented in $TW_s(x)$. As it is made only of wheels, its tree part is 0, or the Lie series $LS[0]$. The wheels part of Cap is a series $\kappa \in CW(x)$ whose coefficients are the yet-unknown κ s:

```

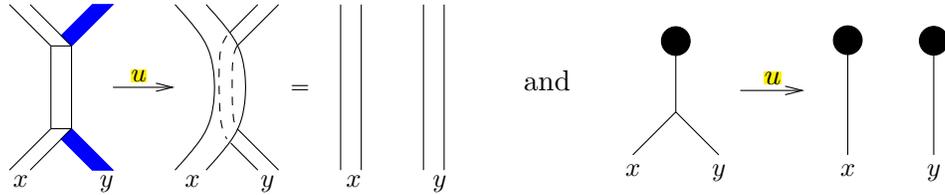
⊙ x = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
    
```

C:VSetup

eq:HardR4

eq:UnzipEq

which come from the two unzip operations,



Solving Equations [\(44\)](#) and [\(45\)](#) degree by degree with the initial condition $\alpha = -y/2 + \dots$ we find that one possible solution, given in the factored presentation, is

$$V = E_f \left(x \rightarrow -\frac{\overline{xy}}{24} + \frac{7 \overline{x x \overline{xy}}}{5760} - \frac{7 \overline{x \overline{xy} y}}{5760} + \frac{\overline{\overline{xy} y y}}{1440} + \dots, \right. \\ \left. y \rightarrow \frac{\overline{x}}{2} - \frac{\overline{xy}}{12} + \frac{\overline{x x \overline{xy}}}{5760} - \frac{\overline{x \overline{xy} y}}{720} + \frac{\overline{\overline{xy} y y}}{720} + \dots; \right. \\ \left. -\frac{\overline{xy}}{48} + \frac{\overline{x x \overline{xy}}}{2880} + \frac{\overline{x \overline{xy} y}}{2880} + \frac{\overline{\overline{xy} x y}}{5760} + \frac{\overline{\overline{xy} y y}}{2880} + \dots \right)_s,$$

C: VCapEqns

C33 The three equations in [\(44\)](#) and [\(45\)](#) are coded as follows:

```

R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0 ;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]] ;
CapEqn = (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
(Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]) ;
    
```

and $Cap = -\overline{xx}/96 + \overline{xxx}/11,520 - \overline{xxxxx}/725,760 + \dots$ ^{C34}. Note that according to [\[WKO3\]](#), Cap is always $\sum a_n \overline{x^n}$, where $\sum a_n \hbar^n = \frac{1}{4} \log\left(\frac{\hbar/2}{\sinh \hbar/2}\right)$ ^{C35}.

We can also write V in the lower-interlaced presentation:

$$V = E_l \left(\begin{aligned} x &\rightarrow -\frac{\overline{xy}}{24} + \frac{\overline{xxy}}{96} + \frac{\overline{xxxxy}}{2880} - \frac{\overline{xyxyy}}{480} + \frac{\overline{xyyy}}{1440} + \dots, \\ y &\rightarrow \frac{\overline{x}}{2} - \frac{\overline{xy}}{12} + \frac{\overline{xxy}}{96} + \frac{\overline{xxxxy}}{960} - \frac{\overline{xyxyy}}{320} + \frac{\overline{xyyy}}{720} + \dots; \\ &\quad -\frac{\overline{xy}}{48} + \frac{\overline{xxxxy}}{2880} + \frac{\overline{xyxyy}}{2880} + \frac{\overline{xyxyy}}{5760} + \frac{\overline{xyyy}}{2880} + \dots \end{aligned} \right), \text{C36}$$

(Cap is the same in both presentations).

Recall from [\[WKO2, Section 4.4\]](#) and from Comment [2.15](#) that the tree part of “our” V , taken in the lower-interlaced presentation, is $\log F^{21}$, where F is the solution of “generalized

computations below

^{C34}We set the initial condition for α in degree 1, then declare that α, β, γ , and κ are the series which solve equations [R4Eqn](#), [UnitarityEqn](#), and [CapEqn](#), and then print the values of V and κ (note the \hbar^{-1} that comes with [R4Eqn](#) — it indicates a degree shift — [R4Eqn](#) in degree k only puts conditions on our unknowns at degree $k - 1$):

```

βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ}, (ħ⁻¹ R4Eqn) ∧ UnitarityEqn ∧ CapEqn];
{V₀@{4}, κ@{6}}
    
```

C:VCapSolu

```
SeriesSolve::ArbitrarilySetting: In degree 1 arbitrarily setting {κs[x] → 0}.
```

[\[WKO4\]](#)/VCapSolution.nb Fuller output:

```
SeriesSolve::ArbitrarilySetting: In degree 3 arbitrarily setting {αs[x, y] → 0}.
```

```
SeriesSolve::ArbitrarilySetting: In degree 5 arbitrarily setting {αs[x, x, x, y] → 0}.
```

```
General::stop: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>
```

$$\left\{ \begin{aligned} \overline{x} &\rightarrow \text{LS} \left[0, -\frac{\overline{xy}}{24}, 0, \frac{\overline{xxxxy}}{5760} - \frac{\overline{xyxyy}}{5760} + \frac{\overline{xyyy}}{1440}, \dots \right], \\ \overline{y} &\rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{xxxxy}}{5760} - \frac{1}{720} \overline{xyxyy} + \frac{1}{720} \overline{xyyy}, \dots \right], \\ \text{CWS} &\left[0, -\frac{\overline{xy}}{48}, 0, \frac{\overline{xxxxy}}{2880} + \frac{\overline{xyxyy}}{2880} + \frac{\overline{xyxyy}}{5760} + \frac{\overline{xyyy}}{2880}, \dots \right], \text{CWS} \left[0, -\frac{\overline{xx}}{96}, 0, \frac{\overline{xxxxx}}{11520}, 0, -\frac{\overline{xxxxx}}{725760}, \dots \right] \end{aligned} \right\}$$

The solutions of [\(44\)](#) and [\(45\)](#) are not unique, and hence occasionally `SeriesSolve` encounters a coefficient whose value is not determined by the equations. When this happens its default action is to set the missing coefficient to 0. In the computation this happened to the coefficient of \overline{x} in κ and to the coefficient of \overline{xyy} in α .

^{C35}Indeed, the series below matches with the computation of κ , above.

```
Series[1/4 Log[ħ/2/sinh[ħ/2]], {ħ, 0, 12}]
```

$$-\frac{\hbar^2}{96} + \frac{\hbar^4}{11520} - \frac{\hbar^6}{725760} + \frac{\hbar^8}{38707200} - \frac{\hbar^{10}}{1916006400} + \frac{691 \hbar^{12}}{62768369664000} + O[\hbar]^{13}$$

^{C36}We could re-compute V in E_l by making some simple modifications to the input lines in [C33](#), but it is easier to use our tools and convert between the two presentations:

```
Λ[V₀]
```

FootC:VCapEqns

C:LambdaV

KV problem” of [AT, Section 5.3] and where the superscript 21 means “interchange the role of x and y ”. Thus using the notation of [AT] a solution to degree 4 of the generalized KV problem is^{C37}

$$\log F = \left(\frac{\overline{y}}{2} + \frac{\overline{xy}}{12} + \frac{\overline{xyy}}{96} - \frac{\overline{xxxy}}{720} + \frac{\overline{xyxy}}{320} - \frac{\overline{xyyy}}{960}, \frac{\overline{xy}}{24} + \frac{\overline{xyy}}{96} - \frac{\overline{xxxy}}{1440} + \frac{\overline{xyxy}}{480} - \frac{\overline{xyyy}}{2880} \right).$$

Next, we’d like to compute a solution of the original Kashiwara-Vergne equations of [KV]. These are the two equations below, written for unknowns $f, g \in FL(x, y)$:

$$x + y - \log e^y e^x = (1 - e^{-\text{ad}x})f + (e^{\text{ad}y} - 1)g, \quad (46) \quad \text{eq:KV1}$$

$$\text{div}_x f + \text{div}_y g = \frac{1}{2} \text{tr}_u \left(\left(\frac{\text{ad}x}{e^{\text{ad}x} - 1} + \frac{\text{ad}x}{e^{\text{ad}x} - 1} - \frac{\text{ad} \text{BCH}(x, y)}{e^{\text{ad} \text{BCH}(x, y)} - 1} \right) (u) \right). \quad (47) \quad \text{eq:KV2}$$

By tracing the definitions of the comparison map κ which appears in [AT, Theorem 5.8], we find that a solution (f, g) of the Kashiwara-Vergne equations can be computed from $\log F$ via the formula

$$(f, g) = \frac{e^{\text{ad}(\log F)} - 1}{\text{ad}(\log F)} (\mathcal{E}(\log F)),$$

where \mathcal{E} denotes the Euler operator, which multiplies every homogeneous element by its degree. To degree 4, we find^{C38} that

$$(f, g) = \left(\frac{\overline{y}}{2} + \frac{\overline{xy}}{6} + \frac{\overline{xyy}}{24} - \frac{\overline{xxxy}}{180} + \frac{\overline{xyxy}}{80} + \frac{\overline{xyyy}}{360}, \frac{\overline{xy}}{12} + \frac{\overline{xyy}}{24} - \frac{\overline{xxxy}}{360} + \frac{\overline{xyxy}}{120} + \frac{\overline{xyyy}}{180} \right).$$

computations below

$$\begin{aligned} & \text{E1} \left[\left\langle \overline{x} \rightarrow \text{LS} \left[0, -\frac{\overline{xy}}{24}, \frac{1}{96} \overline{xyy}, \frac{\overline{xxxy}}{2880} - \frac{1}{480} \overline{xyxy} + \frac{\overline{xyyy}}{1440}, \dots \right], \right. \right. \\ & \quad \left. \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, \frac{1}{96} \overline{xyy}, \frac{1}{960} \overline{xxxy} - \frac{1}{320} \overline{xyxy} + \frac{1}{720} \overline{xyyy}, \dots \right] \right\rangle, \\ & \quad \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \frac{\overline{xxxxy}}{2880} + \frac{\overline{xyxy}}{2880} + \frac{\overline{xyxy}}{5760} + \frac{\overline{xyyy}}{2880}, \dots \right] \end{aligned}$$

C:logF

^{C37}The more authoritative version, of course, is the one printed directly by the computer:

$$\text{logF} = \Lambda[\mathbf{V}_0][[1]] \quad // \quad \text{d}\sigma[\{\mathbf{x}, \mathbf{y}\} \rightarrow \{\mathbf{y}, \mathbf{x}\}]$$

$$\begin{aligned} & \left\langle \overline{x} \rightarrow \text{LS} \left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, \frac{1}{96} \overline{xyy}, -\frac{1}{720} \overline{xxxy} + \frac{1}{320} \overline{xyxy} - \frac{1}{960} \overline{xyyy}, \dots \right], \right. \\ & \quad \left. \overline{y} \rightarrow \text{LS} \left[0, \frac{\overline{xy}}{24}, \frac{1}{96} \overline{xyy}, -\frac{\overline{xxxy}}{1440} + \frac{1}{480} \overline{xyxy} - \frac{\overline{xyyy}}{2880}, \dots \right] \right\rangle \end{aligned}$$

C:atkv

^{C38}With higher authority:

$$\begin{aligned} & \text{atkv} = \text{logF} \quad // \quad \text{EulerE} \quad // \quad \text{adSeries} \left[\frac{e^{\text{ad}} - 1}{\text{ad}}, \text{logF}, \text{tb} \right]; \\ & \quad \{ \mathbf{f} = \text{atkv}_x, \mathbf{g} = \text{atkv}_y \} \end{aligned}$$

$$\begin{aligned} & \left\{ \text{LS} \left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24} \overline{xyy}, -\frac{1}{180} \overline{xxxy} + \frac{1}{80} \overline{xyxy} + \frac{1}{360} \overline{xyyy}, \dots \right], \right. \\ & \quad \left. \text{LS} \left[0, \frac{\overline{xy}}{12}, \frac{1}{24} \overline{xyy}, -\frac{1}{360} \overline{xxxy} + \frac{1}{120} \overline{xyxy} + \frac{1}{180} \overline{xyyy}, \dots \right] \right\} \end{aligned}$$

computations below

We can then verify that (f, g) indeed satisfy Equations (46) and (47), at least to degree 9:

C:KVTest

```

(oo) (h^-1 (LS[x + y] - BCH[y, x] == f - g - Ad[-x][f] + Ad[y][g]) ^
(oo)   div_x[f] + div_y[g] ==
(oo)   1/2 tr_u[adSeries[ad/e^ad-1, x][u] + adSeries[ad/e^ad-1, y][u] - adSeries[ad/e^ad-1, BCH[x, y]][u]]) @
(oo) {6} // Timing

```

```

SeriesSolve::ArbitrarilySetting: In degree 7 arbitrarily setting {as[x, x, x, x, y] -> 0}.
{13.8281, BS[7 True, ...]}

```

Of course, we could have simply solved Equations (46) and (47) directly:

C:KVDirect

```

(oo) {F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
(oo) SeriesSolve[{F, G},
(oo)   h^-1 (LS[x + y] - BCH[y, x] == F - G - Ad[-x][F] + Ad[y][G]) ^
(oo)   div_x[F] + div_y[G] ==
(oo)   1/2 tr_u[adSeries[ad/e^ad-1, x][u] + adSeries[ad/e^ad-1, y][u] - adSeries[ad/e^ad-1, BCH[x, y]][u]]];
(oo) {F, G}

```

```

SeriesSolve::ArbitrarilySetting: In degree 7 arbitrarily setting {as[x, x, x, x, y] -> 0}.
{13.8281, BS[7 True, ...]}
{LS[
  {
    Y/2, X Y/6, 1/24 X Y Y, -1/180 X X X Y + 1/80 X X Y Y + 1/360 X Y Y Y, ...
  },
  LS[
    0, X Y/12, 1/24 X Y Y, -1/360 X X X Y + 1/120 X X Y Y + 1/180 X Y Y Y, ...
  ]
]}

```

Fuller output:
[\[WK04\]/KVDirect.nb](#)

(To the degree shown, the results are the same. But starting at degree 8 they diverge as the solutions are non-unique.)

subsec:tau

3.3. The involution τ and the Twist Equation. Alekseev and Torossian [AT, Section 8.2] construct an involution τ on the set SolKV of solutions of the Kashiwara-Vergne equations. Phrased using the language of [WKO2], Alekseev and Torossian define a map $\tau: \mathcal{A}^w(\uparrow_2) \rightarrow \mathcal{A}^w(\uparrow_2)$ by $\tau(V) := R(1, 2)V^{21}\Theta^{-1/2}$, where $\Theta^s \equiv e^{st}$ and $t = \uparrow + \uparrow \in \mathcal{A}^w(\uparrow_2)$. They then prove that τ restricts to an involution of the set of solutions Equations (44) and (45). It is not known if τ is different from the identity; in other words, it is not known if every V satisfying (44) and (45) also satisfies the “Twist Equation”

$$V = \tau(V). \tag{48} \text{eq:Twist}$$

In topology, the Twist Equation is essential for the compatibility between Z^u and Z^w ; see [WKO2, Section 4.7]. So it is not known if “every Z^w is compatible with some Z^u ”.

Below the dark line we verify that to degree 6, “our” V satisfies the Twist Equation (48).

Following that, we reproduce the results of Albert, Harinck, and Torossian [AHT], who studied the linearizations

$$[x, A] + [y, B] = 0 \quad \text{and} \quad \text{div}_x A + \text{div}_y B = 0 \quad \text{with } A, B \in FL(x, y) \tag{49} \text{eq:Lineari}$$

of Equations (46) and (47) (which are equivalent to (44) and (45)), and the linearization of Equation (48),

$$A(x, y) = B(y, x). \tag{50} \text{eq:Lineari}$$

We find that up to degree 16, the dimensions of the spaces of solutions of (49) and of (49) \wedge (50) are the same and are given by the following table:

deg A, B	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
dimension	1	0	0	0	0	0	0	1	0	1	1	2	2	3	3	5

(51) tab:dims

————— computations below —————

^{C39}We define $\Theta 1[x, y, s]$ to be e^{st} in the E_l presentation in a straightforward manner, then convert it to the E_s presentation, and then print its value in both the E_l and E_s presentations:

```

⊙⊙ Θ1[x_, y_, s_] := E1[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
⊙♥ Θs[x_, y_, s_] := Θ1[x, y, s] // Γ;
{Θ1[x, y, 1], Θs[x, y, 1]}
    
```

```

🖥️ {E1[⟨x̄ → LS[ȳ, 0, 0, 0, ...], ȳ → LS[x̄, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
Es[⟨x̄ → LS[ȳ,  $\frac{xȳ}{2}$ ,  $\frac{1}{6} \overline{xȳ}$  -  $\frac{1}{12} \overline{xȳȳ}$ ,  $\frac{1}{24} \overline{xȳȳȳ}$  -  $\frac{1}{24} \overline{xȳȳȳȳ}$ , ...],
ȳ → LS[x̄,  $-\frac{xȳ}{2}$ ,  $-\frac{1}{12} \overline{xȳȳ}$  +  $\frac{1}{6} \overline{xȳȳȳ}$ ,  $\frac{1}{24} \overline{xȳȳȳȳ}$  -  $\frac{1}{24} \overline{xȳȳȳȳȳ}$ , ...]⟩, CWS[0, 0, 0, 0, ...]]}
    
```

This done, the computation of $\tau(V_0)$ and the verification that it is equal to V_0 to degree 6 is routine:

```

⊙⊙ τV = Rs[x, y] ** (V0 // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1/2];
(V0 ≡ τV) @ {6}
    
```

```

🖥️ BS[7 True, ...]
    
```

^{C40}We solve for series A and B satisfying (49). These equations are linear, so the printed solution is 0. Yet we store messages produced by `LinearSolve` in a stream called `msgs`. As `LinearSolve` progresses, it outputs messages detailing which coefficients were set in an arbitrary manner in each degree, and the dimension of

Assuming that every solution of the KV equations to degree k can be extended to a solution at all degrees (and similarly for $KV \wedge \text{Twist}$)¹⁸, the above table shows the number of degrees of freedom for the solutions of KV (and/or $KV \wedge \text{Twist}$), in each degree.

¹⁸I am not aware that this was ever proven for KV (and/or $KV \wedge \text{Twist}$), yet a similar result holds for Drinfel'd associators; see [Dr1, Dr2, BN2, BN3].

computations below

the space of solutions in each degree can be read from that information:

C:Lineariz

```

(oo) {A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
(oh) msgs = SeriesSolve[{A, B},
  ħ-1 (b[x, A] + b[y, B] ≡ LS[0]) ∧ (divx[A] + divy[B] ≡ CWS[0])];
{A, B}

```

```

SeriesSolve::ArbitrarilySetting: In degree 1 arbitrarily setting {As[y] → 0}.
{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}

```

Next, we read the stream msgs, just to explore its format:

C:msgs

```

(oo) Read[msgs]

```

```

{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}

```

Next we compute A to degree 12, and read only the dimensions information contained in msgs:

C:dims

```

(oo) A@12; Length[Last[#]] & /@ Read[msgs]

```

```

SeriesSolve::ArbitrarilySetting: In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y] → 0}.
SeriesSolve::ArbitrarilySetting: In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y] → 0}.
SeriesSolve::ArbitrarilySetting: In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.
General::stop: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>
{1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}

```

Fuller output:
[WKO4]/dims.nb

Finally we do the same, but now adding Equation (50):

C:dims1

```

(oo) {A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
(oh) msgs1 = SeriesSolve[{A1, B1},
  ħ-1 (b[x, A1] + b[y, B1] ≡ LS[0]) ∧ (divx[A1] + divy[B1] ≡ CWS[0]) ∧
  (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]

```

```

SeriesSolve::ArbitrarilySetting: In degree 1 arbitrarily setting {A1s[y] → 0}.
SeriesSolve::ArbitrarilySetting: In degree 8 arbitrarily setting {A1s[x, x, x, x, y, x, y] → 0}.
SeriesSolve::ArbitrarilySetting: In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y] → 0}.
General::stop: Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>
{1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}

```

Fuller output:
[WKO4]/dims1.nb

ssociators

3.4. Drinfel'd Associators. It pains me to say so little about Drinfel'd associators, but this is a computational paper and everything we need about associators was already said elsewhere; e.g., in Drinfel'd's original papers [Dr1, Dr2], in my [BN2, BN3], and in earlier papers in this series [WKO2, WKO3]. Hence here I will only recall the few things that are necessary in order to understand the computations below.

Recall that the Drinfel'd-Kohno algebra \mathfrak{t}_n is the completed graded Lie algebra with degree 1 generators $\{t_{ij} \equiv t_{ji} : 1 \leq i \neq j \leq n\}$ and relations $[t_{ij}, t_{kl}] = 0$ when i, j, k, l are distinct ("locality relations") and $[t_{ij} + t_{ik}, t_{jk}] = 0$ when i, j, k are distinct ("4T relations")^{C41}. For any fixed $2 \leq k \leq n$ the $k - 1$ elements $\{t_{ik} : 1 \leq i < k\}$ form a free subalgebra FL_{k-1} of \mathfrak{t}_n , and \mathfrak{t}_n is an iterated semi-direct product of these subalgebras:

$$\mathfrak{t}_n \cong ((\dots (FL_1 \rtimes FL_2) \rtimes \dots) \rtimes FL_{n-2}) \rtimes FL_{n-1}. \tag{52}$$

eq:DKSD

Hence as a vector space, \mathfrak{t}_n has a basis with elements ordered pairs (k, w) , where $2 \leq k \leq n$ and w is a Lyndon word in the letters $\{1, \dots, k-1\}$ (which really stand for $\{t_{1k}, \dots, t_{k-1k}\}$)^{C42}

The collection $\{\mathfrak{t}_n\}$ of all Drinfel'd-Kohno algebras forms an "operad" (e.g. [Fr]). We only need to mention a part of that structure here: that for any n and m , there are many maps $\mathfrak{t}_n \rightarrow \mathfrak{t}_m$. Namely, whenever $\{s_i\}_{i=1}^n$ is a collection of disjoint subsets of $\{1, \dots, m\}$ (some of which may be empty), we have a morphism of Lie algebras $\Psi \mapsto \Psi^{s_1, \dots, s_n}$ mapping \mathfrak{t}_n to \mathfrak{t}_m , and defined by its values on the generators of \mathfrak{t}_n as follows:

$$(t_{ij})^{s_1, \dots, s_n} := \sum_{\alpha \in s_i, \beta \in s_j} t_{\alpha\beta}. \tag{C43}$$

computations below

C:4T

^{C41}We verify these relations, using obvious notation:

$$\circledast \{ \mathbf{b}[t[1, 3], t[4, 2]], \mathbf{b}[t[1, 2] + t[1, 3], t[2, 3]] \}$$

$$\text{DK} \{ 0, 0 \}$$

^{C42}Hence for example, $[t_{13}, t_{12}] = -[t_{13}, t_{23}]$ (the bracket of a generator of FL_3 with the generator of FL_2 is an element of FL_3). In computer speak, this is

C:DKExamp

$$\circledast \mathbf{b}[t[1, 3], t[1, 2]]$$

$$\text{DK} [3, -\overline{12}]$$

Note that the head DK represents "a basis element in a Drinfel'd-Kohno algebra", and that the Lyndon word 12 becomes $[t_{13}, t_{23}]$ when interpreted in $FL_3 \subset \mathfrak{t}_3$.

We could make the last output a bit friendlier by turning it into a "Drinfel'd-Kohno Series" (DKS):

C:DKSExamp

$$\circledast \mathbf{b}[t[1, 3], t[1, 2]] // \text{DKS}$$

$$\text{DKS} [0, -\overline{t_{13} t_{23}}, 0, 0, \dots]$$

^{C43}As an example we repeat a single evaluation of a map $\mathfrak{t}_4 \rightarrow \mathfrak{t}_9$ twice. First using a complete and somewhat cumbersome notation, and then using a shortened notation that works only if all indices are single-digit:

C:sigmaExa

$$\circledast \{ t[2, 3]^{\sigma[\{2,4\}, \{1,5\}, \{3,7,8\}, \{9\}]} // \text{DKS}, t[2, 3]^{\sigma[24, 15, 378, 9]} // \text{DKS} \}$$

Note also that by regarding elements of \mathfrak{t}_n as formal exponentials and using the BCH product each \mathfrak{t}_n also acquires a (non-commutative) group structure.^{C44} By convention, when we think of \mathfrak{t}_n as a group, we refer to it as “ $\exp \mathfrak{t}_n$ ”.

We are finally in position to recall the definition of a Drinfel'd associator. With $R = e^{t_{12}/2} \in \exp \mathfrak{t}_2$, a Drinfel'd associator is an element $\Phi \in \exp \mathfrak{t}_3$ which satisfies the “unitarity condition” (53), the pentagon equation (54), and the hexagon equations (55):

$$\text{Unitarity:} \quad \Phi^{321} = \Phi^{-1}, \tag{53}$$

$$\triangleleft: \quad \Phi \cdot \Phi^{1,23,4} \cdot \Phi^{2,3,4} = \Phi^{12,3,4} \cdot \Phi^{1,2,34}, \tag{54}$$

$$\triangleleft_{\pm}: \quad (R^{\pm 1})^{12,3} = \Phi \cdot (R^{\pm 1})^{2,3} \cdot (\Phi^{-1})^{1,3,2} \cdot (R^{\pm 1})^{1,3} \cdot \Phi^{3,1,2}. \tag{55}$$

eq:PhiUnit

eq:PhiPent

eq:PhiHexa

computations below

$$\left\{ \begin{array}{l} \text{DKS} \left[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}}, 0, 0, 0, \dots \right], \\ \text{DKS} \left[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}}, 0, 0, 0, \dots \right] \end{array} \right\}$$

^{C44}For example, in \mathfrak{t}_3 the elements t_{12} and t_{23} do not commute, and hence the product $e^{t_{12}/2}e^{t_{23}/2}$ is messy. Yet by a 4T relation the elements t_{12} and $(t_{12})^{12,3} = t_{13} + t_{23}$ do commute, and hence the product $e^{t_{12}/2} (e^{t_{12}/2})^{12,3}$ is much simpler:

C: BCH4DK

$$\begin{array}{l} \circ \circ \text{ } \mathbf{R} = \text{DKS} [\mathfrak{t}[1, 2] / 2]; \\ \ominus \text{ } \{ \mathbf{R} \star \star \mathbf{R}^{\sigma[2,3]}, \mathbf{R} \star \star \mathbf{R}^{\sigma[12,3]} \} \end{array}$$

$$\left\{ \begin{array}{l} \text{DKS} \left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{23}}}{2}, -\frac{1}{8} \overline{t_{13} t_{23}}, -\frac{1}{48} \overline{\overline{t_{13} t_{23} t_{23}}} + \frac{1}{96} \overline{\overline{t_{13} t_{13} t_{23}}}, \right. \\ \left. -\frac{1}{384} \overline{\overline{\overline{t_{13} t_{23} t_{23} t_{23}}}} + \frac{1}{384} \overline{\overline{\overline{t_{13} t_{13} t_{23} t_{23}}}}, \dots \right], \text{DKS} \left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{13}}}{2} + \frac{\overline{t_{23}}}{2}, 0, 0, 0, \dots \right] \end{array} \right\}$$

A surprising result by Furusho [Fu] (see also [BND1]) states that in the context of $\exp t_n$ the hexagon equations follow from unitarity and the pentagon, provided Φ is initialized to degree 2 by $\Phi = \exp([t_{13}, t_{23}]/24 + \text{higher terms})$.^{C45}

^{C45} Here's an associator Φ_0 , computed to degree 6. The data file [WKO4]/Phi.nb contains a computation of an associator to degree 10, higher than was previously computed [BN2, Br].

```

(oo)  $\Phi_s[2, 1] = \Phi_s[3, 1] = \Phi_s[3, 2] = 0; \Phi_s[3, 1, 2] = 1/24; \Phi_0 = \text{DKS}[3, \Phi_s];$ 
(heart) SeriesSolve[ $\Phi_0, (\Phi_0^{\sigma[3,2,1]} \equiv -\Phi_0) \wedge (\Phi_0 ** \Phi_0^{\sigma[1,2,3,4]} ** \Phi_0^{\sigma[2,3,4]} \equiv \Phi_0^{\sigma[12,3,4]} ** \Phi_0^{\sigma[1,2,3,4]})$ ];
 $\Phi_0 @ \{6\}$ 

```

```

SeriesSolve::ArbitrarilySetting: In degree 3 arbitrarily setting {Phi[3, 1, 1, 2] -> 0}.
SeriesSolve::ArbitrarilySetting: In degree 5 arbitrarily setting {Phi[3, 1, 1, 1, 2] -> 0}.

```

Fuller output:
[WKO4]/Phi.nb

$$\begin{aligned}
 & \text{DKS} \left[0, \frac{1}{24} t_{13} t_{23}, 0, -\frac{7 t_{13} t_{23} t_{23} t_{23}}{5760} + \frac{7 t_{13} t_{13} t_{23} t_{23}}{5760} - \frac{t_{13} t_{13} t_{13} t_{23}}{1440}, \right. \\
 & 0, \frac{31 t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}{967680} - \frac{157 t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{1935360} - \frac{31 t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{387072} - \\
 & \frac{31 t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483840} + \frac{11 t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{290304} + \frac{31 t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{725760} + \\
 & \left. \frac{83 t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{967680} - \frac{13 t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{241920} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{60480}, \dots \right]
 \end{aligned}$$

To be on the safe side, we verify that Φ_0 satisfies the hexagon equations to degree 6:

```

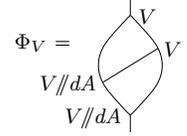
(oo)  $R = \text{DKS}[t[1, 2] / 2];$ 
(heart)  $(R^{\sigma[12,3]} \equiv \Phi_0 ** R^{\sigma[2,3]} ** (-\Phi_0)^{\sigma[1,3,2]} ** R^{\sigma[1,3]} ** \Phi_0^{\sigma[3,1,2]} \wedge$ 
 $(-R)^{\sigma[12,3]} \equiv \Phi_0 ** (-R)^{\sigma[2,3]} ** (-\Phi_0)^{\sigma[1,3,2]} ** (-R)^{\sigma[1,3]} ** \Phi_0^{\sigma[3,1,2]}) @ \{6\}$ 

```

```
BS[7 True, ...]
```


ssociators

3.5. **Associators in \mathcal{A}^w .** We know from [AT, Section 1] that a certain combination of four copies of V makes a solution of the pentagon equation, with values in tder_3 . In the language of [WKO2], this is the statement that V is the Z^w -value of a vertex, that four vertices can make a tetrahedron, and that the Z^w -value Φ_V of a tetrahedron is an associator in \mathcal{A}^w (see the figure on the right). Specifically,



$$\Phi_V = (V//dA)^{12,3}(V//dA)^{1,2}V^{2,3}V^{1,23}, \text{ C46}$$

where we use standard notation: $V^{2,3}$, for example, means “ V with its x strand renamed 2 and its y strand renamed 3” and $V^{1,23}$ means “ V with its x strand renamed 1 and its y strand doubled to become strands 2 and 3”. With the language of Definition 2.2, this is $V^{2,3} = V//d\sigma_2^x//d\sigma_3^y$ and $V^{1,23} = V//d\sigma_1^x//d\Delta_{23}^y$.

Φ_V satisfies the pentagon equation. C47 If our V also satisfies the Twist Equation, then Φ_V also satisfies the hexagon equations (though we do not test that here). Finally, Alekseev and Torossian [AT] prove that if the tree part of Φ_V is written as an exponential $\exp(l\phi)$ of an element ϕ of tder_3 , then in fact $\phi \in \text{sder}_n$, where as in [AT], sder_n is the space of “special derivations in tder_n ”, the derivations which annihilate the sum of all generators on FL_n . C48

computations below

C:PhiV

C46 And here is Φ_V , to degree 4:

$$\begin{aligned} \circledast \mathbf{V}_{12} &= \mathbf{V}_0 // \mathbf{d}\sigma[\{\mathbf{x}, \mathbf{y}\} \rightarrow \{\mathbf{1}, \mathbf{2}\}]; \\ \Phi_V &= (\mathbf{V}_{12} // \mathbf{dA})^{\sigma[12,3]} ** (\mathbf{V}_{12} // \mathbf{dA})^{\sigma[1,2]} ** \mathbf{V}_{12}^{\sigma[2,3]} ** \mathbf{V}_{12}^{\sigma[1,23]} \end{aligned}$$

$$\begin{aligned} \text{Es} \left[\left(1 \rightarrow \text{LS} \left[0, \frac{\overline{23}}{24}, 0, -\frac{\overline{1123}}{1440} + \frac{\overline{71223}}{5760} + \frac{\overline{1233}}{5760} - \frac{\overline{72223}}{5760} + \right. \right. \\ \left. \frac{\overline{72233}}{5760} + \frac{1}{480} \frac{\overline{1213}}{1920} - \frac{\overline{1323}}{1920} + \frac{1}{640} \frac{\overline{123}2}{1152} - \frac{\overline{132}2}{1152} - \frac{\overline{133}2}{1152} - \frac{\overline{233}3}{1440}, \dots \right], \\ 2 \rightarrow \text{LS} \left[0, -\frac{\overline{13}}{24}, 0, \frac{\overline{1113}}{1440} - \frac{\overline{1123}}{1152} + \frac{\overline{71223}}{1920} - \frac{1}{480} \frac{\overline{1132}}{5760} - \frac{\overline{1133}}{5760} + \frac{\overline{1233}}{1152} + \right. \\ \left. \frac{\overline{71213}}{5760} + \frac{19}{5760} \frac{\overline{1323}}{1920} + \frac{\overline{7123}2}{5760} + \frac{\overline{7132}2}{5760} + \frac{\overline{7133}2}{5760} + \frac{\overline{133}3}{1440}, \dots \right], \\ 3 \rightarrow \text{LS} \left[0, \frac{\overline{12}}{24}, 0, -\frac{\overline{1112}}{1440} + \frac{\overline{1123}}{5760} + \frac{\overline{71223}}{5760} + \frac{\overline{71122}}{5760} - \frac{\overline{1132}}{1440} - \frac{\overline{1233}}{1440} + \frac{\overline{1213}}{5760} + \right. \\ \left. \frac{\overline{1323}}{1440} - \frac{\overline{123}2}{1152} - \frac{\overline{7122}2}{5760} - \frac{\overline{7132}2}{5760} - \frac{\overline{133}2}{1440}, \dots \right], \text{CWS}[0, 0, 0, 0, \dots] \end{aligned}$$

C:PentPhiV

C47 Indeed,

$$\circledast \Phi_V ** \Phi_V^{\sigma[1,23,4]} ** \Phi_V^{\sigma[2,3,4]} \equiv \Phi_V^{\sigma[12,3,4]} ** \Phi_V^{\sigma[1,2,34]}$$

$$\text{BS}[5 \text{ True}, \dots]$$

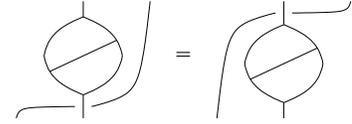
C48 We convert Φ_V to the E_l presentation and take its first (tree) part and call it ϕ , and then we verify that $[x_1, \phi_1] + [x_2, \phi_2] + [x_3, \phi_3] = 0$:

C:Phi_is_s

$$\begin{aligned} \circledast \phi &= (\Phi_V // \Lambda)[[1]]; \\ &(\mathbf{b}[\text{LW@1}, \phi_1] + \mathbf{b}[\text{LW@2}, \phi_2] + \mathbf{b}[\text{LW@3}, \phi_3]) @ \{6\} \end{aligned}$$



The topological meaning of “ $\phi \in \text{sder}_3$ ” is that one may perform a sequence of four *R4* moves to slide a strand underneath a tetrahedron, as shown on the right.



Recall that there is a map $\alpha: \mathfrak{t}_n \rightarrow \mathcal{A}_{\text{prim}}^w(\uparrow_n)$ (equivalently, $\alpha: \mathcal{U}(\mathfrak{t}_n) \rightarrow \mathcal{A}^w(\uparrow_n)$), defined by its values on the generators by sending t_{ij} to a sum of a single arrow from strand i to strand j plus a single arrow from strand j to strand i : $t_{ij} \mapsto i \uparrow j + i \downarrow j$. Using the map α , every Drinfel’d associator becomes an associator in \mathcal{A}^w .^{C49}



In topology, α is the associated graded of the “do nothing” map α which maps ordinary knots to virtual knots. $\uparrow \uparrow \mapsto \uparrow \uparrow + \uparrow \uparrow$ because $\uparrow \uparrow \sim \times \sim \nearrow - \nwarrow \mapsto (\nearrow - \nwarrow) + (\nwarrow - \nearrow) \sim \nwarrow + \nwarrow \sim \uparrow \uparrow + \uparrow \uparrow$. See [WKO1, Section 2.5.5] and [WKO2, Section 3.3].



In Lie theory, the existence of α corresponds to the fact that the invariant metric on $I\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ (represented by an undirected chord) is the sum of the two possible contractions of a space with its dual in $(\mathfrak{g} \times \mathfrak{g}^*) \otimes (\mathfrak{g} \times \mathfrak{g}^*)$ (the two arrows).

[AT] The [AT, Proposition 3.11] version of α is the map $\mathfrak{t}_n \rightarrow \text{sder}_n \subset \text{tder}_n$ taking t_{ij} to $\partial(i \rightarrow x_j, j \rightarrow x_i, (k \neq i, j) \rightarrow 0)$.

computations below



LS[0, 0, 0, 0, 0, 0, ...]

^{C49}Indeed, we define a map DK2Es which takes Drinfel’d-Kohno series to elements of \mathcal{A}^w given in the E_s presentation by applying the built-in αMap , adding 0 wheels, and applying the E_l to E_s conversion Γ . Applying this map to the Drinfel’d associator Φ_0 computed before, we get and associators in \mathcal{A}^w :

C:DK2Es



```
DK2Es[s___][[s_]] := E1[s // alphaMap[s], CWS[0]] // Gamma;
DK2Es[1, 2, 3][Phi_0]
```



$$\begin{aligned} & \text{Es} \left[\left(1 \rightarrow \text{LS} \left[0, \frac{23}{24}, 0, -\frac{1123}{1440} + \frac{71223}{5760} + \frac{1233}{5760} - \frac{72223}{5760} + \right. \right. \right. \\ & \quad \left. \frac{72233}{5760} + \frac{1}{480} \frac{1213}{1213} - \frac{1323}{1920} + \frac{1}{640} \frac{1232}{1232} - \frac{1322}{1152} - \frac{1332}{1152} - \frac{2333}{1440}, \dots \right], \\ & 2 \rightarrow \text{LS} \left[0, -\frac{13}{24}, 0, \frac{1113}{1440} - \frac{1123}{1152} + \frac{71223}{1920} - \frac{1}{480} \frac{1132}{1132} - \frac{1133}{5760} + \frac{1233}{1152} + \right. \\ & \quad \left. \frac{71213}{5760} + \frac{191323}{5760} + \frac{71232}{1920} + \frac{71322}{5760} + \frac{71332}{5760} + \frac{1333}{1440}, \dots \right], \\ & 3 \rightarrow \text{LS} \left[0, \frac{12}{24}, 0, -\frac{1112}{1440} + \frac{1123}{5760} + \frac{71223}{5760} + \frac{71122}{5760} - \frac{1132}{1440} - \frac{1233}{1440} + \frac{1213}{5760} + \right. \\ & \quad \left. \frac{1323}{1440} - \frac{1232}{1152} - \frac{71222}{5760} - \frac{71322}{5760} - \frac{1332}{1440}, \dots \right], \text{CWS}[0, 0, 0, 0, \dots] \end{aligned}$$

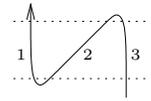
The result matches Φ_V , computed before, to the degree shown. But this is only because both associators are supported in even degrees, and there’s a unique even associator in \mathcal{A}^w up to degree 4. In degree 8 these two associators diverge.

KVandAssoc

3.6. Solving the Kashiwara-Vergne Equations Using a Drinfel'd Associator. Following [WKO3] (in a deeper sense, following [AET]), we know that an element V solving the KV equations (44) and (45) can be computed from a Drinfel'd associator Φ by first computing the invariant $Z_B \equiv Z^u(B)$ of the “buckle” B , shown below both as a knotted trivalent graph and as a product of associators, then puncturing strands 1 and 3 and capping strands 2 and 4 from below, and then regarding the result in $\mathcal{A}^w(\uparrow_2)$ by applying an “Etingof-Kazhdan (EK) isomorphism”:^{C50}

$$B = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \sim \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto Z_B = (\Phi^{-1})^{13,2,4} \Phi^{1,3,2} R^{23} \Phi^{-1} \Phi^{12,3,4} \xrightarrow{\text{puncture, cap, EK}} V.$$

Likewise following [WKO3], we know that $\text{Cap} = \alpha(\nu^{1/4})$, where ν is the Kontsevich integral of the unknot, or the inverse of the associator-combination shown on the right and given by the formula $\alpha(\nu^{-1}) = \Phi // \alpha // dS^2 // dm_2^{32} // dm_1^{21}$.^{C51} (Note



computations below

C:ZB

^{C50}We start with a straightforward computation of Z_B :

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{R} = \text{DKS}[\text{t}[1, 2] / 2]; \\ Z_B = (-\bar{\Phi}_0)^{\sigma[13,2,4]} ** \bar{\Phi}_0^{\sigma[1,3,2]} ** R^{\sigma[2,3]} ** (-\bar{\Phi}_0)^{\sigma[1,2,3]} ** \bar{\Phi}_0^{\sigma[12,3,4]}$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{DKS} \left[\frac{\text{t}_{23}}{2}, -\frac{1}{12} \overline{\text{t}_{13} \text{t}_{23}} - \frac{1}{24} \overline{\text{t}_{14} \text{t}_{24}} + \frac{1}{24} \overline{\text{t}_{14} \text{t}_{34}} + \frac{1}{12} \overline{\text{t}_{24} \text{t}_{34}}, 0, \right. \\ \frac{\overline{\text{t}_{13} \text{t}_{23} \text{t}_{23} \text{t}_{23}}}{5760} + \frac{7 \overline{\text{t}_{14} \text{t}_{24} \text{t}_{24} \text{t}_{24}}}{5760} + \frac{\overline{\text{t}_{14} \text{t}_{34} \text{t}_{24} \text{t}_{24}}}{1920} - \frac{\overline{\text{t}_{14} \text{t}_{34} \text{t}_{34} \text{t}_{24}}}{1920} - \frac{7 \overline{\text{t}_{14} \text{t}_{34} \text{t}_{34} \text{t}_{34}}}{5760} - \\ \frac{\overline{\text{t}_{24} \text{t}_{34} \text{t}_{34} \text{t}_{34}}}{5760} + \frac{\overline{\text{t}_{14} \text{t}_{24} \text{t}_{34} \text{t}_{24}}}{1920} + \frac{\overline{\text{t}_{14} \text{t}_{24} \text{t}_{14} \text{t}_{34}}}{1920} - \frac{\overline{\text{t}_{14} \text{t}_{34} \text{t}_{24} \text{t}_{34}}}{1920} - \frac{1}{720} \overline{\text{t}_{13} \text{t}_{13} \text{t}_{23} \text{t}_{23}} + \\ \frac{1}{720} \overline{\text{t}_{13} \text{t}_{13} \text{t}_{13} \text{t}_{23}} - \frac{7 \overline{\text{t}_{14} \text{t}_{14} \text{t}_{24} \text{t}_{24}}}{5760} + \frac{7 \overline{\text{t}_{14} \text{t}_{14} \text{t}_{34} \text{t}_{34}}}{5760} - \frac{\overline{\text{t}_{14} \text{t}_{24} \text{t}_{34} \text{t}_{34}}}{5760} + \frac{\overline{\text{t}_{14} \text{t}_{14} \text{t}_{14} \text{t}_{24}}}{1440} - \\ \left. \frac{\overline{\text{t}_{14} \text{t}_{14} \text{t}_{14} \text{t}_{34}}}{1440} - \frac{1}{960} \overline{\text{t}_{14} \text{t}_{14} \text{t}_{24} \text{t}_{34}} + \frac{\overline{\text{t}_{14} \text{t}_{24} \text{t}_{24} \text{t}_{34}}}{5760} - \frac{1}{960} \overline{\text{t}_{24} \text{t}_{24} \text{t}_{34} \text{t}_{34}} - \frac{\overline{\text{t}_{24} \text{t}_{24} \text{t}_{24} \text{t}_{34}}}{5760}, \dots \right]$$

In the E_s presentation, “puncture” is $t\eta$. So we puncture strands 1 and 3:

C:VfromPhi

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} Z_B // \text{DK2Es}[1, 2, 3, 4] // t\eta^1 // t\eta^3$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{Es} \left[\left(1 \rightarrow \text{LS} \left[0, -\frac{24}{24}, 0, \frac{7 \overline{2 \ 2 \ 2 \ 4}}{5760} - \frac{\overline{7 \ 2 \ 2 \ 4 \ 4}}{5760} + \frac{\overline{2 \ 4 \ 4 \ 4}}{1440}, \dots \right], \right. \right. \\ \left. \left. 2 \rightarrow \text{LS} [0, 0, 0, 0, \dots], 3 \rightarrow \text{LS} \left[\frac{2}{2}, -\frac{24}{12}, 0, \frac{\overline{2 \ 2 \ 2 \ 4}}{5760} - \frac{1}{720} \overline{2 \ 2 \ 4 \ 4} + \frac{1}{720} \overline{2 \ 4 \ 4 \ 4}, \dots \right], \right. \right. \\ \left. \left. 4 \rightarrow \text{LS} [0, 0, 0, 0, \dots] \right), \text{CWS} [0, 0, 0, 0, \dots] \right]$$

At this point we would normally need to cap and apply EK. But fortunately, strands 2 and 4 carry no arrow heads (as can be seen in the above output), so there is no need to cap them and the EK isomorphisms act by doing nothing. Hence apart from some obvious renaming, the above is already a solution of the KV equations. It matches with the previously-computed V to degree 4 but diverges from it in degree 8 (not shown here). This is consistent with the result in (51), which shows that non-uniqueness starts only in degree 8.

that this computation uses the operation dS^a , which is not easily available in the E_l presentation).

An alternative (yet equivalent) formula for V in terms of Φ follows [AET] more closely. Indeed according to [AET, Theorem 4] and [WKO3] V generates the tangential automorphism of $FL(x, y)$ given explicitly by $(x \mapsto F_x x F_x^{-1}, y \mapsto F_y y F_y^{-1})$, where

$$F = (F_x, F_y) = (\Phi^{-1}(x, -x - y), e^{(x+y)/2} \Phi^{-1}(y, -x - y) e^{-y/2}) \quad (56)$$

(though note that our conventions here agree with the conventions of [WKO3] but slightly differ from the conventions of [AET]).

Below the line we verify Equation (56).

computations below

C:nu

C51 Indeed here is ν^{-1} , followed by a verification that $\nu^{-1} \text{Cap}^4$ is trivial:

```
⊙⊙ vinv = Φ₀ // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

```
Es [⟨x̄ → LS[0, 0, 0, 0, ...]⟩, CWS[0, x̄x̄/24, 0, -xxxx/2880, ...]]
```

C:nucap4

```
⊙⊙ (vinv ** Cap ** Cap ** Cap ** Cap) @ {6}
```

```
Es [⟨x̄ → LS[0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, 0, 0, ...]]
```

C52 We first have to rewrite Φ in terms of $x = t_{12}$ and $y = t_{23}$. To do this we “3” term in Φ , the one involving t_{13} and t_{23} in the factorization (52) (it is the only non trivial term), and apply the appropriate change of variables $t_{13} \rightarrow -x - y$, $t_{23} \rightarrow y$. It is smooth sailing afterwards:

C:Phi1

```
⊙⊙ Φ₁ = (Φ₀)₃ // LieMorphism[LW@1 → -x - y, LW@2 → y]
```

```
LS [0, -x̄ȳ/24, 0, x̄x̄x̄ȳ/1440 - x̄x̄ȳȳ/5760 + x̄ȳȳȳ/1440, ...]
```

C:F

```
⊙⊙ F = ⟨x → LieMorphism[y → -x - y] [-Φ₁],
      y → LS[(x + y) / 2] ~ BCH ~ LieMorphism[x → y, y → -x - y] [-Φ₁] ~ BCH ~ LS[-y / 2]⟩
```

```
Es [⟨x̄ → LS[0, -x̄ȳ/24, 0, 7x̄x̄x̄ȳ/5760 - 7x̄x̄ȳȳ/5760 + x̄ȳȳȳ/1440, ...]⟩,
```

```
    ȳ → LS[x̄/2, -x̄ȳ/12, 0, x̄x̄x̄ȳ/5760 - 1/720 x̄x̄ȳȳ + 1/720 x̄ȳȳȳ, ...]]
```

C:FV

```
⊙⊙ (F ≡ V₀[[1]]) @ {7}
```

```
SeriesSolve::ArbitrarilySetting: In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.
BS [8 True, ...]
```

revolution

3.7. **A Potential S_4 Action on Solutions of KV.** In [BND2], Z. Dancso and I discussed how “the expansion of a tetrahedron” can be interpreted as an associator valued in the appropriate space $\mathcal{A}^u(\Delta) \cong \mathcal{A}^u(\uparrow_3)$ (see also [Th]). The symmetry group of an oriented tetrahedron is the alternating group A_4 , and hence A_4 acts on the set of all associators in $\mathcal{A}^u(\uparrow_3)$ (note that while the action of the permutation group S_3 on $\mathcal{A}^u(\uparrow_3)$ is obvious, its extension to an action of S_4 is non-obvious and is best understood using the isomorphism $\mathcal{A}^u(\Delta) \cong \mathcal{A}^u(\uparrow_3)$). The unitarity equation (53) means that odd permutations map associators to objects whose inverses are associators; with some abuse of language we simply say that “ S_4 acts on the set of associators” (really, it acts on “associators and inverse-associators”). As there are bi-directional relations between associators and solutions of the KV equations, we can expect an action of S_4 on the set of solutions of the KV equations and their inverses.

As mathematicians, Z. Dancso and I only lightly explored this potential action of S_4 ; we wrote down what we think are the formulas inherited from the action on associators, but on the formal level, we’ve verified almost nothing. Yet computer experiments, described below, suggest that our formulas are correct and that they have the properties described below.

The first $\mathbb{Z}/2$ action is the involution τ discussed in Section 3.3. We have nothing further to add.

The second $\mathbb{Z}/2$ action is the involution ρ_2 of \mathcal{A}^u which multiplies every degree d element by $(-1)^d$. Solutions V of the KV equations are not invariant under ρ_2 . Yet if V_0 is the solution computed in this paper then $V_1 := R^{-1/2}V_0$ is invariant under ρ_2 , at least experimentally. Alternatively, V_0 is (experimentally) invariant under $\rho'_2 := R\rho_2$.

A $\mathbb{Z}/3$ action. For $\xi \in \mathcal{A}^w(x, y)$ let $\rho_3(\xi) := \xi // dS^y // d\Delta_{yz}^y // dm_x^{xz} // d\sigma_{yx}^{xy}$, where $d\sigma_{yx}^{xy}$ simply means “swap the labels x and y ”. Then ρ_3 is a trivolution ($(\rho_3)^3 = 1$), and a renormalized version of V_0 , namely $V_2 := V_0 * \Theta^{-1/4} * \exp(\frac{\hat{x}-\hat{y}}{12}) * d\Delta_{xy}^x(Cap^2)$ is, at least experimentally, invariant under the action of ρ_3 .

computations below

C:rho2

C53 Indeed,

```

⊙⊙ ρ₂[V_] := V // (-1)ᵈᵉᵍ;
♥ V₁ = Es[⟨x → LS[0], y → LS[-x/2]⟩, CWS[0]] ** V₀;
  {(V₁ ≡ ρ₂[V₁]) @ {8}, (V₀ ≡ Rs[x, y] ** ρ₂[V₀]) @ {8}}

```

```

SeriesSolve::ArbitrarilySetting: In degree 8 arbitrarily setting {as[x, x, x, x, y, x, y, y] → 0}.
{BS[9 True, ...], BS[9 True, ...]}

```

C:rho3

C54 Indeed for a random ξ_c , $\xi_c // \rho_3 // \rho_3 // \rho_3 = \xi_c$:

```

⊙⊙ ρ₃[ξ_Es] := ξ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
♥ ξ_c = RandomEsSeries[1, {x, y}];
  ξ_c ≡ (ξ_c // ρ₃ // ρ₃ // ρ₃)

```

```

BS[5 True, ...]

```

C:V2

C55 Indeed,

```

⊙⊙ V₂ = V₀ ** Θs[x, y, -1/4] **
♥ Es[⟨x → LS@0, y → LS@0⟩, CWS[cw[x]/12 - cw[y]/12 - (2 Cap[[2]] // tΔ[x, x, y])];
  (V₂ ≡ ρ₃[V₂]) @ {6}

```

```

BS[7 True, ...]

```


4. GLOSSARY OF NOTATION

Icons, then Greek letters, then Latin, and then symbols:




 Links with topology, finite-dimensional Lie theory, and the Alekseev-Torossian paper [AT].




 Human input, multi-line human input, and computer output.



 Source code quotes from the Mathematica packages [FreeLie.m](#) and [AwCalculus.m](#) [WKO4].
 α a map $\mathfrak{t}_n \rightarrow \mathcal{A}_{\text{prim}}^w / \mathcal{A}^u \rightarrow \mathcal{A}^w$ Γ the conversion $TW_l \rightarrow TW_s$ $\Gamma(\lambda)$ $\Gamma_1(\lambda)$ $\Gamma_t(\lambda)$ solution of $e^{-t\partial_\lambda} = C^{\Gamma_t(\lambda)}$ Δ a co-product δ double all strands $\mathcal{A}^w(S) \rightarrow \mathcal{A}^w(S; S)$ η a co-unit Θ $\exp(\uparrow\uparrow + \uparrow\uparrow)$ ι the embedding $CW \rightarrow \mathcal{A}^w$ λ generic element of $FL(S)^S$ Λ the conversion $TW_s \rightarrow TW_l$ $\Lambda(\lambda)$ $\Lambda_1(\lambda)$ $\Lambda_t(\lambda)$ solution of $C^{t\lambda} = e^{-\partial_{\Lambda_t(\lambda)}}$ ν Kontsevich integral of the unknot π_A projection on “framing part” π_T projection on trees π_1 a projection on wheels ρ_2 an involution on \mathcal{A}^w ρ_3 a trivolution on $\mathcal{A}^w(x, y)$ τ an involution on SolKV Φ a Drinfel’d associator Φ_V an associator in \mathcal{A}^w ω generic element of $CW(S)$ $a, \bar{a}, a_i, b, \dots$ generic strand labels a the inclusion usual \leftrightarrow virtual A Abelian lie algebra \mathbf{a} [AT] notation for A ad_y^γ a derivation on $FL(T)$ \overline{AS} the directed AS relation \mathcal{A}^w arrow-diagram spaces $\mathcal{A}_{\text{exp}}^w$ exponentials in \mathcal{A}^w $\mathcal{A}^w(H; T)$ arrow-diagram space on heads-tails skeleton $\mathcal{A}_{\text{exp}}^w(H; T)$ exponentials in $\mathcal{A}^w(H; T)$ B the “buckle” KTG

BCH the Baker-Campbell-Hausdorff series

 BCH_{tb} BCH relative to tb C^λ conjugating generators by exponentials Cap Z^w of a knot-theoretic cap CP the CP relation C_u^γ $C^{(u \rightarrow \gamma)}$ CW cyclic words D a diagram in \mathcal{A}^w $d\Delta$ strand doubling in $\mathcal{A}^w(S)$ $d\Delta_{\text{ma}}$ “strand doubling” in TW_l $d\eta$ strand deletion in $\mathcal{A}^w(S)$ $d\eta_{\text{at}}$ “strand deletion” in TW_l $d\sigma$ strand renaming in $\mathcal{A}^w(S)$ $d\sigma_{\text{at}}$ “strand renaming” in TW_l dA strand adjoint in $\mathcal{A}^w(S)$ dA, dA^S “strand adjoint” in TW_l Elder derivations of FL der [AT] notation for der div $\sum_u \text{div}_u$ div_a a “self-action” map $FL(S) \rightarrow CW(S)$ dm strand stitching in $\mathcal{A}^w(S)$ dms strand antipode in $\mathcal{A}^w(S)$ pidS, dS^S “strand antipode” in TW_l \mathcal{E} the Euler operator \mathcal{E}_f the factored presentation \mathcal{E}_l the lower-interlaced presentation \mathcal{E}_s the split presentation \mathcal{E}_u the upper-interlaced presentation Phe_s a map $FL(T)^H \rightarrow \mathcal{A}_{\text{exp}}^w(H; T)$ Phe the exponential group of \mathfrak{t}_n F solution of the generalized KV equations f, g solution of the original KV equations uvFL free Lie algebra \mathfrak{g} a finite-dimensional Lie algebra frH a set of head labels adh_γ head labels h^{deg} degree-scaling $h\Delta$ head-strand doubling in $\mathcal{A}^w(H; T)$ $h\Delta_{\text{exp}}$ “head-strand doubling” in TW_s $h\eta$ deleting a head-strand in $\mathcal{A}^w(H; T)$ $h\eta_{\text{WHT}}$ “deleting a head-strand” in TW_s $h\sigma$ head-strand renaming in $\mathcal{A}^w(H; T)$

g:B
55
g:BCH
13
g:BCHtb
19
g:C
15
g:Cap
41
g:CP
27
g:Cu
17
g:CW
12
g:D
9
g:dDelta
11
g:ldDelta
23
g:delta
10
g:ldelta
22
g:dsigma
11
g:ldsigma
24
g:dA
10
g:ldA
22
g:der
14
g:atder
14
g:atdiv
18
g:atdivu
18
g:dm
10
g:dS
10
g:ldS
23
g:calE
44
g:Ef
35
g:El
21
g:Es
29
g:Eu
21
g:es
30
g:exptn
50
g:F
43
g:fg
44
g:FL
12
g:fracg
9
g:H
27
g:h
27
g:hdeg
14
g:hDelta
29
g:shDelta
31
g:heta
28
g:sheta
31
g:hsigma
29

$h\sigma$	“head-strand renaming” in TW_s	\mathfrak{g}	$\mathfrak{g} \times \mathfrak{g}^*$
hA	head-strand adjoint in $\mathcal{A}^w(H;T)$	\overrightarrow{IHX}	the directed IHX relation
hA	“head-strand adjoint” in TW_s	j	a “log-Jacobian” $FL \rightarrow CW$
hm	head-strand stitching in $\mathcal{A}^w(H;T)$	J_u	a “partial Jacobian” $FL \rightarrow CW$
hm	“head-strand stitching” in TW_s	l	the lower embedding $FL(S)^S \rightarrow \mathcal{A}^w$
hS	head-strand antipode in $\mathcal{A}^w(H;T)$	lie	[AT] notation for FL
hS	“head-strand antipode” in TW_s	$\mathcal{A}_{\text{prim}}^w$	the primitives in \mathcal{A}^w
$I\mathfrak{g}$	$\mathfrak{g} \times \mathfrak{g}^*$	$\mathcal{A}_{\text{prim}}^w(H;T)$	the primitives in $\mathcal{A}^w(H;T)$
\overrightarrow{IHX}	the directed IHX relation	R	$R(1,2)$
j	a “log-Jacobian” $FL \rightarrow CW$	$R^{\pm 1}(a,b)$	Z^w of a single \pm crossing
J_u	a “partial Jacobian” $FL \rightarrow CW$	R_l^\pm	$R^{\pm 1}$ in TW_l
l	the lower embedding $FL(S)^S \rightarrow \mathcal{A}^w$	R_s^\pm	$R^{\pm 1}$ in TW_s
lie	[AT] notation for FL	$RC^{-\lambda}$	inverse of C^λ
$\mathcal{A}_{\text{prim}}^w$	the primitives in \mathcal{A}^w	RC_u^γ	$RC^{(u \rightarrow \gamma)}$
$\mathcal{A}_{\text{prim}}^w(H;T)$	the primitives in $\mathcal{A}^w(H;T)$	S	a set of strands
R	$R(1,2)$	\mathcal{S}	a symmetric algebra
$R^{\pm 1}(a,b)$	Z^w of a single \pm crossing	sder	“special” derivations
R_l^\pm	$R^{\pm 1}$ in TW_l	\overrightarrow{STU}	a directed STU relation
R_s^\pm	$R^{\pm 1}$ in TW_s	T	a set of tail labels
$RC^{-\lambda}$	inverse of C^λ	t_i	head labels
RC_u^γ	$RC^{(u \rightarrow \gamma)}$	t_{ij}	generators of t_{ij}
S	a set of strands	t_n	the Drinfel’d-Kohno algebra
\mathcal{S}	a symmetric algebra	$t\Delta$	tail-strand doubling in $\mathcal{A}^w(H;T)$
sder	“special” derivations	$t\Delta$	“tail-strand doubling” in TW_s
\overrightarrow{STU}	a directed STU relation	$t\eta$	deleting a tail-strand in $\mathcal{A}^w(H;T)$
T	a set of tail labels	$t\eta$	“deleting a tail-strand” in TW_s
t_i	head labels	$t\sigma$	tail-strand renaming in $\mathcal{A}^w(H;T)$
t_{ij}	generators of t_{ij}	$t\sigma$	“tail-strand renaming” in TW_s
t_n	the Drinfel’d-Kohno algebra	tA	tail-strand adjoint in $\mathcal{A}^w(H;T)$
$t\Delta$	tail-strand doubling in $\mathcal{A}^w(H;T)$	tA	“tail-strand adjoint” in TW_s
$t\Delta$	“tail-strand doubling” in TW_s	TAut	the exponential group of tder
$t\eta$	deleting a tail-strand in $\mathcal{A}^w(H;T)$	tb	tangential bracket
$t\eta$	“deleting a tail-strand” in TW_s	TC	the tails-commute relation
$t\sigma$	tail-strand renaming in $\mathcal{A}^w(H;T)$	tder	tangential derivations
$t\sigma$	“tail-strand renaming” in TW_s	$\text{t}\partial\text{er}$	[AT] notation for tder
tA	tail-strand adjoint in $\mathcal{A}^w(H;T)$	tha	tail-head action in $\mathcal{A}^w(H;T)$
tA	“tail-strand adjoint” in TW_s	tha	“tail-head action” in TW_s
TAut	the exponential group of tder	thm	tail-head stitching in $\mathcal{A}^w(H;T)$
tb	tangential bracket	tm	tail-strand stitching in $\mathcal{A}^w(H;T)$
TC	the tails-commute relation	tm	“tail-strand stitching” in TW_s
tder	tangential derivations	tr_u	a trace map $FL(S) \rightarrow CW(S)$
$\text{t}\partial\text{er}$	[AT] notation for tder		
tha	tail-head action in $\mathcal{A}^w(H;T)$		
tha	“tail-head action” in TW_s		
thm	tail-head stitching in $\mathcal{A}^w(H;T)$		
tm	tail-strand stitching in $\mathcal{A}^w(H;T)$		
tm	“tail-strand stitching” in TW_s		
tr_u	a trace map $FL(S) \rightarrow CW(S)$		

\mathfrak{g}	$\mathfrak{g} \times \mathfrak{g}^*$	[AT] notation for CW
\overrightarrow{IHX}	the directed IHX relation	tail-strand antipode in $\mathcal{A}^w(H;T)$
j	a “log-Jacobian” $FL \rightarrow CW$	“tail-strand antipode” in TW_s
J_u	a “partial Jacobian” $FL \rightarrow CW$	trees and wheels
l	the lower embedding $FL(S)^S \rightarrow \mathcal{A}^w$	domain of E_l
lie	[AT] notation for FL	domain of E_s
$\mathcal{A}_{\text{prim}}^w$	the primitives in \mathcal{A}^w	the upper embedding $FL(S)^S \rightarrow \mathcal{A}^w$
$\mathcal{A}_{\text{prim}}^w(H;T)$	the primitives in $\mathcal{A}^w(H;T)$	unzip operations
R	$R(1,2)$	tail labels
$R^{\pm 1}(a,b)$	Z^w of a single \pm crossing	universal enveloping algebra
R_l^\pm	$R^{\pm 1}$ in TW_l	Z^w of a knot-theoretic vertex
R_s^\pm	$R^{\pm 1}$ in TW_s	head labels
$RC^{-\lambda}$	inverse of C^λ	Z^u of the buckle B
RC_u^γ	$RC^{(u \rightarrow \gamma)}$	the \mathcal{A}^u counterpart of Z^w
S	a set of strands	a (universal) $\mathcal{A}_{\text{exp}}^w$ -valued invariant
\mathcal{S}	a symmetric algebra	postfix operator application,
sder	“special” derivations	“composition done right”
\overrightarrow{STU}	a directed STU relation	a single-arrow diagram
T	a set of tail labels	the stacking product in $\mathcal{A}^w(S)$
t_i	head labels	the “stacking product” in TW_l
t_{ij}	generators of t_{ij}	the stacking product in $\mathcal{A}^w(H;T)$
t_n	the Drinfel’d-Kohno algebra	the “stacking product” in TW_s
$t\Delta$	tail-strand doubling in $\mathcal{A}^w(H;T)$	a product on $\mathcal{A}^w(S)$
$t\Delta$	“tail-strand doubling” in TW_s	the co-product in $\mathcal{A}^w(S)$
$t\eta$	deleting a tail-strand in $\mathcal{A}^w(H;T)$	the co-product in $\mathcal{A}^w(H;T)$
$t\eta$	“deleting a tail-strand” in TW_s	degree-scaling with $h = -1$
$t\sigma$	tail-strand renaming in $\mathcal{A}^w(H;T)$	top-bracket notation
$t\sigma$	“tail-strand renaming” in TW_s	the map $FL(S)^S \rightarrow \text{der}_S$
tA	tail-strand adjoint in $\mathcal{A}^w(H;T)$	set minus, array key removal
tA	“tail-strand adjoint” in TW_s	a disjoint union in $\mathcal{A}^w(S)$
TAut	the exponential group of tder	“disjoint union” in TW_l
tb	tangential bracket	“disjoint union” in TW_s
TC	the tails-commute relation	a union made disjoint
tder	tangential derivations	a disjoint union in $\mathcal{A}^w(H;T)$
$\text{t}\partial\text{er}$	[AT] notation for tder	a skeleton labelled $S = \{1, \dots, n\}$
tha	tail-head action in $\mathcal{A}^w(H;T)$	a cyclic word
tha	“tail-head action” in TW_s	generic element in TW_l
thm	tail-head stitching in $\mathcal{A}^w(H;T)$	generic element in TW_s
tm	tail-strand stitching in $\mathcal{A}^w(H;T)$	element in the domain of E_u
tm	“tail-strand stitching” in TW_s	tangential bracket
tr_u	a trace map $FL(S) \rightarrow CW(S)$	

REFERENCES

- [AHT] L. Albert, P. Harinck, and C. Torossian, *Solution Non Universelle pour le Problème KV-78*, Journal of Lie Theory **18-3** (2008) 617–626, [arXiv:0802.2049](https://arxiv.org/abs/0802.2049). Page(s) 47.
- [AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld’s associators*, Annals of Mathematics **175** (2012) 415–463, [arXiv:0802.4300](https://arxiv.org/abs/0802.4300). Page(s) 3, 4, 6, 7, 9, 10, 11, 13, 14, 15, 18, 19, 25, 39, 41, 44, 47, 53, 54, 59, 60.
- [AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld’s associators, braid groups and an explicit solution of the Kashiwara-Vergne equations*, Publications Mathématiques de L’IHÉS, **112-1** (2010) 143–189, [arXiv:0903.4067](https://arxiv.org/abs/0903.4067). Page(s) 3, 55, 56.
- [BN1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472. Page(s) 9, 11.
- [BN2] D. Bar-Natan, *Non-associative tangles*, in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997. Page(s) 3, 48, 49, 51.
- [BN3] D. Bar-Natan, *On Associators and the Grothendieck-Teichmüller Group I*, Selecta Mathematica, New Series **4** (1998) 183–212. Page(s) 48, 49.
- [BN4] D. Bar-Natan, *The BCH series in terms of Lyndon words*, [mathoverflow question](https://mathoverflow.net/q/116137/8899) and [pensieve entry](https://pennsieve.org/entry/http://mathoverflow.net/q/116137/8899), <http://mathoverflow.net/q/116137/8899> and http://drorbn.net/AcademicPensieve/2012-12/nb/BCH-Lyndon_Question.pdf. Page(s) 14.
- [BN4] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, Acta Mathematica Vietnamica **40-2** (2015) 271–329, [arXiv:1308.1721](https://arxiv.org/abs/1308.1721). Page(s) 1, 3, 4, 6, 9, 12, 13, 15, 16, 17, 18, 27, 28, 29, 31, 32, 33, 34, 39.
- [BND1] D. Bar-Natan and Z. Dancso, *Pentagon and Hexagon Equations Following Furusho*, Proc. of the Amer. Math. Soc. **140-4** (2012) 1243–1250, [arXiv:1010.0754](https://arxiv.org/abs/1010.0754). Page(s) 51.
- [BND2] D. Bar-Natan and Z. Dancso, *Homomorphic expansions for knotted trivalent graphs*, Journal of Knot Theory and its Ramifications **22-1** (2013), [arXiv:1103.1896](https://arxiv.org/abs/1103.1896). Page(s) 57.
- [Br] A. Brochier, *Drinfel’d Associators*, programs and data at <http://abrochier.org/sage.php>. Page(s) 51.
- [CM] F. Casas and A. Murua, *An Efficient Algorithm for Computing the Baker-Campbell-Hausdorff Series and Some of its Applications*, J. of Math. Phys. **50** (2009). Page(s) 14.
- [Dr1] V. G. Drinfel’d, *Quasi-Hopf Algebras*, Leningrad Math. J. **1** (1990) 1419–1457. Page(s) 3, 48, 49.
- [Dr2] V. G. Drinfel’d, *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with $Gal(\mathbb{Q}/\mathbb{Q})$* , Leningrad Math. J. **2** (1991) 829–860. Page(s) 3, 48, 49.
- [EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica, New Series **2** (1996) 1–41, [arXiv:q-alg/9506005](https://arxiv.org/abs/q-alg/9506005). Page(s) 33.
- [Fr] B. Fresse, *Homotopy of Operads and Grothendieck-Teichmüller Groups*, book in progress, <http://math.univ-lille1.fr/~fresse/OperadHomotopyBook/>. Page(s) 49.
- [Fu] H. Furusho, *Pentagon and hexagon equations*, Annals of Mathematics **171-1** (2010) 545–556. Page(s) 51.
- [KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions*, Invent. Math. **47** (1978) 249–272. Page(s) 3, 9, 41, 44.
- [Re] C. Reutenauer, *Free Lie Algebras*, Clarendon Press, Oxford 1993. Page(s) 13.
- [Th] D. Thurston, *The algebra of knotted trivalent graphs and Turaev’s shadow world*, *Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs **4** 337–362, [arXiv:math.GT/0311458](https://arxiv.org/abs/math.GT/0311458). Page(s) 57.
- [WKO1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial*, Alg. and Geom. Top. **16-2** (2016) 1063–1133, [arXiv:1405.1956](https://arxiv.org/abs/1405.1956). Page(s) 1, 3, 4, 6, 7, 9, 54.
- [WKO2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem*, Math. Ann. **367** (2017) 1517–1586, [arXiv:1405.1955](https://arxiv.org/abs/1405.1955). Page(s) 4, 6, 9, 10, 12, 13, 21, 24, 39, 41, 43, 47, 49, 53, 54.

WK03 [WK03] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects III: Double Tree Construction*, <http://drorbn.net/AcademicPensieve/Projects/WK03> (in preparation). Page(s) [1](#), [3](#), [4](#), [6](#), [7](#), [9](#), [43](#), [49](#), [55](#), [56](#).

WK04 [WK04] D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects IV: Some Computations* (self-reference), paper and related files at <http://drorbn.net/AcademicPensieve/Projects/WK04>. The [arXiv:1511.05624](https://arxiv.org/abs/1511.05624) edition may be older. Page(s) [1](#), [3](#), [13](#), [40](#), [43](#), [45](#), [48](#), [51](#), [59](#).

mathematica [Wo] *Wolfram Mathematica 10 Documentation Center*, <http://reference.wolfram.com>. Page(s) [3](#).

Everything below is to be blanked out before the completion of this paper.

RECYCLING

Discussion 4.1. For use in the next section, note that both $\mathcal{A}^w(\uparrow_S)$ and $\mathcal{A}(S; S)$ are associative algebras (the former using the stacking product of Equation (1) and the latter using that of Equation (18)), yet δ is not multiplicative and hence it does not restrict to a Lie morphism on primitives. Instead, on primitives $(\lambda_1; \omega_1), (\lambda_2; \omega_2) \in TW(S)$ we have

$$\delta[l\lambda_1 + i\omega_1, l\lambda_2 + i\omega_2] = [\delta(l\lambda_1 + i\omega_1), \delta(l\lambda_2 + i\omega_2)] + e_s(\partial_{\lambda_1}\lambda_2; \partial_{\lambda_1}\omega_2) - e_s(\partial_{\lambda_2}\lambda_1; \partial_{\lambda_2}\omega_1). \quad (57)$$

Proof of Equation (57).

Proof. E_i and E_f both plant wheels at the top, and as tails commute, they do so in the same manner. So $\omega' = \omega$ and we only need to show Equation (43) at tree level (meaning, modulo wheels). We will show that for every scalar t ,

$$\exp(l(t\lambda)) = \exp_{\#}(e_s(\Gamma(\lambda, t))) // \delta^{-1}; \quad (58)$$

the desired result is the specialization of Equation (58) to $t = 1$. It is clear that Equation (58) holds for some unique $\Gamma_0 = \{a \rightarrow \gamma_{0a}(t)\}$, that $\gamma_{0a}(0) = 0$, and that each coefficient of each $\gamma_{0a}(t)$ depends polynomially on t , and hence it is enough to show that Γ_0 satisfies the differential equation in (6).

MORE.

To Do

- Add some moral explanations near the Lie-theoretic definitions of C , RC , J , j , Γ , and Λ .

Comments by Zsuzsi.

The word “certain” is used too many times in the abstract. I think it would sell better to be more specific there, even at the cost of a longer abstract.

Similarly, I think you could make the first paragraph of the introduction sound less enigmatic.

In the first topology paragraph on page 5, you could say “for S , T , H some sets”, or something along those lines to make it less cryptic?

In general I think the introduction and abstract should cater more to readers not initiated, I think in general if someone doesn’t understand the introduction to a paper they are much less likely to bother reading the rest. (Ps: I do love Figure 1.)

Occasionally I think it might make sense to group the math on one or two full pages and then the corresponding computation on one or two full pages.

FreeLie.m to do.



- * Add “?” help lines and/or tooltip popups.
- * Revise LW and CW I/O; have declarable “generators”.
- * Consider “trees and wheels printing”.

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