Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

FreeLie` implements / extends {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.

Initialization

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150806.

Initialization

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

Initialization

AwCalculus` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150806.

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

$x_1 = LW[1];\ x_2 = LW[2];$

$\{\alpha, \beta, \gamma\} = LS\ /@\ \{x_1 + b[x_1, x_2],\ x_2 - b[x_1, b[x_1, x_2]],\ x_1 + x_2 - 2\, b[x_1, x_2]\}$

alphabetagamma

$\{LS[\overline{1}, \overline{12}, 0, 0, \ldots], LS[\overline{2}, 0, -\overline{1\overline{12}}, 0, \ldots], LS[\overline{1} + \overline{2}, -2\,\overline{12}, 0, 0, \ldots]\}$

BracketExample

$\{b[\alpha, \beta],\ b[\alpha, b[\beta, \gamma]] + b[\beta, b[\gamma, \alpha]] + b[\gamma, b[\alpha, \beta]]\}$

BracketExample

$\{LS[0, \overline{12}, \overline{\overline{12}2}, -\overline{1\overline{1\overline{12}}}, \ldots], LS[0, 0, 0, 0, \ldots]\}$

bch

```
bch = BCH[LW@x, LW@y]
```

bch

$$\mathrm{LS}\left[\overline{x}+\overline{y},\ \frac{\overline{x\,y}}{2},\ \frac{1}{12}\overline{x\,\overline{x\,y}}+\frac{1}{12}\overline{\overline{x\,y}\,y},\ \frac{1}{24}\overline{x\,\overline{\overline{x\,y}\,y}},\ \ldots\right]$$

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]] // TopBracketForm}
```

bch16

$$\left\{38.282645,\ \left\{2181,\ \frac{53\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,y\,x\,y\,y}{1\,089\,728\,640}-\frac{17\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,x\,y\,y\,y}{179\,625\,600}+\frac{389\,x\,x\,x\,x\,y\,x\,y\,x\,x\,y\,x\,x\,y\,x\,x\,y}{1\,320\,883\,200}\right\}\right\}$$

omegas

```
{ω1, ω2} = CWS /@ {CW[1] - 3 CW[2, 1, 1], CW[2] + CW[2, 2]}
```

omegas

$$\left\{\mathrm{CWS}\left[\widehat{1},\ 0,\ -3\,\widehat{112},\ 0,\ \ldots\right],\ \mathrm{CWS}\left[\widehat{2},\ \widehat{22},\ 0,\ 0,\ \ldots\right]\right\}$$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$$\left\{\mathrm{CWS}\left[h\,\widehat{1},\ 0,\ -3\,h^3\,\widehat{112},\ 0,\ \ldots\right],\ \mathrm{CWS}\left[h\,\widehat{2},\ h^2\,\widehat{22},\ 0,\ 0,\ \ldots\right]\right\}$$

TangentialDerivative

```
{λ = ⟨1 → α, 2 → β⟩, γ // Dλ}
```

TangentialDerivative

$$\left\{\left\langle 1\to \mathrm{LS}\left[\overline{1},\ \overline{12},\ 0,\ 0,\ \ldots\right],\ 2\to \mathrm{LS}\left[\overline{2},\ 0,\ -\overline{1\,\overline{12}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{LS}\left[0,\ 0,\ \overline{1\,\overline{12}},\ -\overline{1\,\overline{\overline{12}\,2}},\ \ldots\right]\right\}$$

tb

```
λ1 = λ; λ2 = ⟨1 → β, 2 → γ⟩; tb[λ1, λ2]
```

tb

$$\left\langle 1\to \mathrm{LS}\left[0,\ 0,\ \overline{1\,\overline{12}},\ -\overline{1\,\overline{\overline{12}\,2}},\ \ldots\right],\ 2\to \mathrm{LS}\left[0,\ 0,\ \overline{1\,\overline{12}},\ -\overline{1\,\overline{\overline{12}\,2}},\ \ldots\right]\right\rangle$$

tb2

```
lhs = D_tb[λ1,λ2][ω1]; rhs = b[Dλ1, Dλ2][ω1];
{lhs@{8}, (lhs ≡ rhs)@{8}}
```

tb2

$$\left\{\mathrm{CWS}\left[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 18\,\widehat{11112122}-18\,\widehat{11112212}-36\,\widehat{11121122}+36\,\widehat{11122112},\ \ldots\right],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\right\}$$

TestingGammaODE

$$\mathbf{lhs} = \partial_t \Gamma_t[\lambda];\ \mathbf{rhs} = \lambda\ //\ e^{-t D_\lambda}\ //\ \mathbf{adSeries}\left[\frac{\mathbf{ad}}{e^{\mathbf{ad}}-1},\ \Gamma_t[\lambda]\right];$$

$$\{\Gamma_0[\lambda],\ \mathbf{lhs},\ (\mathbf{lhs} \equiv \mathbf{rhs})@\{6\}\}$$

TestingGammaODE

$$\{\langle 1 \to \mathrm{LS}[0, 0, 0, 0, \ldots], 2 \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle,$$
$$\left\langle 1 \to \mathrm{LS}\left[\overline{1}, \overline{12}, -t\,\overline{1\overline{12}}, \frac{1}{4} t^2\, \overline{1\overline{1\overline{12}}} - t\,\overline{1\overline{\overline{12}2}}, \ldots\right],\right.$$
$$\left. 2 \to \mathrm{LS}\left[\overline{2}, 0, -\overline{1\overline{12}}, -t\,\overline{1\overline{\overline{12}2}}, \ldots\right]\right\rangle, \mathrm{BS}[7\ \mathrm{True}, \ldots]\}$$

TestingGamma

$$\{\gamma\ //\ e^{-t D_\lambda},\ \gamma\ //\ \mathbf{CC}[\Gamma_t[\lambda]]\}$$

TestingGamma

$$\{\mathrm{LS}\left[\overline{1}+\overline{2}, -2\,\overline{12}, -t\,\overline{1\overline{12}}, t\,\overline{1\overline{\overline{12}2}}, \ldots\right], \mathrm{LS}\left[\overline{1}+\overline{2}, -2\,\overline{12}, -t\,\overline{1\overline{12}}, t\,\overline{1\overline{\overline{12}2}}, \ldots\right]\}$$

TestingLambdaODE

$$\mathbf{lhs} = \partial_t \Lambda_t[\lambda];\ \mathbf{rhs} = \lambda\ //\ e^{D_{\Lambda_t[\lambda]}}\ //\ \mathbf{adSeries}\left[\frac{\mathbf{ad}}{e^{\mathbf{ad}}-1},\ \Lambda_t[\lambda],\ \mathbf{tb}\right];$$

$$\{\Lambda_0[\lambda],\ \mathbf{lhs},\ (\mathbf{lhs} \equiv \mathbf{rhs})@\{6\}\}$$

TestingLambdaODE

$$\{\langle 1 \to \mathrm{LS}[0, 0, 0, 0, \ldots], 2 \to \mathrm{LS}[0, 0, 0, 0, \ldots]\rangle,$$
$$\left\langle 1 \to \mathrm{LS}\left[\overline{1}, \overline{12}, t\,\overline{1\overline{12}}, \frac{1}{2} t^2\, \overline{1\overline{1\overline{12}}} + t\,\overline{1\overline{\overline{12}2}}, \ldots\right], 2 \to \mathrm{LS}\left[\overline{2}, 0, -\overline{1\overline{12}}, t\,\overline{1\overline{\overline{12}2}}, \ldots\right]\right\rangle,$$
$$\mathrm{BS}[7\ \mathrm{True}, \ldots]\}$$

TestingLambda

$$\{\gamma\ //\ \mathbf{CC}[t\,\lambda],\ \gamma\ //\ e^{-D_{\Lambda_t[\lambda]}}\}$$

TestingLambda

$$\{\mathrm{LS}\left[\overline{1}+\overline{2}, -2\,\overline{12}, -t\,\overline{1\overline{12}}, -\frac{1}{2} t^2\, \overline{1\overline{1\overline{12}}} + t\,\overline{1\overline{\overline{12}2}}, \ldots\right],$$
$$\mathrm{LS}\left[\overline{1}+\overline{2}, -2\,\overline{12}, -t\,\overline{1\overline{12}}, -\frac{1}{2} t^2\, \overline{1\overline{1\overline{12}}} + t\,\overline{1\overline{\overline{12}2}}, \ldots\right]\}$$

Unclassified aside: an alternative formulation of Λ (on March 1, 2015, this took 61 Seconds):

$$\lambda 2 = \langle 1 \to \mathbf{RandomLieSeries}[\{1, 2\}],\ 2 \to \mathbf{RandomLieSeries}[\{1, 2\}]\rangle;$$

$$\{\mathbf{lhs} = \lambda 2\ //\ \mathbf{EulerE}\ //\ \mathbf{adSeries}\left[\frac{e^{\mathbf{ad}}-1}{\mathbf{ad}},\ \lambda 2\right]\ //\ \mathbf{RC}[-\lambda 2],$$
$$\mathbf{rhs} = \Lambda[\lambda 2]\ //\ \mathbf{EulerE}\ //\ \mathbf{adSeries}\left[\frac{e^{\mathbf{ad}}-1}{\mathbf{ad}},\ \Lambda[\lambda 2],\ \mathbf{tb}\right];\ (\mathbf{lhs} \equiv \mathbf{rhs})@\{8\}\}\ //\ \mathbf{Timing}$$

$$\{54.491149,$$
$$\left\{\left\langle 1 \to \mathrm{LS}\left[-\overline{1} + 2\,\overline{2}, -4\,\overline{12}, \frac{11}{2}\,\overline{1\overline{12}} - \frac{11}{2}\,\overline{\overline{12}2}, -12\,\overline{1\overline{1\overline{12}}} + \frac{121}{6}\,\overline{1\overline{\overline{12}2}} - \frac{47}{6}\,\overline{\overline{\overline{12}2}2}, \ldots\right],\right.\right.$$
$$\left. 2 \to \mathrm{LS}\left[\overline{1} - 2\,\overline{2}, 6\,\overline{12}, -8\,\overline{1\overline{12}} + \frac{33}{2}\,\overline{\overline{12}2}, -\frac{1}{3}\,\overline{1\overline{1\overline{12}}} - \frac{271}{6}\,\overline{1\overline{\overline{12}2}} + \frac{209}{6}\,\overline{\overline{\overline{12}2}2}, \ldots\right]\right\rangle,$$
$$\mathrm{BS}[9\ \mathrm{True}, \ldots]\}\}$$

CCAndRC

{α // CC₁[-γ], α // CC₁[-γ] // RC₁[γ], α // CC₁[-γ] // CC₁[γ]}

CCAndRC

$$\left\{\mathrm{LS}\left[\overline{1},\ 2\,\overline{1\,2},\ -\frac{5}{2}\,\overline{1\,\overline{1\,2}}+\frac{3}{2}\,\overline{\overline{1\,2}\,2},\ \frac{7}{6}\,\overline{1\,\overline{1\,\overline{1\,2}}}-\frac{23}{6}\,\overline{1\,\overline{\overline{1\,2}\,2}}+\frac{2}{3}\,\overline{\overline{\overline{1\,2}\,2}\,2},\ \ldots\right],\right.$$
$$\left.\mathrm{LS}\left[\overline{1},\ \overline{1\,2},\ 0,\ 0,\ \ldots\right],\ \mathrm{LS}\left[\overline{1},\ \overline{1\,2},\ -\overline{1\,\overline{1\,2}},\ 2\,\overline{1\,\overline{1\,\overline{1\,2}}}+\overline{1\,\overline{\overline{1\,2}\,2}},\ \ldots\right]\right\}$$

tru

With[{γ = b[b[LW@v, LW@u], LW@u]}, tr$_u$[γ]] // TopBracketForm

tru

$-\widehat{uv}$

divu

With[{γ = LW@u + b[b[LW@v, LW@u], LW@u]}, div$_u$[γ]] // TopBracketForm

divu

$\widehat{u}-\widehat{uuv}$

Ju

J₁[γ]

Ju

$$\mathrm{CWS}\left[\widehat{1},\ \frac{5\,\widehat{12}}{2},\ -\frac{7\,\widehat{112}}{6}+\frac{7\,\widehat{122}}{6},\ \frac{3\,\widehat{1112}}{8}-\frac{11\,\widehat{1122}}{4}-\frac{3\,\widehat{1212}}{4}+\frac{3\,\widehat{1222}}{8},\ \ldots\right]$$

j

{div[λ]@{5}, j[λ]@{5}}

j

$$\left\{\mathrm{CWS}\left[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ 0,\ 0,\ \ldots\right],\right.$$
$$\left.\mathrm{CWS}\left[\widehat{1}+\widehat{2},\ -\widehat{12},\ -\widehat{112},\ -\widehat{1122}+\widehat{1212},\ -\widehat{11122}+\widehat{11212},\ \ldots\right]\right\}$$

cocycle4j

lhs = j[BCH$_{tb}$[λ1, λ2]]; rhs = j[λ1] + $e^{D_{\lambda1}}$[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}

cocycle4j

$$\left\{\mathrm{CWS}\left[-\widehat{2},\ 0,\ -\frac{7\,\widehat{112}}{3}+\widehat{122},\ \frac{\widehat{1112}}{4}+\frac{7\,\widehat{1122}}{6}-\frac{7\,\widehat{1212}}{3}+\frac{2\,\widehat{1222}}{3},\ \ldots\right],\ \mathrm{BS}[9\,\mathrm{True},\ \ldots]\right\}$$

lhs = j[BCH$_b$[λ1, λ2]]; rhs = j[λ1] + $e^{D_{\lambda1}}$[j[λ2]];
{lhs, (lhs ≡ rhs)}

$$\left\{\mathrm{CWS}\left[-\widehat{2},\ -\frac{3\,\widehat{12}}{2},\ -\frac{37\,\widehat{112}}{12}+\frac{31\,\widehat{122}}{12},\ \frac{\widehat{1112}}{4}-\frac{25\,\widehat{1122}}{24}-\frac{\widehat{1212}}{6}+\frac{\widehat{1222}}{4},\ \ldots\right],\right.$$
$$\mathrm{BS}\left[2\,\mathrm{True},\ -\frac{3\,\widehat{12}}{2}==0,\ -\frac{3\,\widehat{12}}{2}==0\ \&\&\ -\frac{37\,\widehat{112}}{12}+\frac{31\,\widehat{122}}{12}==-\frac{7\,\widehat{112}}{3}+\widehat{122},\right.$$
$$-\frac{3\,\widehat{12}}{2}==0\ \&\&\ -\frac{37\,\widehat{112}}{12}+\frac{31\,\widehat{122}}{12}==-\frac{7\,\widehat{112}}{3}+\widehat{122}\ \&\&$$
$$\left.\left.\frac{\widehat{1112}}{4}-\frac{25\,\widehat{1122}}{24}-\frac{\widehat{1212}}{6}+\frac{\widehat{1222}}{4}==\frac{\widehat{1112}}{4}+\frac{7\,\widehat{1122}}{6}-\frac{7\,\widehat{1212}}{3}+\frac{2\,\widehat{1222}}{3},\ \ldots\right]\right\}$$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

$$\{\text{CWS}[\epsilon\,\widehat{1}+\epsilon\,\widehat{2},\ -\epsilon\,\widehat{12},\ -\epsilon\,\widehat{112},\ 0,\ \ldots],\ \text{BS}[5\ \text{True},\ \ldots]\}$$

Section 2.3 - The [AT]-inspired presentation El of $A^w{}_{exp}$

ElSetup

```
x1 = LW[1]; x2 = LW[2];
{ζa =
   El[⟨1 → LS[x1 + b[x1, x2]], 2 → LS[x2 - b[x1, b[x1, x2]]]⟩, CWS[CW[1] - 3 CW[1, 2, 1]]],
  ζb = El[⟨1 → LS[x2 - b[x1, x2]], 2 → LS[x1 + x2 + b[x2, b[x1, x2]]]⟩,
    CWS[CW[2] - 2 CW[1, 2]]],
  ζc = El[⟨1 → LS[x1 - b[b[x1, x2], b[x1, x2]]], 2 → LS[x2 + 3 b[x1, b[x1, x2]]]⟩,
    CWS[CW[1] - 2 CW[1, 2] + CW[1, 2, 1]]]}
```

ElSetup

$$\{\text{El}[\langle 1\to\text{LS}[\overline{1},\ \overline{1\,2},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{1\,2}},\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ 0,\ -3\,\widehat{112},\ 0,\ \ldots]],$$
$$\text{El}[\langle 1\to\text{LS}[\overline{2},\ -\overline{1\,2},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{1}+\overline{2},\ 0,\ -\overline{\overline{1\,2}\,2},\ 0,\ \ldots]\rangle,$$
$$\text{CWS}[\widehat{2},\ -2\,\widehat{12},\ 0,\ 0,\ \ldots]],\ \text{El}[$$
$$\langle 1\to\text{LS}[\overline{1},\ 0,\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{2},\ 0,\ 3\,\overline{1\,\overline{1\,2}},\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ -2\,\widehat{12},\ \widehat{112},\ 0,\ \ldots]]\}$$

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

$$\{\text{El}[\langle 1\to\text{LS}[2\,\overline{1}+\overline{2},\ 0,\ \tfrac{1}{2}\,\overline{1\,\overline{1\,2}},\ \ldots],\ 2\to\text{LS}[\overline{1}+3\,\overline{2},\ 0,\ \tfrac{5}{2}\,\overline{1\,\overline{1\,2}}-\overline{\overline{1\,2}\,2},\ \ldots]\rangle,$$
$$\text{CWS}[2\,\widehat{1}+\widehat{2},\ -4\,\widehat{12},\ -2\,\widehat{112},\ \ldots]],\ \text{BS}[9\ \text{True},\ \ldots]\}$$

detaExample

```
{ζa // dη^1, ζa // dη^2}
```

detaExample

$$\{\text{El}[\langle 2\to\text{LS}[\overline{2},\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]],$$
$$\text{El}[\langle 1\to\text{LS}[\overline{1},\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[\widehat{1},\ 0,\ 0,\ 0,\ \ldots]]\}$$

dA1

```
{ζd = El[λ, CWS[0]], ζd // dA}
```

dA1

$$\{\text{El}[\langle 1\to\text{LS}[\overline{1},\ \overline{1\,2},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{1\,2}},\ 0,\ \ldots]\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]],$$
$$\text{El}[\langle 1\to\text{LS}[-\overline{1},\ -\overline{1\,2},\ 0,\ 0,\ \ldots],\ 2\to\text{LS}[-\overline{2},\ 0,\ \overline{1\,\overline{1\,2}},\ 0,\ \ldots]\rangle,$$
$$\text{CWS}[-\widehat{1}-\widehat{2},\ \widehat{12},\ \widehat{112},\ \widehat{1122}-\widehat{1212},\ \ldots]]\}$$

dA2

(ζ_d ≡ (ζ_d // dA // dA))@{8}

dA2

BS[9 True, ...]

dA3

lhs = (ζ_a ** ζ_b) // dA; rhs = (ζ_b // dA) ** (ζ_a // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}

dA3

$\{\mathrm{El}[\langle 1 \to \mathrm{LS}[-\overline{1}-\overline{2},\ 0,\ -\frac{1}{2}\overline{1\,\overline{1}\,\overline{2}},\ \ldots],\ 2 \to \mathrm{LS}[-\overline{1}-2\,\overline{2},\ 0,\ \frac{1}{2}\overline{1\,\overline{1}\,\overline{2}}+\overline{\overline{1}\,\overline{2}\,2},\ \ldots]\rangle,$
$\mathrm{CWS}[-\overset{\frown}{2},\ -2\,\overset{\frown}{12},\ -2\,\overset{\frown}{112}-\overset{\frown}{122},\ \ldots]],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$

dS

ζ_d // dS

dS

$\mathrm{El}[\langle 1 \to \mathrm{LS}[\overline{1},\ -\overline{1}\,\overline{2},\ 0,\ 0,\ \ldots],\ 2 \to \mathrm{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{1}\,\overline{2}},\ 0,\ \ldots]\rangle,$
$\mathrm{CWS}[\overset{\frown}{1}+\overset{\frown}{2},\ \overset{\frown}{12},\ -\overset{\frown}{112},\ \overset{\frown}{1122}-\overset{\frown}{1212},\ \ldots]]$

dD1

{ζ_a, ζ_a // dΔ[2, 2, 3]}

dD1

$\{\mathrm{El}[\langle 1 \to \mathrm{LS}[\overline{1},\ \overline{1}\,\overline{2},\ 0,\ 0,\ \ldots],\ 2 \to \mathrm{LS}[\overline{2},\ 0,\ -\overline{1\,\overline{1}\,\overline{2}},\ 0,\ \ldots]\rangle,\ \mathrm{CWS}[\overset{\frown}{1},\ 0,\ -3\,\overset{\frown}{112},\ 0,\ \ldots]],$
$\mathrm{El}[\langle 1 \to \mathrm{LS}[\overline{1},\ \overline{1}\,\overline{2}+\overline{1}\,\overline{3},\ 0,\ 0,\ \ldots],\ 2 \to \mathrm{LS}[\overline{2}+\overline{3},\ 0,\ -\overline{1\,\overline{1}\,\overline{2}}-\overline{1\,\overline{1}\,\overline{3}},\ 0,\ \ldots],$
$3 \to \mathrm{LS}[\overline{2}+\overline{3},\ 0,\ -\overline{1\,\overline{1}\,\overline{2}}-\overline{1\,\overline{1}\,\overline{3}},\ 0,\ \ldots]\rangle,\ \mathrm{CWS}[\overset{\frown}{1},\ 0,\ -3\,\overset{\frown}{112}-3\,\overset{\frown}{113},\ 0,\ \ldots]]\}$

dD2

lhs = (ζ_a ** ζ_b) // dΔ[2, 2, 3]; rhs = (ζ_a // dΔ[2, 2, 3]) ** (ζ_b // dΔ[2, 2, 3]);
{lhs@{3}, (lhs ≡ rhs)@{8}}

dD2

$\{\mathrm{El}[\langle 1 \to \mathrm{LS}[\overline{1}+\overline{2}+\overline{3},\ 0,\ \frac{1}{2}\overline{1\,\overline{1}\,\overline{2}}+\frac{1}{2}\overline{1\,\overline{1}\,\overline{3}},\ \ldots],$
$2 \to \mathrm{LS}[\overline{1}+2\,\overline{2}+2\,\overline{3},\ 0,\ -\frac{1}{2}\overline{1\,\overline{1}\,\overline{2}}-\frac{1}{2}\overline{1\,\overline{1}\,\overline{3}}-\overline{1\,\overline{2}\,\overline{3}}-\overline{\overline{1}\,\overline{2}\,2}-2\,\overline{\overline{1}\,\overline{3}\,2}-\overline{\overline{1}\,\overline{3}\,3},\ \ldots],$
$3 \to \mathrm{LS}[\overline{1}+2\,\overline{2}+2\,\overline{3},\ 0,\ -\frac{1}{2}\overline{1\,\overline{1}\,\overline{2}}-\frac{1}{2}\overline{1\,\overline{1}\,\overline{3}}-\overline{1\,\overline{2}\,\overline{3}}-\overline{\overline{1}\,\overline{2}\,2}-2\,\overline{\overline{1}\,\overline{3}\,2}-\overline{\overline{1}\,\overline{3}\,3},\ \ldots]\rangle,$
$\mathrm{CWS}[\overset{\frown}{1}+\overset{\frown}{2}+\overset{\frown}{3},\ -2\,\overset{\frown}{12}-2\,\overset{\frown}{13},\ -3\,\overset{\frown}{112}-3\,\overset{\frown}{113},\ \ldots]],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$

Section 2.4 - The factored presentation Ef of $A^w{}_{exp}$ and its stronger precursor Es

EsSetup1

```
u = LW@"u"; v = LW@"v";
ξa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
   CWS[CW["u"] - 3 CW["u", "v", "u"]]]
```

EsSetup1

$\mathrm{Es}\left[\left\langle 1\to \mathrm{LS}[\overline{u}, \overline{u\,v}, 0, 0, \ldots], 2\to \mathrm{LS}[\overline{v}, 0, -\overline{u\,\overline{u\,v}}, 0, \ldots], 3\to \mathrm{LS}[\overline{u}, 0, 0, 0, \ldots]\right\rangle, \mathrm{CWS}[\widehat{u}, 0, -3\,\widehat{uuv}, 0, \ldots]\right]$

EsSetup2

```
ξb = RandomEsSeries[0, {1, 2, 3, 4}];
ξb@{2}
```

EsSetup2

$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[-\overline{1}-2\,\overline{2}+2\,\overline{3}-2\,\overline{4}, 2\,\overline{1\,2}+\frac{\overline{1\,3}}{2}+\overline{1\,4}-\frac{\overline{2\,3}}{2}-\frac{\overline{2\,4}}{2}+2\,\overline{3\,4}, \ldots\Big],$

$2\to \mathrm{LS}\Big[2\,\overline{1}-\overline{2}-2\,\overline{3}+\overline{4}, 2\,\overline{1\,2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{1\,4}-\overline{2\,3}-\overline{2\,4}-\frac{\overline{3\,4}}{2}, \ldots\Big],$

$3\to \mathrm{LS}\Big[-\overline{1}+\overline{2}+2\,\overline{4}, -2\,\overline{1\,2}+2\,\overline{1\,3}-\overline{1\,4}-\frac{3\,\overline{2\,3}}{2}+2\,\overline{2\,4}-2\,\overline{3\,4}, \ldots\Big],$

$4\to \mathrm{LS}\Big[-2\,\overline{1}+2\,\overline{2}+2\,\overline{3}+\overline{4}, -\frac{\overline{1\,2}}{2}+\frac{3\,\overline{1\,3}}{2}-2\,\overline{2\,4}+\overline{3\,4}, \ldots\Big]\Big\rangle,$

$\mathrm{CWS}\Big[\widehat{3}-\widehat{4}, \frac{3\,\widehat{11}}{2}+\frac{3\,\widehat{12}}{2}-2\,\widehat{13}+\widehat{14}+\widehat{22}+2\,\widehat{23}-\frac{\widehat{24}}{2}-2\,\widehat{33}-\widehat{34}+\widehat{44}, \ldots\Big]\Big]$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$\Big\{\mathrm{Es}\Big[\Big\langle 3\to \mathrm{LS}\Big[\overline{u}, -\overline{u\,v}, -\overline{u\,\overline{u\,v}}+\frac{1}{2}\,\overline{\overline{u\,v}\,v}, \frac{3}{2}\,\overline{u\,\overline{u\,\overline{u\,v}}}+\overline{u\,\overline{\overline{u\,v}\,v}}-\frac{1}{6}\,\overline{\overline{\overline{u\,v}\,v}\,v}, \ldots\Big],$

$4\to \mathrm{LS}\Big[\overline{u}+\overline{v}, \frac{\overline{u\,v}}{2}, -\frac{23}{12}\,\overline{u\,\overline{u\,v}}-\frac{5}{12}\,\overline{\overline{u\,v}\,v}, \overline{u\,\overline{u\,\overline{u\,v}}}+\frac{13}{24}\,\overline{u\,\overline{\overline{u\,v}\,v}}+\frac{1}{12}\,\overline{\overline{\overline{u\,v}\,v}\,v}, \ldots\Big]\Big\rangle,$

$\mathrm{CWS}\Big[2\,\widehat{u}, -\widehat{uv}, -\frac{3\,\widehat{uuv}}{2}, -\frac{\widehat{uuuv}}{6}+\widehat{uuvv}-\widehat{uvuv}, \ldots\Big]\Big], \mathrm{BS}[9\ \mathrm{True}, \ldots]\Big\}$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$$\{\text{Es}[\langle 1 \to \text{LS}[-2\,\overline{1} + \overline{4},\ -\frac{3\,\overline{14}}{2},\ 20\,\overline{114} - \frac{19}{3}\,\overline{144},\ \ldots],$$
$$4 \to \text{LS}[2\,\overline{1} + \overline{4},\ \overline{14},\ -\frac{31}{2}\,\overline{114} - \frac{13}{6}\,\overline{144},\ \ldots]\rangle,$$
$$\text{CWS}[3\,\widehat{1} - \widehat{4},\ -3\,\widehat{11} + \frac{\widehat{14}}{2} + \widehat{44},\ \frac{71\,\widehat{111}}{4} + \frac{19\,\widehat{114}}{4} - \frac{7\,\widehat{144}}{6} - \frac{2\,\widehat{444}}{3},\ \ldots]],\ \text{BS}[6\ \text{True},\ \ldots]\}$$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$$\{\text{El}[\langle 1 \to \text{LS}[0,\ 0,\ 0,\ \ldots],\ 2 \to \text{LS}[\overline{1},\ 0,\ 0,\ \ldots],\ 3 \to \text{LS}[\overline{1} + \overline{2},\ 0,\ 0,\ \ldots]\rangle,$$
$$\text{CWS}[0,\ 0,\ 0,\ \ldots]],\ \text{BS}[6\ \text{True},\ \ldots]\}$$

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

$$\text{Es}[\langle 1 \to \text{LS}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \ldots]\rangle,\ \text{CWS}[0,\ -\widehat{11},\ 0,\ -\frac{31\,\widehat{1111}}{12},\ 0,\ -\frac{1351\,\widehat{111111}}{360},\ \ldots]]$$

Borromean

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

```
Es[⟨b → LS[0, gr, 1/2 ggr + brg + 1/2 grr,
    -1/2 bbrg + 1/6 gggr + 1/4 ggrr - 1/2 bgbr - 1/2 brgg - 1/2 brrg + 1/6 grrr, ...], g →
   LS[0, -br, 1/2 bbr - bgr - brg + 1/2 brr, -1/6 bbbr - 1/2 bbgr - 1/2 bggr - 1/2 bbrg -
     1/4 bbrr + 1/2 bgrr + 1/2 bgbr + brgr - bgrg - 1/2 brgg + 1/2 brrg - 1/6 brrr, ...],
  r → LS[0, bg, 1/2 bbg + bgr + 1/2 bgg, 1/6 bbbg + 1/2 bbgr +
     1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg, ...]⟩,
 CWS[0, 0, 2 bgr, bbgr - bgbr + bggr - bgrg + bgrr - brgr, ...]]
```

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

CapSetup

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

VCapEqns

```
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0 ;
UnitarityEqn = (V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
```

VCapSolution

```
βs[x] = 1/2; βs[y] = 0;
SeriesSolve[{α, β, γ, κ}, (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V0@{4}, κ@{6}}
```

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

VCapSolution

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

VCapSolution

$$\Big\{\mathrm{Es}\Big[\Big\langle x \to \mathrm{LS}\Big[0, -\frac{\overline{xy}}{24}, 0, \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0, -\frac{\overparen{xy}}{48}, 0, \frac{\overparen{xxxy}}{2880} + \frac{\overparen{xxyy}}{2880} + \frac{\overparen{xyxy}}{5760} + \frac{\overparen{xyyy}}{2880}, \ldots\Big]\Big],$$

$$\mathrm{CWS}\Big[0, -\frac{\overparen{xx}}{96}, 0, \frac{\overparen{xxxx}}{11520}, 0, -\frac{\overparen{xxxxxx}}{725760}, \ldots\Big]\Big\}$$

Sinh

```
Series[1/4 Log[(ℏ/2)/Sinh[ℏ/2]], {ℏ, 0, 12}]
```

Sinh

$$-\frac{\hbar^2}{96} + \frac{\hbar^4}{11520} - \frac{\hbar^6}{725760} + \frac{\hbar^8}{38707200} - \frac{\hbar^{10}}{1916006400} + \frac{691\,\hbar^{12}}{62768369664000} + O[\hbar]^{13}$$

LambdaV

```
Λ[V0]
```

LambdaV

$$\mathrm{El}\Big[\Big\langle x \to \mathrm{LS}\Big[0, -\frac{\overline{xy}}{24}, \frac{1}{96}\,\overline{x\,\overline{xy}}, \frac{\overline{x\,\overline{x\,\overline{xy}}}}{2880} - \frac{1}{480}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, \frac{1}{96}\,\overline{x\,\overline{xy}}, \frac{1}{960}\,\overline{x\,\overline{x\,\overline{xy}}} - \frac{1}{320}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y}, \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0, -\frac{\overparen{xy}}{48}, 0, \frac{\overparen{xxxy}}{2880} + \frac{\overparen{xxyy}}{2880} + \frac{\overparen{xyxy}}{5760} + \frac{\overparen{xyyy}}{2880}, \ldots\Big]\Big]$$

logF

```
logF = Λ[V0][[1]] // dσ[{x, y} → {y, x}]
```

logF

$$\Big\langle x \to \mathrm{LS}\Big[\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, \frac{1}{96}\,\overline{\overline{xy}\,y}, -\frac{1}{720}\,\overline{x\,\overline{x\,\overline{xy}}} + \frac{1}{320}\,\overline{x\,\overline{\overline{xy}\,y}} - \frac{1}{960}\,\overline{\overline{\overline{xy}\,y}\,y}, \ldots\Big],$$

$$y \to \mathrm{LS}\Big[0, \frac{\overline{xy}}{24}, \frac{1}{96}\,\overline{\overline{xy}\,y}, -\frac{\overline{x\,\overline{x\,\overline{xy}}}}{1440} + \frac{1}{480}\,\overline{x\,\overline{\overline{xy}\,y}} - \frac{\overline{\overline{\overline{xy}\,y}\,y}}{2880}, \ldots\Big]\Big\rangle$$

atkv

```
atkv = logF // EulerE // adSeries[(ℯ^ad - 1)/ad, logF, tb];
{f = atkv_x, g = atkv_y}
```

atkv

$$\left\{LS\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\ LS\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\}$$

On March 1, 2015, the following took 379 seconds:

KVTest

```
(ħ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ f - g - Ad[-LW@x][f] + Ad[LW@y][g]) ⋀
    div_x[f] + div_y[g] ≡ 1/2 tr_u[adSeries[ad/(ℯ^ad - 1), LW@x][LW@u] + adSeries[ad/(ℯ^ad - 1), LW@y][
        LW@u] - adSeries[ad/(ℯ^ad - 1), BCH[LW@x, LW@y]][LW@u]]) @{8} // Timing
```

KVTest

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

KVTest

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

KVTest

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

KVTest

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

KVTest

```
{330.441318, BS[9 True, ...]}
```

KVDirect

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs[y] = 1/2;
SeriesSolve[{F, G},
  ħ^-1 (LS[LW@x + LW@y] - BCH[LW@y, LW@x] ≡ F - G - Ad[-LW@x][F] + Ad[LW@y][G]) ⋀
   div_x[F] + div_y[G] ≡ 1/2 tr_u[adSeries[ad/(ℯ^ad - 1), LW@x][LW@u] +
      adSeries[ad/(ℯ^ad - 1), LW@y][LW@u] - adSeries[ad/(ℯ^ad - 1), BCH[LW@x, LW@y]][LW@u]]];
{F,
 G}
```

KVDirect

$$\left\{LS\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\right],\ LS\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\right]\right\}$$

Section 3.3 - The involution τ and the twist equation

Theta

```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

Theta

$$\{El[\langle x\to LS[\overline{y}, 0, 0, 0, \ldots], y\to LS[\overline{x}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],$$
$$Es[\langle x\to LS[\overline{y}, \frac{\overline{xy}}{2}, \frac{1}{6}\overline{x\overline{xy}} - \frac{1}{12}\overline{\overline{xy}y}, \frac{1}{24}x\overline{x\overline{xy}} - \frac{1}{24}x\overline{\overline{xy}y}, \ldots], y\to$$
$$LS[\overline{x}, -\frac{\overline{xy}}{2}, -\frac{1}{12}\overline{x\overline{xy}} + \frac{1}{6}\overline{\overline{xy}y}, \frac{1}{24}x\overline{\overline{xy}y} - \frac{1}{24}\overline{\overline{\overline{xy}y}y}, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]]\}$$

Vtau

```
τV = Rs[x, y] ** (V0 // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1/2];
(V0 ≡ τV)@{6}
```

Vtau

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

Vtau

```
BS[7 True, ...]
```

Linearized

```
{A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
msgs = SeriesSolve[{A, B},
   ħ^-1 (b[LW@x, A] + b[LW@y, B] ≡ LS[0]) ⋀ (divx[A] + divy[B] ≡ CWS[0])];
{A, B}
```

Linearized

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {As[y] → 0}.

Linearized

```
{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}
```

msgs

```
Read[msgs]
```

msgs

```
{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}
```

dims

```
A@12; Length[Last[#]] & /@ Read[msgs]
```

dims

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y, y] → 0}.

dims

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>

dims

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

dims1

```
{A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
msgs1 = SeriesSolve[{A1, B1},
   ħ^-1 (b[LW@x, A1] + b[LW@y, B1] ≡ LS[0]) ⋀
    (div_x[A1] + div_y[B1] ≡ CWS[0]) ⋀ (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]
```

dims1

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {A1s[y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {A1s[x, x, x, x, y, y, x, y, y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y, y] → 0}.

dims1

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims1

```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

Section 3.4 - Drinfel'd Associators

4T

```
{b[t[1, 3], t[4, 2]], b[t[1, 2] + t[1, 3], t[2, 3]]}
```

4T

```
{0, 0}
```

DKExample

```
b[t[1, 3], t[1, 2]]
```

DKExample

```
DK[3, -LW[1, 2]]
```

DKSExample

```
b[t[1, 3], t[1, 2]] // DKS
```

DKSExample

$\mathrm{DKS}\left[0, -\overline{t_{13}\, t_{23}}, 0, 0, \ldots\right]$

sigmaExample

$\left\{t[2, 3]^{\sigma[\{2,4\},\{1,5\},\{3,7,8\},\{9\}]} \mathbin{//} \mathrm{DKS},\ t[2, 3]^{\sigma[24,15,378,9]} \mathbin{//} \mathrm{DKS}\right\}$

sigmaExample

$\left\{\mathrm{DKS}\left[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}}, 0, 0, 0, \ldots\right],\ \mathrm{DKS}\left[\overline{t_{13}} + \overline{t_{17}} + \overline{t_{18}} + \overline{t_{35}} + \overline{t_{57}} + \overline{t_{58}}, 0, 0, 0, \ldots\right]\right\}$

BCH4DK

```
R = DKS[t[1, 2] / 2];
{R ** R^σ[2,3], R ** R^σ[12,3]}
```

BCH4DK

$$\left\{\mathrm{DKS}\left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{23}}}{2}, -\frac{1}{8}\overline{t_{13}\, t_{23}}, -\frac{1}{48}\overline{\overline{t_{13}\, t_{23}}\, t_{23}} + \frac{1}{96}\overline{t_{13}\, \overline{t_{13}\, t_{23}}}, -\frac{1}{384}\overline{\overline{\overline{t_{13}\, t_{23}}\, t_{23}}\, t_{23}} + \frac{1}{384}\overline{t_{13}\, \overline{\overline{t_{13}\, t_{23}}\, t_{23}}}, \ldots\right], \mathrm{DKS}\left[\frac{\overline{t_{12}}}{2} + \frac{\overline{t_{13}}}{2} + \frac{\overline{t_{23}}}{2}, 0, 0, 0, \ldots\right]\right\}$$

Phi

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1/24; Φ0 = DKS[3, Φs];
SeriesSolve[Φ0,
  (Φ0^σ[3,2,1] ≡ -Φ0) ⋀ (Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34])];
Φ0@
  {6}
```

Phi

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

Phi

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

Phi

$$\mathrm{DKS}\Big[0,\ \frac{1}{24}\overline{t_{13}t_{23}},\ 0,\ -\frac{7\,t_{13}t_{23}t_{23}}{5760}+\frac{7\,t_{13}t_{13}t_{23}t_{23}}{5760}-\frac{t_{13}t_{13}t_{13}t_{23}}{1440},$$

$$0,\ \frac{31\,t_{13}t_{23}t_{23}t_{23}t_{23}}{967\,680}-\frac{157\,t_{13}t_{13}t_{23}t_{23}t_{13}t_{23}}{1\,935\,360}-\frac{31\,t_{13}t_{23}t_{13}t_{23}t_{23}t_{23}}{387\,072}-$$

$$\frac{31\,t_{13}t_{13}t_{23}t_{23}t_{23}t_{23}}{483\,840}+\frac{11\,t_{13}t_{13}t_{13}t_{23}t_{13}t_{23}}{290\,304}+\frac{31\,t_{13}t_{13}t_{23}t_{13}t_{23}t_{23}}{725\,760}+$$

$$\frac{83\,t_{13}t_{13}t_{13}t_{23}t_{23}t_{23}}{967\,680}-\frac{13\,t_{13}t_{13}t_{13}t_{13}t_{23}t_{23}}{241\,920}+\frac{t_{13}t_{13}t_{13}t_{13}t_{13}t_{23}}{60\,480},\ \ldots\Big]$$

Hexagons

```
R = DKS[t[1, 2]/2];
(R^σ[12,3] ≡ Φ0 ** R^σ[2,3] ** (-Φ0)^σ[1,3,2] ** R^σ[1,3] ** Φ0^σ[3,1,2] ⋀
   (-R)^σ[12,3] ≡ Φ0 ** (-R)^σ[2,3] ** (-Φ0)^σ[1,3,2] ** (-R)^σ[1,3] ** Φ0^σ[3,1,2])@{6}
```

Hexagons

```
BS[7 True, ...]
```

Section 3.5 - Associators in $\mathcal{A}^w$

PhiV

```
V12 = V0 // dσ[{x, y} → {1, 2}];
ΦV = (V12 // dA)^σ[12,3] ** (V12 // dA)^σ[1,2] ** V12^σ[2,3] ** V12^σ[1,23]
```

PhiV

Es[

$$\Big\langle 1 \to \mathrm{LS}\Big[0, \frac{\overline{23}}{24}, 0, -\frac{\overline{1\overline{1\overline{23}}}}{1440}+\frac{7\,\overline{1\overline{2\overline{23}}}}{5760}+\frac{\overline{1\overline{\overline{23}3}}}{5760}-\frac{7\,\overline{2\overline{2\overline{23}}}}{5760}+\frac{7\,\overline{2\overline{\overline{23}3}}}{5760}+\frac{1}{480}\overline{\overline{12}\,\overline{13}}-\frac{\overline{\overline{13}\,\overline{23}}}{1920}+\frac{1}{640}\overline{\overline{1\overline{23}}2}-\frac{\overline{\overline{\overline{13}2}2}}{1152}-\frac{\overline{\overline{\overline{13}3}2}}{1152}-\frac{\overline{\overline{\overline{23}3}3}}{1440}, \ldots\Big],$$

$$2 \to \mathrm{LS}\Big[0, -\frac{\overline{13}}{24}, 0, \frac{\overline{1\overline{1\overline{13}}}}{1440}-\frac{\overline{1\overline{1\overline{23}}}}{1152}+\frac{7\,\overline{1\overline{2\overline{23}}}}{1920}-\frac{1}{480}\overline{1\overline{\overline{13}2}}-\frac{\overline{1\overline{\overline{13}3}}}{5760}+\frac{\overline{1\overline{\overline{23}3}}}{1152}+\frac{7\,\overline{\overline{12}\,\overline{13}}}{5760}+\frac{19\,\overline{\overline{13}\,\overline{23}}}{5760}+\frac{7\,\overline{\overline{1\overline{23}}2}}{1920}+\frac{7\,\overline{\overline{\overline{13}2}2}}{5760}+\frac{7\,\overline{\overline{\overline{13}3}2}}{5760}+\frac{\overline{\overline{\overline{13}3}3}}{1440}, \ldots\Big],$$

$$3 \to \mathrm{LS}\Big[0, \frac{\overline{12}}{24}, 0, -\frac{\overline{1\overline{1\overline{12}}}}{1440}+\frac{\overline{1\overline{1\overline{23}}}}{5760}+\frac{7\,\overline{1\overline{2\overline{23}}}}{5760}+\frac{7\,\overline{1\overline{\overline{12}2}}}{5760}-\frac{\overline{1\overline{\overline{13}2}}}{1440}-\frac{\overline{1\overline{\overline{23}3}}}{1440}+\frac{\overline{\overline{12}\,\overline{13}}}{5760}+\frac{\overline{\overline{13}\,\overline{23}}}{1440}-\frac{\overline{\overline{1\overline{23}}2}}{1152}-\frac{7\,\overline{\overline{\overline{12}2}2}}{5760}-\frac{7\,\overline{\overline{\overline{13}2}2}}{5760}-\frac{\overline{\overline{\overline{13}3}2}}{1440}, \ldots\Big]\Big\rangle, \mathrm{CWS}[0, 0, 0, 0, \ldots]\Big]$$

PentPhiV

```
ΦV ** ΦV^σ[1,23,4] ** ΦV^σ[2,3,4] ≡ ΦV^σ[12,3,4] ** ΦV^σ[1,2,34]
```

PentPhiV

```
BS[5 True, ...]
```

Phi_is_sder

```
ϕ = (ΦV // Λ)[[1]];
(b[LW[1], ϕ1] + b[LW[2], ϕ2] + b[LW[3], ϕ3]) @ {6}
```

Phi_is_sder

```
LS[0, 0, 0, 0, 0, 0, ...]
```

DK2Es

```
DK2Es[s___][ζ_] := El[ζ // αMap[s], CWS[0]] // Γ;
DK2Es[1, 2, 3][Φ0]
```

DK2Es

Es[

$$\Big\langle 1 \to \mathrm{LS}\Big[0,\ \frac{\overline{2\,3}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1440}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{5760}-\frac{7\,\overline{2\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{2\,\overline{\overline{2\,3}\,3}}}{5760}+\frac{1}{480}\,\overline{\overline{1\,2}\,\overline{1\,3}}-\frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1920}+$$
$$\frac{1}{640}\,\overline{\overline{1\,\overline{2\,3}}\,2}-\frac{\overline{\overline{\overline{1\,3}\,2}\,2}}{1152}-\frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1152}-\frac{\overline{\overline{\overline{2\,3}\,3}\,3}}{1440},\ \ldots\Big],$$

$$2 \to \mathrm{LS}\Big[0,\ -\frac{\overline{1\,3}}{24},\ 0,\ \frac{\overline{1\,\overline{1\,\overline{1\,3}}}}{1440}-\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1152}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{1920}-\frac{1}{480}\,\overline{1\,\overline{\overline{1\,3}\,2}}-\frac{\overline{1\,\overline{\overline{1\,3}\,3}}}{5760}+\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1152}+$$
$$\frac{7\,\overline{\overline{1\,2}\,\overline{1\,3}}}{5760}+\frac{19\,\overline{\overline{1\,3}\,\overline{2\,3}}}{5760}+\frac{7\,\overline{\overline{1\,\overline{2\,3}}\,2}}{1920}+\frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760}+\frac{7\,\overline{\overline{\overline{1\,3}\,3}\,2}}{5760}+\frac{\overline{\overline{\overline{1\,3}\,3}\,3}}{1440},\ \ldots\Big],$$

$$3 \to \mathrm{LS}\Big[0,\ \frac{\overline{1\,2}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{1440}+\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{5760}-\frac{\overline{1\,\overline{\overline{1\,3}\,2}}}{1440}-\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1440}+$$
$$\frac{\overline{\overline{1\,2}\,\overline{1\,3}}}{5760}+\frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1440}-\frac{\overline{\overline{1\,\overline{2\,3}}\,2}}{1152}-\frac{7\,\overline{\overline{\overline{1\,2}\,2}\,2}}{5760}-\frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760}-\frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1440},\ \ldots\Big]\Big\rangle,\ \mathrm{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

The computation below takes a a couple of hours and yields “BS[8 True,False,...]”:

```
TrueQ[DK2Es[1, 2, 3][Φ0] ≡ ΦV]@{8}
```

BS[8 True, False, ...]

Section 3.6 - Solving the Kashiwara-Vergne Equations Using a Drinfel'd Associator

ZB

```
R = DKS[t[1, 2] / 2];
Z_B = (-Φ0)^σ[13,2,4] ** Φ0^σ[1,3,2] ** R^σ[2,3] ** (-Φ0)^σ[1,2,3] ** Φ0^σ[12,3,4]
```

ZB

$$\mathrm{DKS}\Big[\frac{t_{23}}{2},\ -\frac{1}{12}\,t_{13}t_{23}-\frac{1}{24}\,t_{14}t_{24}+\frac{1}{24}\,t_{14}t_{34}+\frac{1}{12}\,t_{24}t_{34},\ 0,$$
$$\frac{t_{13}t_{23}t_{23}t_{23}}{5760}+\frac{7\,t_{14}t_{24}t_{24}t_{24}}{5760}+\frac{t_{14}t_{34}t_{24}t_{24}}{1920}-\frac{t_{14}t_{34}t_{34}t_{24}}{1920}-\frac{7\,t_{14}t_{34}t_{34}t_{34}}{5760}-$$
$$\frac{t_{24}t_{34}t_{34}t_{34}}{5760}+\frac{t_{14}t_{24}t_{34}t_{24}}{1920}+\frac{t_{14}t_{24}t_{14}t_{34}}{1920}-\frac{t_{14}t_{34}t_{24}t_{34}}{1920}-\frac{1}{720}\,t_{13}t_{13}t_{23}t_{23}+$$
$$\frac{1}{720}\,t_{13}t_{13}t_{13}t_{23}-\frac{7\,t_{14}t_{14}t_{24}t_{24}}{5760}+\frac{7\,t_{14}t_{14}t_{34}t_{34}}{5760}-\frac{t_{14}t_{24}t_{34}t_{34}}{5760}+\frac{t_{14}t_{14}t_{14}t_{24}}{1440}-$$
$$\frac{t_{14}t_{14}t_{14}t_{34}}{1440}-\frac{1}{960}\,t_{14}t_{14}t_{24}t_{34}+\frac{t_{14}t_{24}t_{24}t_{34}}{5760}-\frac{1}{960}\,t_{24}t_{24}t_{34}t_{34}-\frac{t_{24}t_{24}t_{24}t_{34}}{5760},\ \ldots\Big]$$

VfromPhi

```
Z_B // DK2Es[1, 2, 3, 4] // tη^1 // tη^3
```

VfromPhi

$$\mathrm{Es}\Big[\Big\langle 1\to \mathrm{LS}\Big[0,\ -\frac{24}{24},\ 0,\ \frac{72224}{5760}-\frac{72244}{5760}+\frac{2444}{1440},\ \ldots\Big],$$
$$2\to \mathrm{LS}[0,0,0,0,\ldots],\ 3\to \mathrm{LS}\Big[\frac{2}{2},\ -\frac{24}{12},\ 0,\ \frac{2224}{5760}-\frac{1}{720}\,22 44+\frac{1}{720}\,2444,\ \ldots\Big],$$
$$4\to \mathrm{LS}[0,0,0,0,\ldots]\Big\rangle,\ \mathrm{CWS}[0,0,0,0,\ldots]\Big]$$

The computation below takes a few hours and yields "BS[8 True,False,...]":

```
V_B = Z_B // DK2Es[1, 2, 3, 4] // tη^1 // tη^3 // hη^2 // hη^4 // hσ[{1, 3} → {x, y}] //
  tσ[{2, 4} → {x, y}]; TrueQ[V_B[[1]] ≡ V0[[1]]] @ {8}
```

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

BS[8 True, False, ...]

nu

```
νinv = Φ0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

nu

$$\mathrm{Es}\Big[\langle x\to \mathrm{LS}[0,0,0,0,\ldots]\rangle,\ \mathrm{CWS}\Big[0,\ \frac{xx}{24},\ 0,\ -\frac{xxxx}{2880},\ \ldots\Big]\Big]$$

nucap4

```
(vinv ** Cap ** Cap ** Cap ** Cap) @ {6}
```

nucap4

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

nucap4

```
Es[⟨x → LS[0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, 0, 0, ...]]
```

Section 3.7 - A Potential S_4 Action on Solutions of KV

rho2

```
ρ2[V_] := V // (-1)^deg;
V1 = Es[⟨x → LS[0], y → LS[-LW@x / 2]⟩, CWS[0]] ** V0;
{(V1 ≡ ρ2[V1]) @ {8}, (V0 ≡ Rs[x, y] ** ρ2[V0]) @ {8}}
```

rho2

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

rho2

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

rho2

```
{BS[9 True, ...], BS[9 True, ...]}
```

rho3

```
ρ3[ℰ_Es] := ℰ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
ξc = RandomEsSeries[1, {x, y}];
ξc ≡ (ξc // ρ3 // ρ3 // ρ3)
```

rho3

```
BS[5 True, ...]
```

V2

```
V2 = V0 ** Θs[x, y, -1 / 4] **
    Es[⟨x → LS@0, y → LS@0⟩, CWS[CW[x] / 12 - CW[y] / 12] - (2 Cap⟦2⟧ // tΔ[x, x, y])];
(V2 ≡ ρ3[V2]) @ {6}
```

V2

```
BS[7 True, ...]
```