```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⁀, ⁀}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,
  hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

```
lhs = Rs[1, 2] ** Rs[1, 3] ** Rs[2, 3]; rhs = Rs[2, 3] ** Rs[1, 3] ** Rs[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

$$\left\{\text{Es}\left[\left\langle 1 \to \text{LS}[0, 0, 0, \ldots], 2 \to \text{LS}\left[\overline{1}, 0, 0, \ldots\right], 3 \to \text{LS}\left[\overline{1}+\overline{2}, \frac{\overline{12}}{2}, \frac{1}{12}\,\overline{1\,\overline{12}}+\frac{1}{12}\,\overline{\overline{12}\,2}, \ldots\right]\right\rangle, \text{CWS}[0, 0, 0, \ldots]\right], \text{BS}[6\,\text{True}, \ldots]\right\}$$

```
x = LW["x"]; y = LW["y"]; z = LW["z"];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V₀ = Es[⟨x → α, y → β⟩, γ];
```

```
V₀
```

$$\text{Es}\left[\left\langle \overline{x} \to \text{LS}\left[\overline{x}\,\alpha\text{s}[x]+\overline{y}\,\alpha\text{s}[y], \overline{xy}\,\alpha\text{s}[x, y], \overline{x\,\overline{xy}}\,\alpha\text{s}[x, x, y]+\overline{\overline{xy}\,y}\,\alpha\text{s}[x, y, y], \overline{x\,\overline{x\,\overline{xy}}}\,\alpha\text{s}[x, x, x, y]+\overline{x\,\overline{\overline{xy}\,y}}\,\alpha\text{s}[x, x, y, y]+\overline{\overline{\overline{xy}\,y}\,y}\,\alpha\text{s}[x, y, y, y], \ldots\right], \overline{y} \to \text{LS}\left[\overline{x}\,\beta\text{s}[x]+\overline{y}\,\beta\text{s}[y], \overline{xy}\,\beta\text{s}[x, y], \overline{x\,\overline{xy}}\,\beta\text{s}[x, x, y]+\overline{\overline{xy}\,y}\,\beta\text{s}[x, y, y], \overline{x\,\overline{x\,\overline{xy}}}\,\beta\text{s}[x, x, x, y]+\overline{x\,\overline{\overline{xy}\,y}}\,\beta\text{s}[x, x, y, y]+\overline{\overline{\overline{xy}\,y}\,y}\,\beta\text{s}[x, y, y, y], \ldots\right]\right\rangle, \text{CWS}\left[\widehat{x}\,\gamma\text{s}[x]+\widehat{y}\,\gamma\text{s}[y], \widehat{xx}\,\gamma\text{s}[x, x]+\widehat{xy}\,\gamma\text{s}[x, y]+\widehat{yy}\,\gamma\text{s}[y, y], \widehat{xxx}\,\gamma\text{s}[x, x, x]+\widehat{xxy}\,\gamma\text{s}[x, x, y]+\widehat{xyy}\,\gamma\text{s}[x, y, y]+\widehat{yyy}\,\gamma\text{s}[y, y, y], \widehat{xxxx}\,\gamma\text{s}[x, x, x, x]+\widehat{xxxy}\,\gamma\text{s}[x, x, x, y]+\widehat{xxyy}\,\gamma\text{s}[x, x, y, y]+\widehat{xyxy}\,\gamma\text{s}[x, y, x, y]+\widehat{xyyy}\,\gamma\text{s}[x, y, y, y]+\widehat{yyyy}\,\gamma\text{s}[y, y, y, y], \ldots\right]\right]$$

```
(Rs[x, z] // dΔ[x, x, y])
```

$$\text{Es}\left[\left\langle \overline{x} \to \text{LS}[0,0,0,0,\ldots],\ \overline{y} \to \text{LS}[0,0,0,0,\ldots],\ \overline{z} \to \text{LS}\left[\overline{x}+\overline{y},0,0,0,\ldots\right]\right\rangle,\ \text{CWS}[0,0,0,0,\ldots]\right]$$

```
V₀ ** (Rs[x, z] // dΔ[x, x, y])
```

$$\text{Es}\left[\left\langle \overline{x} \to \text{LS}\left[\overline{x}\,\alpha\text{s}[x]+\overline{y}\,\alpha\text{s}[y],\ \overline{xy}\,\alpha\text{s}[x,y],\ \overline{x\overline{xy}}\,\alpha\text{s}[x,x,y]+\overline{\overline{xy}y}\,\alpha\text{s}[x,y,y],\right.\right.$$
$$\left.\overline{x\overline{x\overline{xy}}}\,\alpha\text{s}[x,x,x,y]+\overline{x\overline{\overline{xy}y}}\,\alpha\text{s}[x,x,y,y]+\overline{\overline{\overline{xy}y}y}\,\alpha\text{s}[x,y,y,y],\ldots\right],$$
$$\overline{y} \to \text{LS}\left[\overline{x}\,\beta\text{s}[x]+\overline{y}\,\beta\text{s}[y],\ \overline{xy}\,\beta\text{s}[x,y],\ \overline{x\overline{xy}}\,\beta\text{s}[x,x,y]+\overline{\overline{xy}y}\,\beta\text{s}[x,y,y],\right.$$
$$\left.\overline{x\overline{x\overline{xy}}}\,\beta\text{s}[x,x,x,y]+\overline{x\overline{\overline{xy}y}}\,\beta\text{s}[x,x,y,y]+\overline{\overline{\overline{xy}y}y}\,\beta\text{s}[x,y,y,y],\ldots\right],$$
$$\left.\overline{z} \to \text{LS}\left[\overline{x}+\overline{y},0,0,0,\ldots\right]\right\rangle,\ \text{CWS}\left[\widehat{x}\,\gamma\text{s}[x]+\widehat{y}\,\gamma\text{s}[y],\ \widehat{xx}\,\gamma\text{s}[x,x]+\widehat{xy}\,\gamma\text{s}[x,y]+\widehat{yy}\,\gamma\text{s}[y,y],\right.$$
$$\widehat{xxx}\,\gamma\text{s}[x,x,x]+\widehat{xxy}\,\gamma\text{s}[x,x,y]+\widehat{xyy}\,\gamma\text{s}[x,y,y]+\widehat{yyy}\,\gamma\text{s}[y,y,y],$$
$$\widehat{xxxx}\,\gamma\text{s}[x,x,x,x]+\widehat{xxxy}\,\gamma\text{s}[x,x,x,y]+\widehat{xxyy}\,\gamma\text{s}[x,x,y,y]+$$
$$\left.\widehat{xyxy}\,\gamma\text{s}[x,y,x,y]+\widehat{xyyy}\,\gamma\text{s}[x,y,y,y]+\widehat{yyyy}\,\gamma\text{s}[y,y,y,y],\ldots\right]\Big]$$

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];

R4Eqn = V₀ ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V₀;
UnitarityEqn = V₀ ** (V₀ // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn =
  (V₀ ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡ (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

### R4Eqn

$$\mathrm{BS}\Big[2\,\mathrm{True},\ 0 == -\frac{\overline{xy}}{2} - \overline{xy}\,\alpha s[y] + \overline{xy}\,\beta s[x],$$

$$0 == -\frac{\overline{xy}}{2} - \overline{xy}\,\alpha s[y] + \overline{xy}\,\beta s[x]\ \&\&\ 0 == \frac{1}{12}\overline{x\overline{xy}} + \frac{1}{12}\overline{\overline{xy}y} + \frac{1}{2}\overline{\overline{xy}y}\,\alpha s[y] -$$
$$\frac{1}{2}\overline{x\overline{xy}}\,\alpha s[x]\,\alpha s[y] + \frac{1}{2}\overline{\overline{xy}y}\,\alpha s[y]^2 - \overline{x\overline{xy}}\,\alpha s[x,y] - \frac{1}{2}\overline{x\overline{xy}}\,\beta s[x] + \frac{1}{2}\overline{x\overline{xy}}\,\beta s[x]^2 -$$
$$\frac{1}{2}\overline{\overline{xy}y}\,\beta s[x]\,\beta s[y] + \overline{\overline{xy}y}\,\beta s[x,y],\ 0 == -\frac{\overline{xy}}{2} - \overline{xy}\,\alpha s[y] + \overline{xy}\,\beta s[x]\ \&\&$$
$$0 == \frac{1}{12}\overline{x\overline{xy}} + \frac{1}{12}\overline{\overline{xy}y} + \frac{1}{2}\overline{\overline{xy}y}\,\alpha s[y] - \frac{1}{2}\overline{x\overline{xy}}\,\alpha s[x]\,\alpha s[y] + \frac{1}{2}\overline{\overline{xy}y}\,\alpha s[y]^2 -$$
$$\overline{x\overline{xy}}\,\alpha s[x,y] - \frac{1}{2}\overline{x\overline{xy}}\,\beta s[x] + \frac{1}{2}\overline{x\overline{xy}}\,\beta s[x]^2 - \frac{1}{2}\overline{\overline{xy}y}\,\beta s[x]\,\beta s[y] + \overline{\overline{xy}y}\,\beta s[x,y]\ \&\&$$
$$0 == -\frac{1}{24}\overline{x\overline{\overline{xy}y}} - \frac{1}{12}\overline{x\overline{\overline{xy}y}}\,\alpha s[y] - \frac{1}{12}\overline{\overline{\overline{xy}y}y}\,\alpha s[y] + \frac{1}{4}\overline{x\overline{\overline{xy}y}}\,\alpha s[x]\,\alpha s[y] -$$
$$\frac{1}{6}\overline{x\,x\overline{xy}}\,\alpha s[x]^2\,\alpha s[y] - \frac{1}{4}\overline{\overline{\overline{xy}y}y}\,\alpha s[y]^2 + \frac{1}{3}\overline{x\overline{\overline{xy}y}}\,\alpha s[x]\,\alpha s[y]^2 - \frac{1}{6}\overline{\overline{\overline{xy}y}y}\,\alpha s[y]^3 +$$
$$\frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\alpha s[x,y] - \frac{1}{2}\overline{x\,x\overline{xy}}\,\alpha s[x]\,\alpha s[x,y] + \frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\alpha s[y]\,\alpha s[x,y] -$$
$$\overline{x\,x\overline{xy}}\,\alpha s[x,x,y] - \overline{x\overline{\overline{xy}y}}\,\alpha s[x,y,y] + \frac{1}{12}\overline{x\,x\overline{xy}}\,\beta s[x] + \frac{1}{12}\overline{x\overline{\overline{xy}y}}\,\beta s[x] -$$
$$\frac{1}{4}\overline{x\,x\overline{xy}}\,\beta s[x]^2 + \frac{1}{6}\overline{x\,x\overline{xy}}\,\beta s[x]^3 + \frac{1}{4}\overline{x\overline{\overline{xy}y}}\,\beta s[x]\,\beta s[y] - \frac{1}{3}\overline{x\overline{\overline{xy}y}}\,\beta s[x]^2\,\beta s[y] +$$
$$\frac{1}{6}\overline{\overline{\overline{xy}y}y}\,\beta s[x]\,\beta s[y]^2 - \frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\beta s[x,y] + \frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\beta s[x]\,\beta s[x,y] -$$
$$\frac{1}{2}\overline{\overline{\overline{xy}y}y}\,\beta s[y]\,\beta s[x,y] + \overline{x\overline{\overline{xy}y}}\,\beta s[x,x,y] + \overline{\overline{\overline{xy}y}y}\,\beta s[x,y,y],\ \ldots\Big]$$