FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS IV: SOME COMPUTATIONS

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Abstract. In the previous three papers in this series, [WKO1]–[WKO3], Z. Dancso and I studied a certain theory of "homomorphic expansions" of "w-knotted objects", a certain class of knotted objects in 4-dimensional space. When all layers of interpretation are stripped off, what remains is a study of a certain number of equations written in a family of spaces $\mathcal{A}^w$, closely related to degree-completed free Lie algebras and to degree-completed spaces of cyclic words.

The purpose of this paper is to introduce mathematical and computational tools that enable explicit computations (up to a certain degree) in these $\mathcal{A}^w$ spaces and to use these tools to solve the said equations and verify some properties of their solutions, and as a consequence, to carry out the computation (up to a certain degree) of certain knot-theoretic invariants discussed in [WKO1]–[WKO3] and in my related paper [BN3].

Contents

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Key words and phrases. w-knots, w-tangles, Kashiwara-Vergne, associators, double tree, Mathematica, free Lie algebras.

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1. Introduction

Within the previous three papers in this series [WKO1]–[WKO3][1] a number of intricate equations written in various graded spaces related to free Lie algebras and to spaces of cyclic words were examined in detail, for good reasons that were explained there and elsewhere. The purpose of this paper is to introduce mathematical tools (on the upper parts of pages) and computational tools (on the lower parts of pages, below the long dividing line[C1]) that allow for the explicit solution of these equations, at least up to a certain degree.

The equations we have in mind arise in other papers and appear throughout this paper. Yet to help our impatient readers orient themselves, here's a "flash summary" of the most important equations and their topological and algebraic significance:

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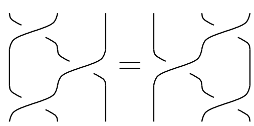
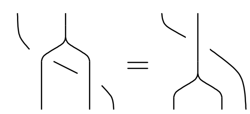
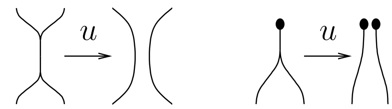
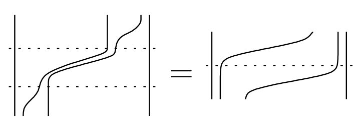
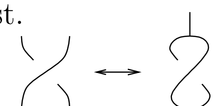
 $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ the key to knot invariants	 $R^{23}R^{13}V = R^{12,3}$ [AT]: $F(x+y) = \log e^x e^y$	 $VV^* = 1$; $VC^{12} = C^1C^2$ [AT]: $j(F) \in \mathrm{im}(\tilde\delta)$
	together, "the Kashiwara-Vergne equations"	
Pentagon.  $\Phi\Phi^{1,23,4}\Phi^{234} = \Phi^{12,3,4}\Phi^{1,2,34}$ "Drinfel'd associators"	Twist.  $\Theta = V^{-1}RV^{21}$ compatibility of associators with Kashiwara-Vergne	Buckle. MORE

Why bother? What do limited explict computations add, given that these intricate equations are known to be soluble, and given that the conceptual framework within which these

[1]Also within my [BN3], and within papers by Alekseev, Enriquez, and Torossian [AT, AET], and within Kashiwara's and Vergne's [KV], and also within many older papers about Drinfel'd associators (e.g. Drinfel'd's [Dr1, Dr2] and my [BN2].

[C1]If you are not interested in the actual computations, it is safe to ignore the parts of pages below the long dividing line and restrict to "strict" mathematics, which is always above that line.
The programs described in this paper were written in Mathematica [Wo] and are available at [WKO4]. Before starting with any computations, download the packages `FreeLie.m` and `AwCalculus.m` and type within Mathematica:

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

The last line above declares that by default we wish the computer to print series within graded spaces (such as free Lie algebras) to degree 4.

equations make sense is reasonably well understood [WKO1]–[WKO3]? My answers are three:

1. Personally, my belief in what I can't compute decays quite rapidly as a function of the complexity involved. Even if the overall picture is clear, the details will surely go wrong, and sooner or later, something bigger than a detail will go wrong. Even a limited computation may serve as a wonderful sanity check. In situations such as ours, where many signs and conventions need to be decided and may well go wrong, even a low-degree computation increases my personal confidence level by a great degree. Given computations that work to degree 6 (say), it is hard to imagine that a detail was missed or that conventions were established in an inconsistent manner. In fact, if the computer programs are clear enough and are shown to work, these programs become the authoritative declerations of the details and conventions.
2. The computational tools introduced here may well be used in other contexts where free Lie algebras and/or cyclic words arise.
3. The papers [WKO1, WKO2] (and likewise [BN3]) are about equations, but even more so, about the construction of certain knot and tangle invariants. With the tools presented here, the invariants of arbitrary knotted objects of the types studied in [WKO1, WKO2, BN3] may be computed.

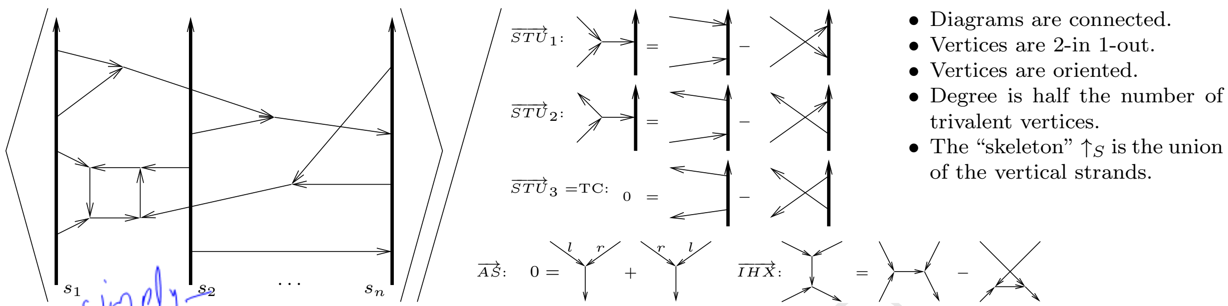
The equations of [WKO1]–[WKO3] always involve group-like, or "exponential" elements, and are written in some spaces of "arrow diagrams" that go under the umbrella name $\mathcal{A}^w$. Hence a crucial first step is to find convenient presentations for the group-like elements $\mathcal{A}^w_{\exp}$ in $\mathcal{A}^w$-spaces. It turns out that there are (at least) two such presentations, each with its own advantages and disadvantages. Hence in Section 2 we recall $\mathcal{A}^w$ briefly (2.1), discuss the "AT" and the "KBH" presentations of $\mathcal{A}^w_{\exp}$ (2.2 and 2.3), and describe how to convert between the two presentations (2.4).

MORE: summaries of the remaining sections.

2. Group-like elements in $\mathcal{A}^w$

subsec:Aw

2.1. **A brief review of $\mathcal{A}^w$.** Let $S = \{s_1, s_2, \dots\}$ be a finite set of "strand labels". The space $\mathcal{A}^w(\uparrow_S)$ is the graded vector space[2] of diagrams made of (vertical) "strands" labeled by the elements of S, and "arrows" as summarized by the following picture:



In topology, elements of $\mathcal{A}^w(\uparrow_S)$ are closely related to (finite type invariants of) knotted 2-dimensional tubes in $\mathbb{R}^4$ ([WKO1]–[WKO3], [BN3]). In Lie theory, they represent "universal" $\mathfrak{g}$-invariant tensors in $\mathcal{U}(I\mathfrak{g})^{\otimes S}$, where $I\mathfrak{g} := \mathfrak{g} \ltimes \mathfrak{g}^*$[3] and $\mathfrak{g}$ is some finite dimensional Lie algebra ([WKO1]–[WKO3]). Several significant Lie theoretic problems (e.g., the Kashiwara-Vergne problem, [KV, AT, WKO2]) can be interpreted as problems about $\mathcal{A}^w(\uparrow_S)$.

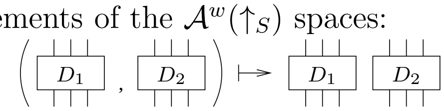
SortedForm

Comment 2.1. Using the $\overrightarrow{STU}_2$ relation one may sort the skeleton vertices in every $D \in \mathcal{A}^w(\uparrow_S)$ so that along every skeleton component all arrow heads appear ahead of all arrow tails, and by a diagrammatic analogue of the PBW theorem (compare [BN1, Theorem 8]), this sorted form is unique modulo $\overrightarrow{STU}_1$, TC, $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations.

Operations

Definition 2.2. A number of operations are defined on elements of the $\mathcal{A}^w(\uparrow_S)$ spaces:

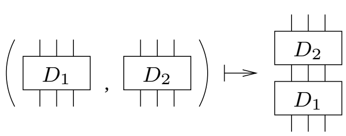
1. If S_1 and S_2 are disjoint, then given $D_1 \in \mathcal{A}^w(\uparrow_{S_1})$ and $D_2 \in \mathcal{A}^w(\uparrow_{S_2})$, their union $D_1 \cup D_2 \in \mathcal{A}^w(\uparrow_S)$ is obtained by placing them side by side as illustrated on the right.



   In topology, $\cup$ corresponds to the disjoint union of 2-links. In Lie theory, it corresponds to the map $\mathcal{U}(I\mathfrak{g})^{\otimes S_1} \otimes \mathcal{U}(I\mathfrak{g})^{\otimes S_2} \to \mathcal{U}(I\mathfrak{g})^{\otimes (S_1 \cup S_2)}$.

2. Given $D_1 \in \mathcal{A}^w(\uparrow_S)$ and $D_2 \in \mathcal{A}^w(\uparrow_S)$, their product $D_1 D_2 \in \mathcal{A}^w(\uparrow_S)$ is obtained by "stacking D_2 on top of D_1":

$$(D_1, D_2) = \quad (1)$$



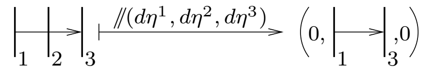
$$= D_1 D_2.$$

eq:TubePro

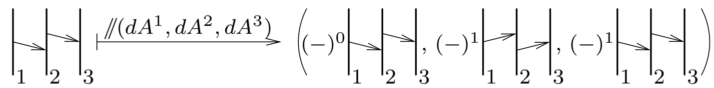
   In topology, the stacking product corresponds to the concatanation operation on knotted tubes, akin to the standard stacking product of tangles. In Lie theory, it comes from the algebra structure of $\mathcal{U}(I\mathfrak{g})^{\otimes S}$.

[2] For simplicity we always work over $\mathbb{Q}$.

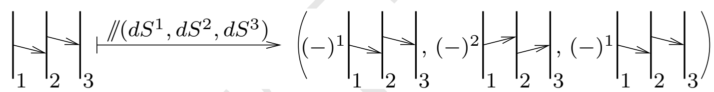
[3] In earlier papers we have used the order $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$.

3. Given $D \in \mathcal{A}^w(\uparrow_S)$ and $s \in S$, $D /\!/ d\eta^s$ is the result of deleting strand s from D and mapping it to 0 if any arrow connects to s, as illustrated on the right.


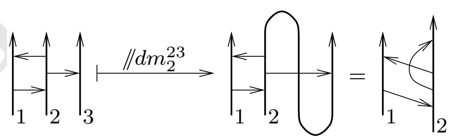
In topology, $d\eta^s$ is the removal of one component from a 2-link. In Lie theory it corresponds to the co-unit $\eta\colon \mathcal{U}(I\mathfrak{g}) \to \mathbb{Q}$.

4. Given $D \in \mathcal{A}^w(\uparrow_S)$ and $s \in S$, $D /\!/ dA^s$ is the result of "flipping over stand s and multiplying by $(-)$ sign for each arrow whose head connects to s", as illustrated above.


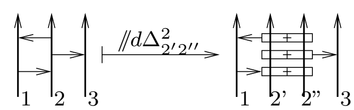
In topology, dA^s is the reversal of the 1D orientation of a knotted tube [WKO2]. In Lie theory, it is the antipode of $\mathcal{U}(I\mathfrak{g})$ combined with the sign reversal $\varphi \to -\varphi$ acting on the $\mathfrak{g}^*$ factor of $I\mathfrak{g}$. When elements of $\mathcal{U}(I\mathfrak{g})$ are interpreted as differential operators acting on functions on $\mathfrak{g}$, dA corresponds to the L^2 adjoint.

5. Similarly, $D /\!/ dS^s$ is the result of "flipping over stand s and multiplying by a $(-)$ sign for each arrow head or tail that connects to s", as illustrated above[4].


In topology, dS^s is the reversal of both the 1D and the 2D orientation of a knotted tube [WKO2]. In Lie theory, it is the antipode of $\mathcal{U}(I\mathfrak{g})$.

6. Given $D \in \mathcal{A}^w(\uparrow_S)$, given $a, b \in S$, and given $c \notin S \setminus \{a, b\}$, $D /\!/ dm^{ab}_c$ is the result of "concatanating strands a and b and calling the resulting strand c, as illustrated on the right.


In topology, dm^{ab}_c is the "internal concatenation" of two tubes within a single 2-link, akin to the "capping" operation that combines two strands of an ordinary tangle into a single "longer" one. In Lie theory, it is an "internal product", $\mathcal{U}(I\mathfrak{g})^{\otimes n} \to \mathcal{U}(I\mathfrak{g})^{\otimes(n-1)}$ which "merges" two factors within $\mathcal{U}(I\mathfrak{g})^{\otimes n}$.

7. Given $D \in \mathcal{A}^w(\uparrow_S)$, given $a \in S$, and given $b, c \notin S \setminus a$, $D /\!/ d\Delta^a_{bc}$ is the result of "doubling" strand a, calling the resulting "daughter strands" b and c, and summing over all ways of lifting the arrows that were connected to a to either b or c (so if there are k arrows connected to a, $D /\!/ d\Delta^a_{bc}$ is a sum of 2^k diagrams).


In topology, $d\Delta$ is the operation of "doubling" one component in a 2-link. In Lie theory, it is the co-product $\Delta\colon \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(I\mathfrak{g})^{\otimes 2}$ acting on the a factor in $\mathcal{U}(I\mathfrak{g})^{\otimes S}$, extended by the identity acting on all other factors.

8. Finally, the operation $d\sigma^a_b\colon \mathcal{A}(\uparrow_S) \to \mathcal{A}(\uparrow_{S \setminus \{a\} \cup \{b\}}$ does nothing but renaming the strand a to b (assuming $a \in S$ and $b \notin S \setminus \{a\}$). 2.2

We note that the product operation $(D_1, D_2) \mapsto D_1 D_2$ can be implemented using the union operation $\cup$, the strand-concatanation operation dm, and some renaming — namely, if $\bar{S} = \{\bar{s} \colon s \in S\}$ is some set of "temporary" labels disjoint from S but in a bijection with

[4]The letter S is used here for both "a set of strands" and "an operation similar to an antipode". Hopefully no confusion will arise.

S, then

$$D_1 D_2 = \left(D_1 \cup \left(D_2 /\!\!/ \prod_s d\sigma^s_{\bar{s}}\right)\right) /\!\!/ \prod_s dm^{s\bar{s}}_s. \tag{2}$$

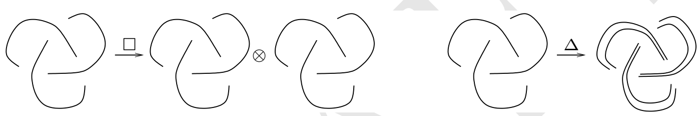
Therefore below we will sometime omit the implementation of $(D_1, D_2) \mapsto D_1 D_2$ provided all other operations are implemented.

We note that $\mathcal{A}^w(\uparrow_S)$ is a co-algebra, with the co-product $\Box(D)$, for a diagram D representing an element of $\mathcal{A}^w(\uparrow_S)$, being the sum of all ways of dividing D between a "left co-factor" and a "right co-factor" so that connected components of $D \backslash \uparrow_S$ (D with its skeleton removed) are kept intact (compare with [BN1, Definition 3.7]).

Definition 2.3. An element Z of $\mathcal{A}^w(\uparrow_S)$ is "group-like" if $\Box(Z) = Z \otimes Z$. We denote the set of group-like elements in $\mathcal{A}^w(\uparrow_S)$ by $\mathcal{A}^w_{\exp}(\uparrow_S)$.

We leave it for the reader to verify that all the operations defined above restrict to operations $\mathcal{A}^w_{\exp} \to \mathcal{A}^w_{\exp}$.

In topology, $\Box$ is the operation of "cloning" an entire 2-link. It is not to be confused with $d\Delta$; one dimension down and in just one component, the pictures are:



In Lie theory, $\Box$ is not the co-product $\Delta\colon \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(I\mathfrak{g})^{\otimes 2}$. Rather, given two finite dimensional Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_1$, $\Box$ corresponds to the map

$$\Box\colon \mathcal{U}(I(\mathfrak{g}_1 \oplus \mathfrak{g}_2))^{\otimes S} \to \mathcal{U}(I\mathfrak{g}_1)^{\otimes S} \otimes \mathcal{U}(I\mathfrak{g}_2)^{\otimes S}.$$

We seek to have efficient descriptions of the elements of $\mathcal{A}^w_{\exp}(\uparrow_S)$ and efficient means of computing the above operations on such elements.

Let $\mathcal{P}^w(\uparrow_S)$ denote the set of primitives of $\mathcal{A}^w(\uparrow_S)$: these are the elements $\zeta \in \mathcal{A}^w(\uparrow_S)$ satisfying $\Box(\zeta) = \zeta \otimes 1 + 1 \otimes \zeta$. Let $FL(S)$ denote the degree-completed free Lie algebra with generators S,[C2] and let $CW(S)$ denote the degree-completed vector space spanned by non-empty cyclic words on the alphabet S.[C3] In [WKO2, Proposition 3.14] we have shown that there is a short exact sequence of vector spaces

$$0 \to CW(S) \to \mathcal{P}^w(\uparrow_S) \to FL(S)^S \to 0,$$

where $FL(S)^S$ denotes the set of all functions $S \to FL(S)$, and hence $\mathcal{P}^w(\uparrow_S) \simeq FL(S)^S \oplus CW(S)$ (not canonically!). Often in bi-algebras there is a bijection given by $\zeta \mapsto e^\zeta$ between primitive elements ζ and group-like elements e^ζ. Hence we may expect to be able to represent elements of $\mathcal{A}^w_{\exp}(\uparrow_S)$ as formal exponentials of combinations of "trees" (elements of $FL(S)^S$) and "wheels" (elements of $CW(S)$)[5]:

[5]We use the set-theoretic notation "$\times$" rather than the linear-algebraic "$\otimes$" in Equation (3) to emphasize that the two sides of that equation are only expected to be set-theoretically isomorphic. The left-hand-side, in fact, is not even a linear space in a natural way.

$$\mathcal{A}^w_{\exp}(\uparrow_S) \sim TW(S) := FL(S)^S \times CW(S) = \left\{ (\lambda;\, \omega) \colon \begin{array}{c} \lambda = \{s \to \lambda_s\}_{s\in S},\ \lambda_s \in FL(S) \\ \omega \in CW(S) \end{array} \right\}. \quad (3)$$

eq:expectation

We implement Equation (3) in a more-or-less straightfoward way in Section 2.2 and in a less straighforward but somewhat stronger way in Section 2.3.

com:WhyTwo *Comment* 2.4. Why are there two presentations to elements of $\mathcal{A}^w_{\exp}$?
Answer 1. Because $\mathcal{A}^w$ is a bi-algebra in two ways, with the same co-product $\Box$. The first is by using the product of Equation (1), topologically corresponding to the pairwise concatanation of one set of $|S|$ knotted tubes in $\mathbb{R}^4$ with another set of $|S|$ knotted tubes in $\mathbb{R}^4$ (see [WKO1]). The second comes from [BN3]: a tube τ in $\mathbb{R}^4$ leads to a {balloon, hoop} pair, where the hoop is obtained by pushing the longtitude of τ off τ, and the balloon by capping τ on one end. And given two knots in $\mathbb{R}^4$, each consisiting of $|S|$ balloons and $|S|$ hoops, they can be multiplied by multiplying the hoops in pairs using the "π_1 product" and separately multiplying the balloons in pairs using the "π_2 product".
Answer 2. Roughly speaking, $\mathcal{A}^w$ is a combinatorial model of (tensor powers of a completion of) the universal enveloping algebra $\mathcal{U}(I\mathfrak{g})$ of the semi-direct product $I\mathfrak{g} = \mathfrak{g} \ltimes \mathfrak{g}^*$, for any finite-dimensional Lie algebra $\mathfrak{g}$, and where $\mathfrak{g}^*$ is taken as an Abelian Lie algebra and $\mathfrak{g}$ acts on $\mathfrak{g}^*$ using the co-adjoint action.

C3 In computer talk, generators of $FL(S)$ are always single-character "Lyndon words" (e.g. [Re]); in our case we set x_i to be the single-character word "i", for $i = 1, 2$. And then α, β, and γ to be the Lie series $x_1 + [x_1, x_2]$, $x_2 - [x_1, [x_1, x_2]]$, and $x_1 + x_2 - 2[x_1, x_2]$ (elements of FL are infinite series, in general, but these examples are finite):

```
x1 = LW[1]; x2 = LW[2];
{α, β, γ} =
  MakeLieSeries /@ {x1 + b[x1, x2], x2 - b[x1, b[x1, x2]], x1 + x2 - 2 b[x1, x2]}
```

$\{\mathrm{LS}[\overline{1}, \overline{12}, 0, 0], \mathrm{LS}[\overline{2}, 0, -\overline{1\overline{12}}, 0], \mathrm{LS}[\overline{1} + \overline{2}, -2\,\overline{12}, 0, 0]\}$

Note that as we requested earlier, our example series are printed to degree 4. Note also that they are printed using "top bracket" notation, which is easier to read when many brackets are nested.
We then compute $[\alpha, \beta]$ and verify the Jacobi identity for α, β, and γ:

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

$\{\mathrm{LS}[0, \overline{12}, \overline{\overline{12}2}, -\overline{1\overline{1\overline{12}}}], \mathrm{LS}[0, 0, 0, 0]\}$

C3 Cyclic words in computer talk:

```
{ω1, ω2} = MakeCWSeries /@ {CW["1"] - 3 CW["121"], CW["2"] + CW["22"]}
```

$\{\mathrm{CWS}[\overset{\frown}{1}, 0, -3\,\overset{\frown}{121}, 0], \mathrm{CWS}[\overset{\frown}{2}, \overset{\frown}{22}, 0, 0]\}$

By PBW, $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$, and hence group-like elements in $\mathcal{U}(I\mathfrak{g})$ can either be written in "mixed form", as exponentials of elements of $\mathfrak{g} \ltimes \mathfrak{g}^*$, or in "factored form", as product of an exponential in $\mathcal{S}(\mathfrak{g}^*)$ with an exponential in $\mathcal{U}(\mathfrak{g})$. Very roughly speaking, the "mixed form" corresponds to the "AT presentation" below, and the "factored form" to the "KBH presentation" below.

The reality is a bit more delicate, though. $\mathcal{A}^w$ is only a model of the $\mathfrak{g}$-invariant part of $\mathcal{U}(I\mathfrak{g})$, and the notions of being group-like in $\mathcal{A}^w$ and in $\mathcal{U}(I\mathfrak{g})$ do not match. Yet the flavour remains — in the AT presentation arrow tails ("elements of $\mathfrak{g}^*$") mix with arrow heads ("$\mathfrak{g}$"), while in the KBH presentation heads and tails are kept apart. com:WhyTwo 2.4

subsec:AT

2.2. **The AT presentation E_l of $\mathcal{A}^w_{\exp}$.** In this section we use notation from [WKO2, Section 3.2] (which when relevant follows [AT]) without further mention. We also extend the notation a bit — whereas in [WKO2, AT] a set of generators $\{x_1, \dots, x_n\}$ is fixed and is always indexed by the integers $\{1, \dots, n\}$, we allow an arbitrary finite set S of indices. Hence $\mathfrak{a}_n$, $\mathfrak{tder}_n$, $\mathfrak{tr}_n$ (etc.) of [WKO2, AT] are replace by $\mathfrak{a}_S$, $\mathfrak{tder}_S$, $\mathfrak{tr}_S$ (etc.) here.

Given a pair $(\lambda;\omega) \in TW(S) = FL(S)^S \times CW(S) = (\mathfrak{a}_S \oplus \mathfrak{tder}_S) \times \mathfrak{tr}_S$ we set

$$E_l(\lambda;\omega) \coloneqq \exp(l\lambda)\exp(\iota\omega), \qquad \left(\begin{matrix}\text{“}E_l\text{” for “}\underline{\text{E}}\text{xponentiation} \\ \text{after using } \underline{l}\text{”}\end{matrix}\right)$$

where $l\colon FL(S)^S = \mathfrak{a}_S \oplus \mathfrak{tder}_S \to \mathcal{A}^w(\uparrow_S)$ is the “lower” splitting[6] of trees into $\mathcal{A}^w(\uparrow_S)$, where ι is the obvious inclusion of wheels ($= CW(S) = \mathfrak{tr}_S$) into $\mathcal{A}^w(\uparrow_S)$, and where exponentiation is taken using the stacking product (1) of $\mathcal{A}^w(\uparrow_S)$. It follows from the results of [WKO2, Section 3.2] that $E_l\colon TW(S) \to \mathcal{A}^w_{\exp}(\uparrow_S)$ is a set-theoretic bijection. Hence the operations of Definition 2.2 induce corresponding operations on $TW(S)$. We list these within the proposition below.

prop:ElOps

Proposition 2.5. *The bijection E_l intertwines the following operations with the operations in Definition 2.2:*

1. *If $S_1 \cap S_2 = \emptyset$ and $(\lambda_i;\omega_i) \in TW(S_i)$,*
$$E_l(\lambda_1;\omega_1) \cup E_l(\lambda_2;\omega_2) = E_l(\lambda_1 \cup \lambda_2; \omega_1 + \omega_2), \tag{4}$$
eq:ElCup
where $\cup\colon FL(S_1)^{S_1} \times FL(S_2)^{S_2} \to FL(S_1 \cup S_2)^{S_1 \cup S_2}$ is the union operation of functions (or, in computer science language, the concatanation of associative arrays) followed by the inclusions $FL(S_i) \to FL(S_1 \cup S_2)$, and $\omega_1 + \omega_2$ is defined using the inclusions $CW(S_i) \to CW(S_1 \cup S_2)$.
2. *If $(\lambda_i;\omega_i) \in TW(S)$,*
$$E_l(\lambda_1;\omega_1)E_l(\lambda_2;\omega_2) = E_l(\mathrm{BCH}_{tb}(\lambda_1,\lambda_2); \omega_1 + e^{\lambda_1}(\omega_2)). \tag{5}$$
eq:ElProdu
Here we employ the BCH *formula using the “tangential bracket” $[\cdot,\cdot]_{tb}$ that $FL(S)^S$ inherits via the isomorphism $FL(S) = \mathfrak{a}_S \oplus \mathfrak{tder}_S$ (alternatively, it is the bracket inherited from the stacking-product-commutator of $\mathcal{A}_w$)[7]:*
$$\mathrm{BCH}_{tb}(\lambda_1,\lambda_2) = \lambda_1 + \lambda_1 + \frac{[\lambda_1,\lambda_2]_{tb}}{2} + \frac{[\lambda_1,[\lambda_1,\lambda_2]_{tb}]_{tb} + [[\lambda_1,\lambda_2]_{tb},\lambda_2]_{tb}}{12} + \dots$$
Also, $e^{\lambda_1}(\omega_2)$ is defined by exponentiating the action of $\mathfrak{tder}_S$ on $\mathfrak{tr}_S$ (taking the action of $\mathfrak{a}_S$ to be trivial).
3. *If $(\lambda;\omega) \in TW(S)$ and $s \in S$,*
$$E_l(\lambda;\omega)/\!\!/d\eta^s = E_l((\lambda\backslash s;\omega)/\!\!/(s \to 0)), \tag{6}$$
eq:ElEta
where $\lambda\backslash s$ denotes the function λ with the element s removed from its domain (in computer talk, “remove the key s”), and $(s \to 0)$ denotes the substitution $s = 0$, which is defined on both FL and CW and maps $FL(S) \to FL(S\backslash s)$ and $CW(S) \to CW(S\backslash s)$.

[6] We could have equally well used the “upper” splitting u, setting $E_u(\lambda;\omega) \coloneqq \exp(\iota\omega)\exp(u\lambda)$, with only minor modifications to the formulas that follow.

[7] $[\cdot,\cdot]_{tb}$ is a non-trivial modification of the obvious component-wise bracket of $FL(S)^S = \bigoplus_S FL(S)$.

4. *For a single $s \in S$, I don't know a simple description of the operation dA^s in E_l language. Yet the composition $dA^S := \prod_{s \in S} dA^s$ is manageable:*

$$E_l(\lambda;\,\omega)/\!\!/dA^S = E_l(-\lambda;\, e^{\lambda}(\omega) - j(\lambda)). \tag{7}$$

Here j is the Alekseev-Torossian "logarithm of the Jacobian" [AT, Section 5.1] *(extended by 0 on $\mathfrak{a}_S$): $j(\lambda) = \frac{e^\lambda - 1}{\lambda}(\operatorname{div}\lambda)$, where $\operatorname{div}\colon \mathfrak{tder}_S \to \mathfrak{tr}_S$ is the divergence functional and λ acts on $\mathfrak{tr}_S$ as before.*

5. *For a single $s \in S$, I don't know a simple description of the operation dS^s in E_l language. Yet the composition $dS^S := \prod_{s \in S} dS^s$ is manageable:*

$$E_l(\lambda;\,\omega)/\!\!/dS^S = E_l(-\lambda/\!\!/(-1)^{\deg};\, (e^{\lambda}(\omega) - j(\lambda))/\!\!/(-1)^{\deg}), \tag{8}$$

where in general $h^{\deg}$ denotes the operations $FL \to FL$ and $CW \to CW$ which multiply any degree k element by h^k.

6. *I don't know a simple description of the operation dm^{ab}_c in E_l language. Yet note that Equation* (2) *implies that "applying dm to all strands" is manageable, being the stacking product described in* (5)*.*

7. *We have*

$$E_l(\lambda;\,\omega)/\!\!/d\Delta^a_{bc} = E_l((\lambda\backslash a) \cup (b \to \lambda_a,\, c \to \lambda_a)/\!\!/(a \to b + c);\, \omega/\!\!/(a \to b + c)), \tag{9}$$

where $(a \to b + c)$ denotes the obvious replacement of the generator a with the sum $b + c$. It represents morphisms $FL(S) \to FL((S\backslash a) \cup \{b, c\})$, $FL(S)^H \to FL((S\backslash a) \cup \{b, c\})^H$ (for some set H), and $CW(S) \to CW((S\backslash a) \cup \{b, c\})$.

8. *We have*

$$E_l(\lambda;\,\omega)/\!\!/d\sigma^a_b = E_l(((\lambda\backslash a) \cup (b \to \lambda_a))/\!\!/(a \to b);\, \omega/\!\!/(a \to b)), \tag{10}$$

where $(a \to b)$ denotes the obvious "generator renaming" morphisms $FL(S) \to FL((S\backslash a) \cup b)$, $FL(S)^H \to FL((S\backslash a) \cup b)^H$ (for some set H), and $CW(S) \to CW((S\backslash a) \cup b)$.

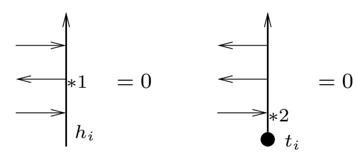
Proof. Equations (4), (6), (9), and (10) are trivial and were stated only to introduce notation. The tree-level part of Equation (5) follows from the fact that l is a morphism of Lie algebras (see within the proof of [WKO2, Proposition 3.14]). The wheels part of Equation (5) follows from [WKO2, Remark 3.19]. Equation (7) follows from the observation that dA^S is the adjoint map $*$ of [WKO2, Definition 3.21] and then from [WKO2, Proposition 3.22]. Equation (8) is the easily-established fact that on $\mathcal{A}^w$, $dS^S = (-1)^{\deg} dA^S$. □

2.3. **The KBH presentation E_s of $\mathcal{A}^w_{\exp}$.** Following [BN3], in the "split" presentation E_s of $\mathcal{A}^w_{\exp}$ arrow heads are treated separately from arrow tails in diagrams such as the one near the beginning of Section 2.1. This presentation of $\mathcal{A}^w_{\exp}$ is more complicated than the previous one, yet it is also more powerful, and in some sense, it is made of simpler ingredients. For E_s we first enlarge the collection of spaces $\{\mathcal{A}^w(S)\}$ to a somewhat bigger collection $\{\mathcal{A}^w(H;T)\}$ on which a larger class of operations act. The new operations are more "atomic" than the old ones, in the sense that each of the operations of Definition 2.2 is a composition of 2-3 of the new operations. The advantage is that the new operations all have reasonably simple descriptions as operations on the group-like subsets $\{\mathcal{A}^w_{\exp}(H;T)\}$, and hence even the few operations whose description in the E_l presentation was ommitted in Proposition 2.5 can be fully described and computed in the E_s presentation.

A sketch of our route is as follows: In Section 2.3.1, right below, we describe the spaces $\{\mathcal{A}^w(H;T)\}$. In Section 2.3.2 we describe the zoo of operations acting on $\{\mathcal{A}^w(H;T)\}$. Section 2.3.3 is the tofu of the matter — we describe the operations of the previous section in terms of spaces $\{TW(H;T)\}$ of trees and wheels, whose elements are in a bijection with the group like elements of $\{\mathcal{A}^w(H;T)\}$. Finally in Section 2.3.4 we explain how the system of spaces $\{\mathcal{A}^w(S)\}$ includes into the system $\{\mathcal{A}^w(H;T)\}$ and how the operations of the former are expressed in terms of the latter, concluding the description of E_s.

2.3.1. *The family* $\{\mathcal{A}^w(H;T)\}$. Let $H = \{h_1, h_2, \dots\}$ be some finite set of "head labels" and let $T = \{t_1, t_2, \dots\}$ be some finite set of "tail labels" (these sets need not be of the same cardinality). Let $\mathcal{A}^w(H;T)$ be $\mathcal{A}^w(\uparrow_{H\sqcup T})$[8] moded out by the following further relations:

- If an arrow tail lands anywhere on a head strand ($*1$ on the right), the whole diagram is zero.
- The CP relation: If an arrow head is the lowest vertex on a tail strand ($*2$ on the right), the whole diagram is zero.



Comment 2.6. Using these two relations one may show that $\mathcal{A}^w(\uparrow_{H\sqcup T})$ is isomorphic to the set of arrow diagrams in which only arrow heads land on the head strands (obvious, by the first relation) and in which only arrow tails meet the tail strands (use $\overrightarrow{STU}_2$ to slide any arrow head on a tail strand until it's near the bottom, then use the second relation; see also Comment 2.1), still modulo $\overrightarrow{AS}$, $\overrightarrow{IHX}$, $\overrightarrow{STU}_1$ and TC.

In topology (see [BN3]), head strands correspond to "hoops", or based knotted circles, and tail strands correspond to balloons, or based knotted spheres. The two relations and the isomorphism above are also meaningful [BN3].

[8] We will often use sets of labels H and T that are *not* disjoint. The notation "$H\sqcup T$" stands for the union of H and T, made disjoint by brute force; for example, by setting $H \sqcup T \coloneqq (\{h\}\times H)\cup(\{t\}\times T)$, where h and t are two distinct labels chosen in advance to indicate "heads" and "tails". In practice we will keep referring to the images of the elements of H within $H\sqcup T$ as h_i rather than (h, h_i), and likewise for the t_i's. We will mostly avoid the confusion that may arise when $H\cap T \neq \emptyset$ by labeling operations as "head operations" which will always refer to labels in $H \hookrightarrow H\sqcup T$ or as "tail operations", when referring to labels in $T \hookrightarrow H \sqcup T$.

In Lie theory head strands represent $\mathcal{U}(\mathfrak{g})$ and tail strands represent the (right) Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}^*) \simeq \mathcal{S}(\mathfrak{g}^*)$. The evaluation $\mathfrak{g}^* \to 0$ induces a surjection of $\mathcal{U}(I\mathfrak{g})$ onto the first of these spaces whose kernel is "any word containing a letter in $\mathfrak{g}^*$", explaining the first relation above. The second relation is the definition of the Verma module.

Operations

2.3.2. *Operations on* $\{\mathcal{A}^w(H;T)\}$.

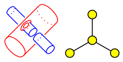
Operations

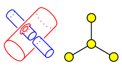
Definition 2.7. Just as in Definition 2.2 [def:Operations], there are several operations that are defined on $\mathcal{A}^w(H;T)$. In brief, these are:

1. A union operation $\cup\colon \mathcal{A}^w(H_1;T_1) \otimes \mathcal{A}^w(H_2;T_2) \to \mathcal{A}^w(H_1 \cup H_2; T_1 \cup T_2)$, defined when $H_1 \cap H_2 = T_1 \cap T_2 = \emptyset$, with obvious topological (compare with "$*$" of [BN3, Figure 5] [KBH-fig:Connected Sum]) and Lie theoretic meanings.
2. A "stacking" product # can be defined on $\mathcal{A}^w(H;T)$ by concatenating all pairs of equally-labeled head strands and then merging all pairs of equally-labeled tail strands in a pair of diagrams $D_1, D_2 \in \mathcal{A}^w(H;T)$. The "merging" of tail strands is described in more detail as the operation tm below. In fact, it may be better to define # using a formula similar to Equation (2) [eq:multiplem] and the operations hm, tm, $h\sigma$, and $t\sigma$ defined below:
$$D_1 \# D_2 = \left(D_1 \cup \left(D_2 /\!\!/ \prod_{x \in H} h\sigma^x_{\bar{x}} /\!\!/ \prod_{u \in T} t\sigma^t_{\bar{t}} \right) \right) /\!\!/ \prod_{x\in H} hm^{x\bar{x}}_x /\!\!/ \prod_{u \in T} tm^{u\bar{u}}_u. \tag{11}$$
eq:AHTStac
Warning. Restricted to $\mathcal{A}^w(S;S)$ the product # does not agree with the stacking product $\cdot$ of $\mathcal{A}^w(\uparrow_S)$.
In topology, # is the concatenation of hoops followed by the merging of balloons; this is not the same as the concatenation of knotted tubes. In Lie theory, # corresponds to the componentwise product of $\mathcal{U}(\mathfrak{g})^{\otimes H} \otimes \mathcal{S}(\mathfrak{g}^*)^{\otimes T}$. Even when H and T are both singletons, this is not the same as the product of $\mathcal{U}(I\mathfrak{g})$, even though linearly $\mathcal{U}(I\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}^*)$.
3. If $x \in H$ and $u \in T$, the operations $h\eta^x$ and $t\eta^u$ drop the head-strand x or the tail-strand u similarly to the operation η^s of Definition 2.2 [def:Operations].
4. hA^x reverses the head-strand x while multiplying by a (-1) factor for every arrow head on x. tA^u is the identity.
5. $tS^x = hA^x$ while tS^u multiplies by a factor of (-1) for every arrow tail on u (by TC, there's no need to reverse u).
6. The operation hm^{xy}_z is defined similarly to m^{ab}_c of Definition 2.2 [def:Operations]. Likewise for tm^{uv}_w, except in this case, the tail-strands u and v must first be cleared of all arrow-heads using the process of Comment 2.6 [com:PureForm]. Once u and v carry only arrow-tails, all these tail can be put on a new tail-strand w in some arbitrary order (which doesn't matter, by TC). Note that $tm^{uv}_w = tm^{vu}_w$, so tm is "meta-commutative".
In topology, tm^{uv}_w is the "merging of balloons" operation of [BN3, Section 3.1] [KBH-subsec:MMAOperations], which in itself is analogues to the (commutative) multiplication of π_2.
In Lie theory, tm^{uv}_w is the product of $\mathcal{S}(\mathfrak{g}^*)$. Note that tail strands more closely represent the Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g})$ whose isomorphism with $\mathcal{S}(\mathfrak{g}^*)$ involves "sliding all $\mathfrak{g}$-letters in a $\mathcal{U}(I\mathfrak{g})$-word to the left and then canceling them". This is analogoues to the process of cancelling arrow-heads which is a pre-requisite to the deifnition of tm^{uv}_w.
7. $h\Delta^x_{yz}$ and $t\Delta^u_{vw}$ are defined similarly to Δ^a_{bc}.

8. $h\sigma^x_y$ and $t\sigma^u_v$ are defined similarly to σ^a_b.
9. **New!** Given a tail $u \in T$, a "new" tail label $v \notin T\setminus u$ and a head $x \in H$ the operation $thm^{ux}_v\colon \mathcal{A}^w(H;T) \to \mathcal{A}^w(H\setminus x; (T\setminus u)\cup\{v\})$ is the obvious "tail-strand head-strand concatenation" — similarly to m^{ab}_c, concatenate the strand u to the strand x putting u before x, and call the resulting "new" strand v. Note that for this to be well defined, v must be a tail strand.[9]

   In practice, thm^{ux}_v is never used on its own, but the combination $h\Delta^x_{xx'}/\!\!/thm^{ux'}_u$ (where x' is a temporary label) is very useful. Hence we set $tha^{ux}\colon \mathcal{A}^w(H;T)\to\mathcal{A}^w(H;T)$ ("tail by head action on u by x") to be that combination. In words, this is "double the strand x and put one of the copies on top of u".[10]

   
   In topology, tha is the action of hoops on balloons as in [BN3, Section 3.1], which is similar to the action of π_1 on π_2. In Lie theory, it is the right action of $\mathcal{U}(\mathfrak{g})$ on the Verma module $\mathcal{U}(I\mathfrak{g})/\mathfrak{g}\mathcal{U}(I\mathfrak{g})$, or better, the action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{S}(\mathfrak{g}^*)$ induced from the co-adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.


Exercise 2.8. In the cases when we did not state the topological or Lie theoretical meaning of an operation in Definition 2.7, find what it is.

sec:AHTExp

2.3.3. *Group-like elements in $\{\mathcal{A}^w(H;T)\}$.* For any fixed finite sets H and T there is a co-product $\Box\colon \mathcal{A}^w(H;T)\otimes\mathcal{A}^w(H;T)$ defined just as in the case of $\mathcal{A}^w(\uparrow_S)$ (Definition 2.3), and along with the product $\#$ (and obvious units and co-units), $\mathcal{A}^w(H;T)$ is a graded connected co-commutative bi-algebra. Hence it makes sense to speak of the group-like elements $\mathcal{A}^w_{\exp}(H;T)$ within $\mathcal{A}^w(H;T)$, and they are all $\#$-exponentials of primitives in $\mathcal{A}^w(H;T)$. The primitives $\mathcal{P}^w(H;T)$ in $\mathcal{A}^w(H;T)$ are connected diagrams and hence they are trees and wheels. As in Comment 2.6, the trees must have their roots on head strands and their leafs on tail strands, and the wheels must have all their "legs" on tail strands. As tails commute, we may think of the trees as abstract trees with leafs labeled by labels in T and roots in H, and the wheels are abstract cyclic words with letters in T. Hence canonically $\mathcal{P}^w(H;T)\simeq FL(T)^H\oplus CW(T)$ and hence there is a bijection

$$E_s\colon TW(H;T) := FL(T)^H \oplus CW(T) \xrightarrow{\sim} \mathcal{A}^w_{\exp}(H;T) \qquad (12)$$ eq:EsHT

defined by

$$(\lambda\colon H\to FL(T);\, \omega\in FL(T)) \mapsto \exp_\#\left(e_s(\lambda;\omega)\right), \qquad (13)$$ eq:esHT

where $e_s(\lambda;\omega)$ is the sum over $x\in H$ of planting λ_x with its root on strand x and its leafs on the strands in T so that the labels match but at an arbitrary order on any T strand, plus the result of planting ω on just the T strands so that the labels match but at an arbitrary order on any T strand.

Together, Equations (12) and (13) make the E_s presentation of $\mathcal{A}^w_{\exp}(H;T)$. It is easy to verify that the operations in Definition 2.7 intertwine $\Box$ and hence map group-like elements to group-like elements and hence they induce operations on $TW(H;T)$. These are summarized within the proposition below.

[9]Note also that the analogeous operation htm^{xu}_v "put x before u to get a tail v" is 0 and hence we can safely ignore it, and that thm^{xu}_y and htm^{xu}_y, defined in the same way as thm^{ux}_v and htm^{xu}_v except to produce a head strand y, are not well defined because they do not respect the CP relation.

[10]Note that $thm^{ux}_v = tha^{ux}/\!\!/h\eta^x/\!\!/t\sigma^u_v$ so we lose no generality by considering tha^{ux} instead of thm^{ux}_v.

Proposition 2.9. *In the KBH presentation E_s the operations of Definition 2.7 act as follows:*[11]*:*

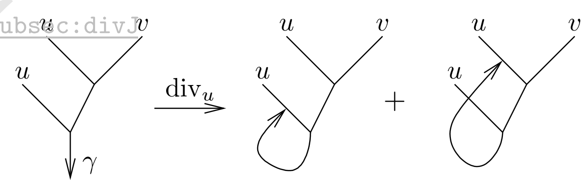
1. $E_s(\lambda_1;\,\omega_1) \cup E_s(\lambda_2;\,\omega_2) = E_s(\lambda_1 \cup \lambda_2;\,\omega_1 + \omega_2)$ (14)
2. $E_s(\lambda_1;\,\omega_1) \# E_s(\lambda_2;\,\omega_2) = E_s\left((x \to \mathrm{BCH}(\lambda_{1x}, \lambda_{2x}))_{x \in H};\,\omega_1 + \omega_2\right)$ (15)
3. $E_s(\lambda;\,\omega) /\!\!/ h\eta^x = E_s(\lambda \backslash x;\,\omega)$ (16)
   $E_s(\lambda;\,\omega) /\!\!/ t\eta^u = E_s(\lambda /\!\!/ (u \to 0);\,\omega /\!\!/ (u \to 0))$ (17)
4. $E_s(\lambda;\,\omega) /\!\!/ hA^x = E_s((\lambda \backslash x) \cup (x \to -\lambda_x);\,\omega)$ (18)
   $tA^u = I$ (19)
5. $hS^x = hA^x$, (20)
   $E_s(\lambda;\,\omega) /\!\!/ tS^u = E_s(\lambda /\!\!/ (u \to -u);\,\omega /\!\!/ (u \to -u))$ (21)
6. $E_s(\lambda;\,\omega) /\!\!/ hm_z^{xy} = E_s((\lambda \backslash \{x, y\}) \cup (z \to \mathrm{BCH}(\lambda_x, \lambda_y));\,\omega)$ (22)
   $E_s(\lambda;\,\omega) /\!\!/ tm_w^{uv} = E_s(\lambda /\!\!/ (u, v \to w);\,\omega /\!\!/ (u, v \to w))$ (23)
7. $E_s(\lambda;\,\omega) /\!\!/ d\Delta_{yz}^x = E_s((\lambda \backslash x) \cup (y \to \lambda_x, z \to \lambda_x);\,\omega)$ (24)
   $E_s(\lambda;\,\omega) /\!\!/ t\Delta_{vw}^u = E_s(\lambda /\!\!/ (u \to v + w);\,\omega /\!\!/ (u \to v + w))$ (25)
8. $E_s(\lambda;\,\omega) /\!\!/ d\sigma_y^x = E_s((\lambda \backslash x) \cup (y \to \lambda_x);\,\omega)$ (26)
   $E_s(\lambda;\,\omega) /\!\!/ t\sigma_v^u = E_s(\lambda /\!\!/ (u \to v);\,\omega /\!\!/ (u \to v))$ (27)

Something somewhere's got to have some substance, and in our case, that's tha^{ux}. For $u \in T$ and $\gamma \in FL(T)$ the operation $RC_u^\gamma\colon FL(T) \to\colon FL(T)$ and the functional $J_u\colon FL(T) \to CW(T)$ were defined in [BN3] and are reviewed in the two definition below. With these,

9. $E_s(\lambda;\,\omega) /\!\!/ tha^{ux} = E_s\left(\lambda /\!\!/ RC_u^{\lambda_x};\,(\omega + J_u(\lambda_x)) /\!\!/ RC_u^{\lambda_x}\right).$ (28)

Definition 2.10. (Compare [BN3, Section 4.2]) Given $u \in T$ and $\gamma \in FL(T)$ let $C_u^{-\gamma}$ denote the automorphism of $FL(T)$ defined by mapping the generator u to its "conjugate" $e^{-\gamma} u e^{\gamma} = e^{-\operatorname{ad}\gamma}(u)$. Let RC_u^γ be the inverse of $C_u^{-\gamma}$ (which is *not* C_u^γ).

Definition 2.11. (Compare [BN3, Section 5.1]) Given $u \in T$ and let $\operatorname{div}_u\colon FL(T) \to CW(T)$ be the functional defined by the picture on the right (more details in [BN3]). Given also $\gamma \in FL(T)$, set



$$J_u(\gamma) := \int_0^1 ds\ \operatorname{div}_u(\gamma /\!\!/ RC_u^{s\gamma}) /\!\!/ C_u^{-s\gamma}.$$

Proof of Proposition 2.9. The first 8 assertions (14 operations) are very easy. The main challenge to the reader should be to gather her concentration for the 14-times repeatitive task of unwrapping definitions. If you are ready to cut corners, only go over (14), (22), (23), (24), and (25). Let us turn to the proof of the last assertion, Equation (28). That proof is in fact in [BN3], or at least can be assembled from pieces already in [BN3]. Yet the assembly would be a bit delicate, and hence a proof is reproduced below which refers back to [BN3] only at one technical point.

By inspecting the definition of tha^{ux}, it is clear that there is *some* assignment $\gamma \mapsto R_u^\gamma$ that assigns an operator $R_u^\gamma\colon FL(T) \to FL(T)$ to every $\gamma \in FL(T)$ and that there is some functional $K_u\colon FL(T) \to CW(T)$, for which a version of Equation (28) holds:

[11]Here we no longer state conditions such as $H_1 \cap H_2 = \emptyset$, $u \in T$, $(\lambda;\,\omega) \in TW(H;T)$. They are the same as in Definition 2.7, and more impotantly, they are "what makes sense".

$$E_s(\lambda;\omega)/\!\!/tha^{ux} = E_s\left(\lambda/\!\!/R_u^{\lambda_x};\,(\omega+K_u(\lambda_x))/\!\!/R_u^{\lambda_x}\right) \tag{29}$$

Indeed, tha^{ux} acts on $E_s(\lambda;\omega)$ by placing a copy of $\exp(\lambda_x)$ at the top of the tail strand u, and then re-writing the result without having any heads on strand u so as to invert E_s back again. The re-writing is done by sliding the heads of $\exp(\lambda_x)$ down to the bottom of strand u, where they cancel by CP. Every time a head slides past a tail we get a contribution from $\overrightarrow{STU}_2$. Sometimes a head of a λ_x will slide against a tail of another λ_x, whose head will have to slide down too, leading to a rather complicated iterative process. Nevertheless, these contributions are the same for every tail on strand u, namely for every occurence of the variable u in $FL(T)^H$ and/or in $CW(T)$. This explains the terms $\lambda/\!\!/R_u^{\lambda_x}$ and $\omega/\!\!/R_u^{\lambda_x}$ in Equation (29). We note that the degree 0 part of the operator $R_u^{\lambda_x}$ is the identity, and hence it is invertible.

But yet another type of term arises in the process — sometimes a head of some tree will slide against a tail of its own, and then the contribution arising from $\overrightarrow{STU}_2$ will be a wheel. Hence there is an additional contribution to the output, some $L_u(\lambda_x)$ which clearly can depend only on u and λ_x. Using the invertibility of $R_u^{\lambda_x}$ to write $L_u(\lambda_x)=K_u(\lambda_x)/\!\!/R_u^{\lambda_x}$ we completely reproduce Equation (29).

We now need to show that R_u^γ and $K_u(\gamma)$ are RC_u^γ and $J_u(\gamma)$ of Definitions 2.10 and 2.11. Tracing again through the discussion in the previous two paragraphs, we see that at any fixed degree, R_u^γ and $K_u(\gamma)$ depend polynomially on the coefficients of γ, and hence it is legitimate to study their variation with respect to γ. It is also easy to verify that $R_u^0=RC_u^0=I$ and that $K_u(0)=J_u(0)=0$, and hence it is enough to show that, with an indeterminate scalar τ,

$$\frac{d}{d\tau}R_u^{\tau\gamma}=\frac{d}{d\tau}RC_u^{\tau\gamma}\qquad\text{and}\qquad\frac{d}{d\tau}K_u(\tau\gamma)=\frac{d}{d\tau}J_u(\tau\gamma). \tag{30}$$

Let us compute the left-hand-sides of the above equations. If τ is an infinitesimal (so $\tau^2=0$), or more precisely, computing the above left-hand-sides at $\tau=0$, we can re-trace the process described in the two paragraphs following Equation (29) keeping in mind that with $\lambda_x=\tau\gamma$ the $\overrightarrow{STU}_2$ relation can only by applied once (or else terms proportional to τ^2 will arise). The result is

$$\left.\frac{d}{d\tau}R_u^{\tau\gamma}\right|_{\tau=0}=\mathrm{ad}_u^\gamma\qquad\text{and}\qquad\left.\frac{d}{d\tau}K_u(\tau\gamma)\right|_{\tau=0}=\mathrm{div}_u(\gamma), \tag{31}$$

where $\mathrm{ad}_u^\gamma\colon FL(T)\to FL(T)$ is the derivation which maps the generator u of $FL(T)$ to $[\gamma,u]$ and annihilates all other generators of $FL(T)$ (compare [BN3, Definition 10.5]) and where $\mathrm{div}_u(\gamma)$ is the same as in Definition 2.11.

Moving on to general τ, we note that the operations hm and tha satisfy

$$hm_z^{xy}/\!\!/tha^{uz}=tha^{ux}/\!\!/tha^{uy}/\!\!/hm_z^{xy} \tag{32}$$

(stitching strands x and y and then stitching a copy of the result to u is the same as stitching a copy of x to u, then a copy of y, and then stitching x to y; compare [BN3, Equation (6)]). Applying the operators on the two sides of Equation (32) to $E_s(\lambda;\omega)$ (assuming H and T are such that it makes sense), then expanding using (22) and (29), and then ignoring the wheels in the resulting equality, we find that R_u satisfies

$$R_u^{\mathrm{BCH}(\lambda_x,\lambda_y)}=R_u^{\lambda_x}/\!\!/R_u^{\lambda_y/\!\!/R_u^{\lambda_x}} \tag{33}$$

(compare [BN3, Equation (16)]). Similarly, looking only at the wheel part of (32) we get

$$K_u(\mathrm{BCH}(\lambda_x,\lambda_y))/\!\!/R_u^{\mathrm{BCH}(\lambda_x,\lambda_y)} = K_u(\lambda_x)/\!\!/R_u^{\lambda_x}/\!\!/R_u^{\lambda_y/\!\!/R_u^{\lambda_x}} + K_u(\lambda_y/\!\!/R_u^{\lambda_x})/\!\!/R_u^{\lambda_y/\!\!/R_u^{\lambda_x}},$$

which, composing on the right with $R_u^{\mathrm{BCH}(\lambda_x,\lambda_y)}$ and using (33), is equivalent to

$$K_u(\mathrm{BCH}(\lambda_x,\lambda_y)) = K_u(\lambda_x)/\!\!/R_u^{\lambda_x} + K_u(\lambda_y/\!\!/R_u^{\lambda_x})/\!\!/C_u^{-\lambda_x} \tag{34}$$

(compare [BN3, Equation (19)]).

Equations (33) and (34) hold for any λ, and hence for any λ_x and λ_y. Specializing to $\lambda_x=\tau\gamma$ and $\lambda_y=\epsilon\gamma$, where ϵ is some new indeterminate scalar, and using the fact that $\mathrm{BCH}(\tau\gamma,\epsilon\gamma)=(\tau+\epsilon)\gamma$, Equations (33) and (34) become

$$R_u^{(\tau+\epsilon)\gamma} = R_u^{\tau\gamma}/\!\!/R_u^{\epsilon\gamma/\!\!/R_u^{\tau\gamma}} \qquad\text{and}\qquad K_u((\tau+\epsilon)\gamma) = K_u(\tau\gamma)/\!\!/R_u^{\tau\gamma} + K_u(\epsilon\gamma/\!\!/R_u^{\tau\gamma})/\!\!/C_u^{-\tau\gamma}.$$

Now differetiating with respect to ϵ at $\epsilon=0$ and using Equation (31) with τ replaced with ϵ, we get

$$\frac{d}{d\tau}R_u^{\tau\gamma} = R_u^{\tau\gamma}/\!\!/\,\mathrm{ad}_u^{\gamma/\!\!/R_u^{\tau\gamma}} \qquad\text{and}\qquad \frac{d}{d\tau}K_u(\tau\gamma) = \mathrm{div}_u(\gamma/\!\!/R_u^{\tau\gamma})/\!\!/C_u^{-\tau\gamma}.$$

The first of these equations is the same equation that is satisfied by RC_u (see [BN3, Lemma 10.7], with $\delta\gamma$ proportional to γ), and hence $R_u=RC_u$. By a simple change of variables, $J_u(\tau\gamma)=\int_0^\tau dt\ \mathrm{div}_u(\gamma/\!\!/RC_u^{t\gamma})/\!\!/C_u^{-t\gamma}$, and hence $\frac{d}{d\tau}J_u(\tau\gamma)=\mathrm{div}_u(\gamma/\!\!/RC_u^{\tau\gamma})/\!\!/C_u^{-\tau\gamma}$ (compare with the formula for the full differential of J, [BN3, Proposition 10.10]). Comparing with the above formula for the derivative of K_u, we find that $K_u=J_u$. □

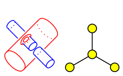
2.3.4. *The inclusion* $\{\mathcal{A}^w(\uparrow_S)\}\hookrightarrow\{\mathcal{A}^w(H;T)\}$. The following definition and proposition imply that there is no loss in studying the spaces $\mathcal{A}^w(H;T)$ rather than the spaces $\mathcal{A}^w(\uparrow_S)$.

Definition 2.12. Let $\delta\colon\mathcal{A}^w(\uparrow_S)\to\mathcal{A}^w(S;S)$ be the composition of the "double every strand" map $\prod_{s\in S}\Delta^s_{hs,ts}\colon\mathcal{A}^w(\uparrow_S)\to\mathcal{A}^w(\uparrow_{hS\sqcup tS})$ with the projection $\mathcal{A}^w(\uparrow_{hS\sqcup tS})\to\mathcal{A}^w(S;S)$ (as an exception to the rule of Footnote 8 we temporarily highlight the distinction between head and tail labels by afixing them with the prefixes h and t).

Proposition 2.13. *δ is a vector space isomorphism*[12]*. The inverse of δ on $D\in\mathcal{A}^w(S;S)$ is given by the process*

1. *Write D with only arrow heads on the head strands and only arrow tails on the tail strands. By Comment 2.6 this produces a well-defined element D' of $\mathcal{A}^w(\uparrow_{hS\sqcup tS})$.*
2. *Concatenate all the head-tail pairs of strands in D' by putting each head ahead of its corresponding tail: $\delta^{-1}D=D'/\!\!/\prod_s m_s^{hs,ts}$.*

Proof. $\delta^{-1}/\!\!/\delta=I$ by inspection, and $\delta/\!\!/\delta^{-1}$ is clearly the identity on diagrams sorted to have heads ahead of tails as in Comment 2.1. □



In topology, δ agrees with the δ of [BN3, Section 2.2]. In Lie theory, it agrees with the linear (non-multiplicative) isomorphism $\mathcal{U}(I\mathfrak{g})\simeq\mathcal{U}(\mathfrak{g})\otimes\mathcal{S}(\mathfrak{g}^*)$ and with similar isomorphisms considered by Etingof and Kazhdan within their work on the quantization of Lie bialgebras [EK] (albeit only when the Lie bialgebras in question are cocommutative).

The next proposition shows how the operations of defined on the $\mathcal{A}^w(\uparrow_S)$-spaces in Definition 2.2 can be written in terms of the "head and tail" operations of Definition 2.7, thus completing the description of the E_s presentation.

[12]See also Discussions 2.15 and 2.16.

Proposition 2.14. *1. If S_1 and S_2 are disjoint and $D_1 \in \mathcal{A}^w(\uparrow_{S_1})$ and $D_2 \in \mathcal{A}^w(\uparrow_{S_2})$, then $\delta(D_1 \cup D_2) = \delta(D_1) \cup \delta(D_2)$.*

2. Let $D_1, D_2 \in \mathcal{A}^w(\uparrow_S)$. Then $\delta(D_1D_2)$ can be written in terms of $\delta(D_1)$ and $\delta(D_2)$ using its description in terms of $\cup$, $d\sigma$, and dm in Equation (2) and using the formulas for $\cup$, $d\sigma$, and dm that appear above and below.

3. $d\eta^s /\!\!/ \delta = \delta /\!\!/ h\eta^s /\!\!/ t\eta^s$.

4. $dA^s /\!\!/ \delta = \delta /\!\!/ hA^s /\!\!/ tA^s /\!\!/ tha^{ss}$.

5. $dS^s /\!\!/ \delta = \delta /\!\!/ hS^s /\!\!/ tS^s /\!\!/ tha^{ss}$.

6. $dm^{ab}_c /\!\!/ \delta = \delta /\!\!/ tha^{ab} /\!\!/ hm^{ab}_c /\!\!/ tm^{ab}_c$.

7. $d\Delta^a_{bc} /\!\!/ \delta = \delta /\!\!/ h\Delta^a_{bc} /\!\!/ t\Delta^a_{bc}$.

8. $d\sigma^a_b /\!\!/ \delta = \delta /\!\!/ h\sigma^a_b /\!\!/ t\sigma^a_b$.

Proof. The only difficulty is with items 4–6. Item 4 is easier to understand in the form $\delta^{-1} /\!\!/ dA^s = hA^s /\!\!/ tA^s /\!\!/ tha^{ss} /\!\!/ \delta^{-1}$. Indeed, δ^{-1} plants heads ahead of tails on strand s. Applying dA^s reverses that strand (and adds some signs). This reversal can be achieved by reversing the head part (with signs), then the tail part (with signs), and then by swapping the two parts across each other. The first reversal is hA^s, the second is tA^s, and the swap is tha^{ss} followed by δ^{-1}. Item 5 is proven in exactly the same way, and item 6 is proven in a similar way, where the right hand side traces the schematics $(ha\,ta\,hb\,tb) \xrightarrow{tha} (ha\,hb\,ta\,tb) \xrightarrow{hm/\!\!/tm} ((ha\,hb)(ta\,tb))$. □

Discussion 2.15. It is easy to verify that $\delta\colon \mathcal{A}^w(\uparrow_S) \to \mathcal{A}^w(S;S)$ intertwines the co-algebra structures on its domain and its range, and hence it restricts to an isomorphism $\delta\colon \mathcal{A}^w_{\exp}(\uparrow_S) \to \mathcal{A}^w_{\exp}(S;S)$. Therefore $E_s /\!\!/ \delta^{-1}$ is a bijection between $TW(S)$ and $\mathcal{A}^w_{\exp}(\uparrow_S)$. Proposition 2.14 now tells us how to write all the "d" operations of Definition 2.2 as "h" and "t" operations, and Proposition 2.9 tells us how to write these as operations on $TW(S)$. Overall $E_s /\!\!/ \delta^{-1}$ is a complete presentation of $\mathcal{A}^w_{\exp}(\uparrow_S)$.

Discussion 2.16. For use in the next section, note that both $\mathcal{A}^w(\uparrow_S)$ and $\mathcal{A}(S;S)$ are associative algebras (the former using the stacking product of Equation (1) and the latter using that of Equation (11)), yet δ is not multiplicative and hence it does not restrict to a Lie morphism on primitives. Instead, on primitives $(\lambda_1;\omega_1), (\lambda_2;\omega_2) \in TW(S)$ we have

$$\delta[l\lambda_1+\iota\omega_1,\ l\lambda_2+\iota\omega_2] = [\delta(l\lambda_1+\iota\omega_1),\ \delta(l\lambda_2+\iota\omega_2)] + e_s(\partial_{\lambda_1}\lambda_2;\ \partial_{\lambda_1}\omega_2) - e_s(\partial_{\lambda_2}\lambda_1;\ \partial_{\lambda_2}\omega_1), \quad (35)$$

where ∂_λ denotes the tangential derivation in $\mathfrak{tder}_S$ corresponding to λ under the identification $FL(S)^S \simeq \mathfrak{a}_S \oplus \mathfrak{tder}_S$. Note that as in [AT], these derivations also act on $CW(S)$.

Proof of Equation (35).

MORE

2.4. **The conversion between the AT and the KBH presentations.** We now have two presentations for elements of $\mathcal{A}^w_{\exp}$, and we wish to be able to convert between the two. In other words, given $\lambda = \{s \to \lambda_s\}_{s\in S} \in FL(S)^S$ and $\omega \in CW(S)$, we wish to find λ' and ω' such that $E_l(\lambda;\omega) = E_s(\lambda';\omega') /\!\!/ \delta^{-1}$.

Given $(\lambda;\omega)$ as above and a scalar t, let $\Gamma(\lambda,t) = \{s \to \gamma_s(t)\} \in FL(S)^S$ be the unique solution of the system of ordinary differential equations

$$\forall s \in S, \quad \frac{d\gamma_s(t)}{dt} = \gamma_s(t) /\!\!/ e^{-t\partial_\lambda} /\!\!/ \frac{\operatorname{ad}\gamma_s(t)}{e^{\operatorname{ad}\gamma_s(t)}-1}; \qquad \gamma_s(0) = 0. \quad (36)$$

Let $\Gamma(\lambda) := \Gamma(\lambda, 1)$.

Theorem 2.17. $\omega' = \Gamma(\lambda)$ *and* $\omega' = \omega$. *Namely,*

$$E_l(\lambda;\,\omega) = E_s(\Gamma(\lambda);\,\omega) /\!\!/ \delta^{-1} \tag{37}$$

Proof. E_l and E_s both plant wheels at the top, and as tails commute, they do so in the same manner. So $\omega' = \omega$ and we only need to show Equation (37) at tree level (meaning, modulo wheels). We will show that for every scalar t,

$$\exp(l(t\lambda)) = \exp_{\#}(e_s(\Gamma(\lambda, t))) /\!\!/ \delta^{-1}; \tag{38}$$

the desired result is the specialization of Equation (38) to $t = 1$. It is clear that Equation (38) holds for some unique $\Gamma_0 = \{s \to \gamma_{0s}(t)\}$, that $\gamma_{0s}(0) = 0$, and that each coefficient of each $\gamma_{0s}(t)$ depends polynomialy on t, and hence it is enough to show that Γ_0 satisfies the differential equation in (36).

MORE.

MORE.

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Everything below is to be blanked out before the completion of this paper. sectionIntroduction This paper being a third in a series [WKO1, WKO2], as well as a continuation of [AT, AET] and of [BN3], we will forgo a description of the context and the motivations and forgo the precise definitions, and instead jump right into the heart of the matter — the equations we seek to solve, and the spaces in which they are written. Our fundamental quantities are

- $R = \exp(\upuparrows)$, the Z^w-value of a crossing, a member of the space $\mathcal{A}^w(\uparrow_2)$ defined in [WKO1] and reviewed in Section 2.4 below.
- V, the Z^w-value of a vertex, a member of $\mathcal{A}^w(\uparrow_3)$.
- $C \in \mathcal{A}^w(\uparrow)$, the Z^w-value of a cap.
- A Drinfel'd associator Φ and a braiding element for u-braids $\Theta = \exp(\frac{1}{2}\upuparrows)$.

:Equations subsectionThe Equations

- Reidemeister 4, R4 $$R_{23}R_{13}V = VR_{12,3} \qquad (39)$$

sec:Spaces subsectionThe Spaces

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