FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: THE DOUBLE TREE CONSTRUCTION

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Abstract. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological solution to the Kashiwara-Vergne problem. In particular we recover via a topological argument the Alkeseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of *2-dimensional foams* and various associated features in four-dimensioanl space. We use a topological construction which we name the double tree construction to show that every *expansion* (also known as *universal finite type invariant*) of parenthesized braids extends first to an expansion of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is the same as a *Drinfel'd associator* Φ, and an expansion for the aforementioned w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem [KV] as reformulated by Alekseev and Torossian [AT]. Hence our result provides a topological framework for the result of [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof that the said formula works — namely that the equations satisfied by V follow from the equations satisfied by Φ.

Contents

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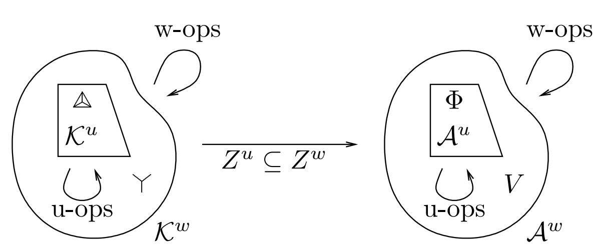
1. Introduction

1.1. **Executive Summary.** This section is a large-scale overview of the main result of this paper and the idea behind its proof; it is followed by a more detailed introduction.

A *homomorphic expansion* for a class of topological objects $\mathcal{K}$ is an invariant $Z\colon \mathcal{K} \to \mathcal{A}$ whose target space $\mathcal{A}$ is canonically associated with $\mathcal{K}$ (its *associated graded*). Homomorphic expansions satisfy a certain universality property, and respect operations which exist on $\mathcal{K}$, and therefore also on $\mathcal{A}$. Such invariants are often hard to find, and when they are found, they are often intimately connected with deep mathematics:

- For many classes of knotted objects in 3-dimensional spaces homomorphic expansions don't exist — for example, one would have loved ordinary tangles to have homomorphic expansions, but they don't.
- Yet a certain class $\mathcal{K}^u$ of knotted objects in 3-space, *parenthesized tangles*, or nearly-equivalently, *knotted trivalent graphs* – which we adopt in this paper and denote by *sKTG* – do have homomorphic expansions. A homomorphic expansion $Z^u\colon \mathcal{K}^u \to \mathcal{A}^u$ is defined by its values on a couple of elements of $\mathcal{K}^u$ which generate $\mathcal{K}^u$ using the operations $\mathcal{K}^u$ is equipped with. The most interesting of these generators is the tetrahedron ⟁, and $\Phi = Z^u$(⟁) turns out to be equivalent to a *Drinfel'd associator*.
- A certain class $\mathcal{K}^w$ of graphs, called *w-foams* and denoted *wTF^o* in the paper – the name is based on a conjectured equivalence to a class of 2-dimensional *welded* knotted tubes in 4-dimensional space – also has homomorphic expansions. The most interesting generator of $\mathcal{K}^w$ is the *vertex* $\curlyvee$, and if $Z^w\colon \mathcal{K}^w \to \mathcal{A}^w$ is a homomorphic expansion, then it turns out that $V = Z^w(\curlyvee)$ is equivalent to a solution of the *Kashiwara-Vergne problem* in Lie theory.

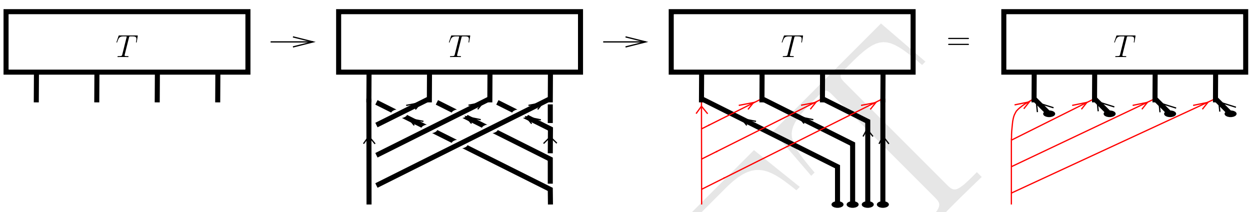
Roughly speaking, $\mathcal{K}^u$ is a part of $\mathcal{K}^w$ and $\mathcal{A}^u$ is a part of $\mathcal{A}^w$, as in the figure on the right (more precisely, there are natural maps $a\colon \mathcal{K}^u \to \mathcal{K}^w$ and $\alpha\colon \mathcal{A}^u \to \mathcal{A}^w$). The main purpose of this paper is to prove the following theorem, whose precise version is stated later as Theorem 1.1:



Theorem. Any homomorphic expansion Z^u for $\mathcal{K}^u$ extends uniquely to a homomorphic expansion Z^w for $\mathcal{K}^w$, and therefore, any Drinfel'd associator Φ yields a solution V of the Kashiwara-Vergne problem.

The proof of this theorem is based on the natural idea: we simply show that the generators of $\mathcal{K}^w$ can be explicitly expressed using the generators of $\mathcal{K}^u$ and the operations of $\mathcal{K}^w$, and that the resulting explicit formulas for $Z^w(\curlyvee)$ (and for Z^w of the other generators) satisfies all the required relations.

The devil is in the details. It is in fact impossible to express the generators of $\mathcal{K}^w$ in terms of the generators of $\mathcal{K}^u$ — to do that, one first has to pass to a larger space $\tilde{\mathcal{K}}^w$ (in the paper $\widetilde{wTF}$) that has more objects and more operations, and in which the desired explicit expressions do exist. But even in $\tilde{\mathcal{K}}^w$ these expressions are complicated, and are best described within a certain "double tree construction" which also provides the framework for the verification of relations. Here's an unexplained summary; the explanations make the bulk of this paper:



1.2. **Detailed Introduction.** This paper is the third in a sequence [WKO1, WKO2, WKO3] studying finite type invariants of w-knotted objects, and contains the strongest result: a topological construction for a homomorphic expansion of *w-foams* from the Kontsevich integral. This in particular implies the Kashiwara-Vergne Theorem, more precisely, the [AET] formula for solutions of the Kashiwara-Vergne equations in terms of Drinfel'd associators. Readers familiar with finite type invariants in general should have no trouble reading [WKO2] and this paper without having read [WKO1]. The setup and main results of [WKO2] are used heavily in this paper. Reproducing all necessary details would be lengthy, so we only include concise summaries for readers already familiar with the content, and otherwise refer to specific sections of [WKO2] throughout.

The Kashiwara-Vergne conjecture (KV for short) — proposed in 1978 [KV] and proven in 2006 by Alekseev and Meinrenken [AM] — asserts that solutions exist for a certain set of equations in the space of "tangential automorphisms" of the free lie algebra on two generators. For a precise statement we refer the reader to [WKO2] or [AT]. The existence of such solutions has has strong implications in Lie theory and harmonic analysis, in particular it implies the Duflo isomorphism, which was shown to be knot-theoretic in [BLT].

In [AT] Alekseev and Torossian give another proof for the KV conjecture based on a deep connection with Drinfel'd associators. In turn, Drinfel'd's theory of associators [Dr] can be interpreted as a theory of well-behaved universal finite type invariants of parenthesized tangles[1] [LM, BN2], or of knotted trivalent graphs [Da]. In [AET] Alekseev, Enriquez and Torossian gave an explicit formula for solutions of the Kashiwara-Vergne equations in terms of Drinfel'd associators.

In [WKO2] we re-interpreted the Kashiwara-Vergne conjecture as the problem of finding a "homomorphic" universal finite type invariant of a class of knotted trivalent tubes in 4-dimansional space (called w-tangled foams), and explained the connection to Drilfel'd

[1] "q-tangles" in [LM], "non-associative tangles" in [BN2].

associators in terms of a relationship between 3-dimensional and 4-dimensional topology. Another topological interpretation for the KV problem has recently emerged in [AKKN].

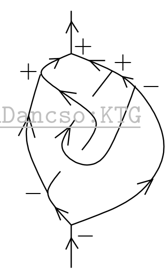
In this paper we present a topological construction for a homomorphic universal finite type invariant of w-tangled foams, thereby giving a new, topological proof for the KV conjecture. This construction also leads to an explicit formula for solutions, which we prove agrees with [AET].

1.2.1. *Topology.* We begin by describing a chain of maps from "parenthesized braids" to "(signed) knotted trivalent graphs" to "w-tangled foams":

$$\mathcal{K} := \{ uPaB \xrightarrow{\text{cl}} sKTG \xrightarrow{a} \widetilde{wTF} \}.$$

Let us first briefly elaborate on each of these spaces and maps.

Parenthesized braids are braids whose ends are ordered along two lines, the "bottom" and the "top", along with parenthetizations of the endpoints on the bottom and on the top. Two examples are shown in Figure 1. Parentehesized braids form a category whose objects are parenthetizations, morphisms are the parenthesized braids themselves, and composition is given by stacking. In addition to stacking, there are several operations defined on parenthesized braids: strand addition, removal and doubling. A detailed introduction to parenthesized braids is in [BN1].



Trivalent graphs are oriented graphs with three edges meeting at each vertex and whose vertices are equipped with a cyclic orientation of the incident edges. A knotted trivalent graph (KTG) is a framed embedding of a trivalent graph into $\mathbb{R}^3$. KTGs are studied from a finite type invariant point of view in [BND1]. In this paper we use a version of KTGs that was introduced and studied in [WKO2, Section 4.6], namely trivalent tangles with one or two ends and with some extra combinatorial information: trivalent vertices are equipped with a marked "distinguished edge" and signs. We call this space *sKTG* (for signed KTGs), as in [WKO2]. An example is shown on the right. The space *sKTG* is also equipped with several operations: tangle insertion, sticking a 1-tangle onto an edge of another tangle, disjoint union of 1-tangles, edge unzip, and edge orientation switch (see [WKO2, Section 4.6] for details).

The space $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4], and will be introduced in detail in Section 2. It can be described as a circuit algebra (similar to a planar algebra but with non-planar connections allowed, see [WKO2, Section 2.4]) generated by certain features (various kinds of crossings and vertices, as well as "caps") modulo certain relations ("Reidemeister moves") and equipped with a number of auxiliary

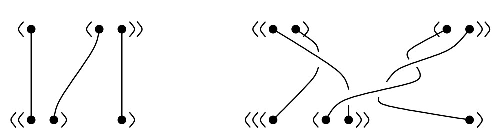
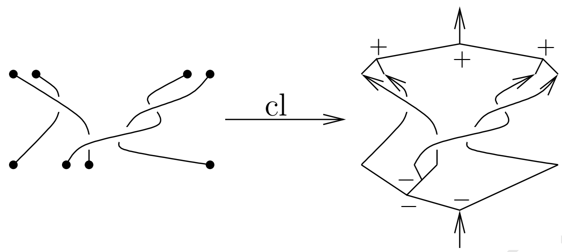


Figure 1. Two examples of parenthesized braids. Note that by convention the parenthetization can be read from the distance scales between the endpoints of the braid, and so we are going to omit the parentheses in the future.

operations beyond circuit algebra composition. This Reidemeister theory conjecturally represents knotted tubes in 4-dimensional space with singular *foam vertices*, caps, and attached one-dimensional strings.

The map $\mathrm{cl} : uPaB \to sKTG$ is the "closure map". Given a parenthesized braid, close up its top and bottom each by gluing a binary tree according to the parentetization; this produces a *sKTG* with the convention that all strands are oriented upwards, bottom vertices are negative, and top vertices are positive. An example is shown below.



The map $a : sKTG \to \widetilde{wTF}$ arises combinatorially from the fact that all *sKTG* diagrams can be interpreted as elements of $\widetilde{wTF}$, and all $sKTG$ Reidemeister moves are also imposed in $\widetilde{wTF}$. Topologically, it is an extended version of Satoh's tubing map, described in Remark 3.1.1 of [WKO2].

1.2.2. *Algebra.* The chain of maps $\mathcal{K}$ is an example of a general "algebraic structure", as discussed in [WKO2, Section 2.1]. An algebraic structure consists of a collection of objects belonging to a number of "spaces" or "different kinds", and operations that may be unary, binary, multinary or nullary, between these spaces. In this case there are many spaces (or kinds of objects): for example, parenthesized braids with specified bottom and top parenthetizations form one space, so do knottings of a given trivalent graph (skeleton). There is also a large collection of operations, consisting of all the internal operations of *uPaB*, *sKTG* and $\widetilde{wTF}$, as well as the maps a and cl.

In Sections 2.1 to 2.3 of [WKO2] we discuss associated graded structures and expansions for general algebraic structures. For any algebraic structure (think braids, or tangles with tangle composition), one allows formal linear compositions of elements of the same *kind* (think, same skeleton). Associated graded structures are taken with respect to the filtration by powers of the *augmentation ideal*. For the spaces *uPaB*, *sKTG* and $\widetilde{wTF}$, the associated graded spaces $\mathcal{A}^{hor}$, $\mathcal{A}^u$ and $\mathcal{A}^{sw}$ are the spaces of "horizontal chord diagrams on parenthesized strands", "chord diagrams on trivalent skeleta", and "arrow diagrams", as described in [BN1], [WKO2, Section 4.6], and Section 2 of this paper, respectively. As a result, the associated graded structure of $\mathcal{K}$ is

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{\mathrm{cl}} \mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and a, respectively. More specifically, cl is the "closure of chord diagrams", and α is "replacing each chord with the sum of its two possible oriantations", see [WKO2, Section 3.3].

An expansion [WKO2, Section 2.3] is a filtration-respecting map from an algebraic structure to its associated graded structure, whose associated graded map is the identity. In knot theory, expansions are also called universal finite type invariants. A homomorphic expansion is an expansion which behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the associated graded structure; for a detailed definition and introduction see [WKO2, Section 2.3]. Hence, a homomorphic expansion $Z : \mathcal{K} \to \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u, and Z^w for *uPaB*, *sKTG* and $\widetilde{wTF}$, respectively, so that the following diagram commutes:

$$\begin{array}{ccccc} uPaB & \xrightarrow{\text{cl}} & sKTG & \xrightarrow{a} & \widetilde{wTF} \\ \downarrow{\scriptstyle Z^b} & & \downarrow{\scriptstyle Z^u} & & \downarrow{\scriptstyle Z^w} \\ \mathcal{A}^{hor} & \xrightarrow{\text{cl}} & \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array} \tag{1}$$

We recall (see [BN1]) that a homomorphic expansion Z^b for parenthesized braids is determined by a "horizontal chord associator" $\Phi = Z^b(|\!/|)$. A homomorphic expansion Z^u of *sKTG* is also determined[2] by a Drinfel'd associator (horizontal chords or not; see [WKO2, Section 4.6]), so the significance of the left commutative square is to force the associator corresponding to Z^u to be a horizontal chord associator. In turn, Z^w is determined by a solution F (a close cousin of $V = Z^w(\nearrow\!\nwarrow)$) to the Kashiwara-Vergne problem (see [WKO2, Section 4.4 – 4.5]). The goal of this paper is to prove the following theorem, which, via the correspondence above, implies the KV conjecture:

Theorem 1.1. (1) *Assuming that $Z : \mathcal{K} \to \mathcal{A}$ exists, it is determined*[3] *by Z^u.*

(2) *There is a formula for V in terms of the Drinfel'd associator Φ associated to Z^u:*

$$V = C_1^{-1} C_2^{-1} C_{(12)} \varphi \left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \Phi(a_{23}, a_{43}) \right),$$

the notation will be explained later. This agrees with the formula proven in [AET].

(3) *Every Z^b extends to a Z.*

The key to the proof of the theorem is to show that the generator $\nearrow\!\nwarrow$ of $\widetilde{wTF}$ can be expressed in terms of the generator $|\!/|$ of *uPaB* and the operations of $\mathcal{K}$. Assuming that Z exists, this yields a formula for V in terms of Φ.

1.3. **Paper Structure.** In Section 2 we provide an overview of the space wTF^o of (oriented) w-foams and its extension with strings $\widetilde{wTF}$. We provide a brief review of definitions and crucial facts from [WKO2], and details of the extension. We prove that homomorphic expansions for wTF^o extend uniquely to homomorphic expansions for $\widetilde{wTF}$.

Section 3 makes up the bulk of the paper and is devoted to the proof of Theorem 1.1. In Section 3.1 we prove part (1) in a diagrammatic quotient called the *tree level*. Section 3.2 upgrades this to a complete proof of part (1). In Section 3.3 we deduce the formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators, proving part (2). In Section 3.4 we prove statement (3): the hardest part.

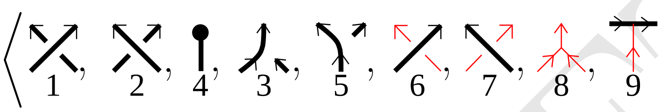
2. The spaces $\widetilde{wTF}$ and $\mathcal{A}^{sw}$ in more detail

As we mentioned in the introduction, $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4]. It can be introduced as a planar algebra or as a circuit algebra; we will do the latter as it is simpler and more concise. Circuit algebras are defined

[2]With the exception of some minor normalization, see [WKO2, Section 4.6], in particular Lemma 4.14 and the paragraph following it.

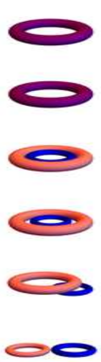
[3]In fact, almost entirely determined by Z^b, with the exception of the minor normalization of Z^u which is not determined by an associator.

in [WKO2, Section 2.4]; in short, they are similar to planar algebras but without the planarity requirement for "connecting strands". As in [WKO2], each generator and relation of $\widetilde{wTF}$ has a local topological interpretation. Recall [WKO2, Sections 1.2, 3.4, 4.1] that wTF^o diagrams represent certain ribbon knotted tubes with foam vertices in $\mathbb{R}^4$, and the circuit algebra wTF^o is conjecturally a Reidemeister theory for this space (i.e., there is a surjection δ from the circuit algebra wTF^o to ribbon knotted tubes with foam vertices, and δ is conjectured to be an isomorphism). The space $\widetilde{wTF}$ extends wTF^o by adding one-dimensional strands to the picture. Note that one dimensional strands cannot be knotted in $\mathbb{R}^4$, however, they can be knotted *with* the two-dimensional tubes. In figures two-dimensional tubes will be denoted by thick lines and one dimensional strings by thin red lines. With this in mind, we define $\widetilde{wTF}$ as a circuit algebra defined in terms of generators and relations, and with some extra operations beyond circuit algebra composition. Each generator, relation and operation has a local topological interpretation which provides much of the intuition behind the proofs. However, the corresponding Reidemeister theorem is only conjectural.

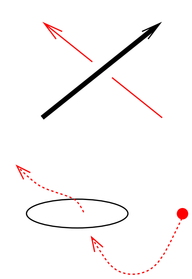
$\widetilde{wTF} = \mathrm{CA}\Big\langle$ 

$\Big|$ relations as in Section 2.2 $\Big|$ auxiliary operations as in Section 2.3 $\Big\rangle$

2.1. **The generators of $\widetilde{wTF}$.** We begin by discussing the local topological meaning of each generator shown above.



The first five generators are as described in [WKO2, Sections 4.1.1], we briefly recall their descriptions here. Knotted (more precisely, braided) tubes in $\mathbb{R}^4$ can equivalently be thought of as movies of flying rings in $\mathbb{R}^3$. The two crossings stand for movies where two rings trade places by the ring of the under strand flying through the ring of the over strand. The dotted end represents a tube "capped off" by a disk. Generators 4 and 5 stand for singular "foam vertices", and will be referred to as the positive and negative vertex, respectively. The positive vertex represents the movie shown on the left: the right ring approaches the left ring from below, flies inside it and merges with it. The negative vertex represents a ring splitting and the inner ring flying out below and to the right. To be completely precise, $\widetilde{wTF}$ as a circuit algebra has more vertex generators than shown above: the vertices appear with all possible orientations of the strands. However, all other versions can be obtained from the ones shown above using "orientation switch" operations (to be discussed in Section 2.3).



The thin red strands denote one dimensional strings in $\mathbb{R}^4$, or "flying points in $\mathbb{R}^3$". The crossings between the two types of strands (generators 6 and 7) represent "points flying through rings". For example, the picture on the left shows generator 6, where "the point on the right approaches the ring on the left from below, flies through the ring and out to the left above it". This explains why there are no generators with a thick strand crossing under a thin red strand: a ring cannot fly through a point.

Generator 9 is a trivalent vertex of 1-dimensional strings in $\mathbb{R}^4$. Finally, the last generator is a *mixed vertex:* a one-dimensional string attached to the wall of a

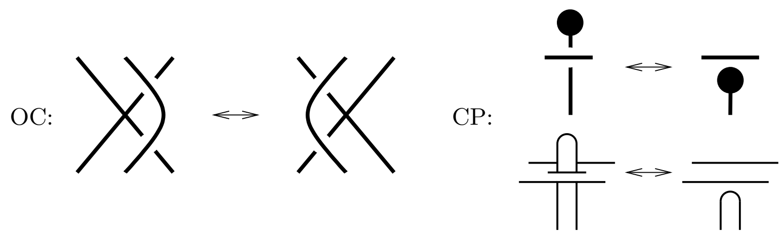
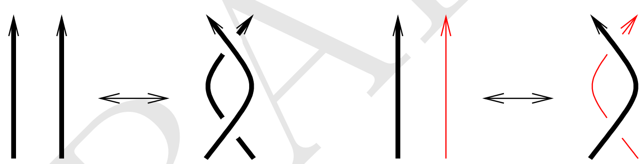


Figure 2. The OC and CP relations.

2-dimensional tube. All generators should be shown in all possible strand orientation combinations; we are suppressing this to save space.

2.2. **The relations.** As a list, the relations for $\widetilde{wTF}$ are the same as the relations for wTF^o [WKO2, Section 4.5]: {R1^s, R2, R3, R4, OC, CP}. Recall that R1^s is the weak (framed) version of the Reidemeister 1 move; R2 and R3 are the usual second and third Reidemeister moves; R4 allowes moving a strand over or under a vertex. OC stands for *Overcorssings Commute*, CP for *Cap Pullout*: these two relations are shown in Figure 2, for a detailed explanation see [WKO2, Section 4.1.2].

In $\widetilde{wTF}$ all relations should be interpreted in all possible combinations of strand types and orientations (tube or string), for example the lower strand of the R2 relation can either be thick black or thin red, as shown below:

R2: 

Similarly, any of the lower strands of the R3, R4, and OC relations may be thin red.

As in wTF^o, the relations all have local topological meaning and conjecturally $\widetilde{wTF}$ is a Reidemeister theory for ribbon knotted tubes in $\mathbb{R}^4$ with caps, singular foam vertices and attached strings. For example, Reidemeister 2 with a thin red bottom strand is imposed because a point flying in through a ring and then immediately flying back out is isotopic to not having any interaction between the point and ring at all.

It is easy to verify that all relations represent local isotopies of welded (ribbon knotted) tubes in $\mathbb{R}^4$ with singular vertices and attached strings. What is not clear at this stage is that this is a complete Reidemeister theory, that is, whether this is a complete set of relations. For more detail on this see [WKO2, Section 1.2].

2.3. **The operations.** Like wTF^o, $\widetilde{wTF}$ is equipped with a set of auxiliary operations in addition to the circuit algebra structure.

The first of these is orientation reversal. For the thin (red) strands, this simply means reversing the direction of the strand. For the thick strands (tubes), orientation switch comes in two versions. Recall from [WKO2, Section 3.4] that in the topological interpretation of wTF^o, each tube is oriented as a 2-dimensional surface, and also has a distinguished "core": a line along the tube which is oriented as a 1-dimensional manifold and determines the "direction" or "1-dimensional orientation" of the tube. Both of these are determined by the direction of the strand in the circuit algebra, via Satoh's tubing map.

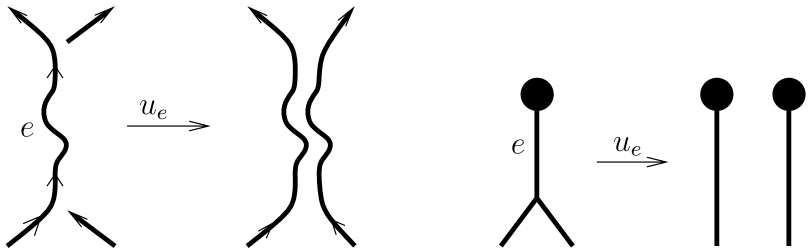
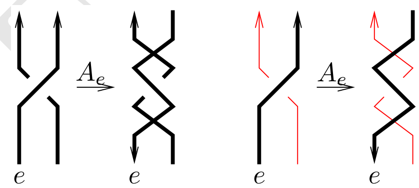


Figure 3. Unzip and disc unzip.

Topologically, the operation "orientation switch", denoted S_e for a given strand e, acts by reversing both the (1-dimensional) direction and the (2-dimensional) orientation of the tube e. Diagrammatically, this corresponds to simply reversing the direction of the corresponding strand e.



The "adjoint" operation, denoted A_e, on the other hand only reverses the (1-dimensional) direction of the tube e, not the orientation as a surface. Diagrammatically, this manifests itself as reversing the strand direction and adding two virtual crossings on either side of each crossing where e crosses *over* another strand, as shown on the right (note that the strand below e may be thick or thin). Note that virtual crossings don't appear when e crosses under another strand. For more details on orientations and orientation switches, see [WKO2, Sections 3.4 and 4.1.3].

The unzip operation u_e doubles the strand e using the blackboard framing, and then attaches the ends of the doubled strand to the connecting ones, as shown in Figure 3. We restrict unzip to strands whose two ending vertices are of different signs. (For the definition of crossing and vertex signs, see [WKO2, Sections 3.4 and 4.1].) Topologically, the blackboard framing of the diagram induces a framing of the corresponding tube in $\mathbb{R}^4$ via Satoh's tubing map, and unzip is the act of "pushing the tube off of itself slightly in the framing direction". Note that unzips preserve the ribbon property.

A related operation, *disc unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagram world, in the direction of the blackboard framing), as before. An example is shown in Figure 3; see [WKO2, Section 4.1.3] for details on framings and unzips.

So far all the operations we have introduced had already existed in wTF^o. There is also a new operation is called "puncture", denoted p_e, which diagrammatically simply turns the thick black strand e into a thin red one. The corresponding topological picture is "puncturing a tube", i.e., removing a small disk from it and retracting the rest to its core. Any crossings where e passes under another strand are not affected, while crossings in which e is the over strand turn into virtual crossings.

For simplicity, we place a restriction on which strands can be punctured, namely at each (fully thick black) vertex puctures are only allowed for one of the three meeting strands, as shown on the left of Figure 4. More general puctures could be allowed in a theory with more than one kind of "string to tube" vertex. The right of the same figure shows that when puncturing one of the thick strands of a mixed vertex, the puncture "spreads". Topologically, this is because the mixed vertex represents a string attached to a tube, so when puncturing e, the entire tube retracts to its core. Finally, a capped tube disappears when punctured.

p

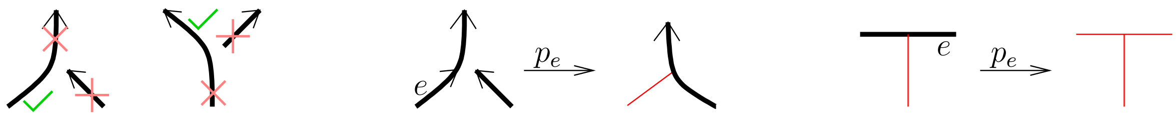


Figure 4. Puncture operations: the picture on the left shows which edges are puncture-able at each vertex. The middle and right pictures show the effect of puncture operations.

fig:punctu

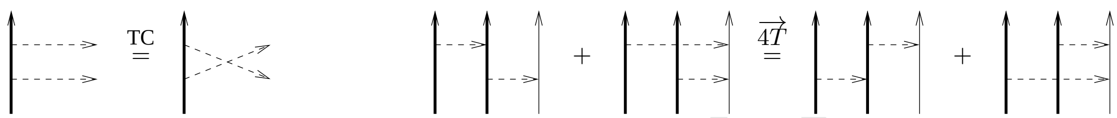
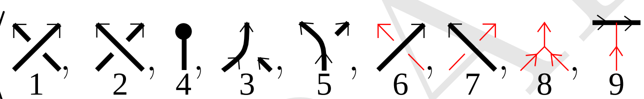


Figure 5. The TC and $\overrightarrow{4T}$ relations. Note that the 3rd strand in each term of the $\overrightarrow{4T}$ relation is ambiguous: it can be either thick black or thin red, the relation applies in either case.

fig:TCand4

In summary,

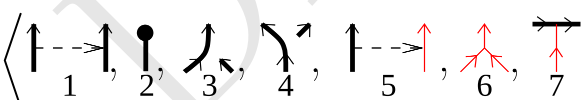
$\widetilde{wTF} = \mathrm{CA}\Big\langle$



$\Big|\ \mathrm{R1}^s, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}, \mathrm{OC}, \mathrm{CP}\ \Big|\ S_e, A_e, u_e, d_e, p_e \Big\rangle$

2.4. **The associated graded structure $\mathcal{A}^{sw}$.** As in [WKO2], the space $\widetilde{wTF}$ is filtered by powers of the augmentation ideal and its associated graded space, denoted $\mathcal{A}^{sw}$, is a "space of arrow diagrams on foam skeletons with strings". As a circuit algebra, $\mathcal{A}^{sw}$ is presented as follows:

$\widetilde{wTF} = \mathrm{CA}\Big\langle$



$\Big|$ relations as below $\Big|$ auxiliary operations as below $\Big\rangle$.

Generators 1 and 5 are called single arrows and they are of degree one, while all others are "skeleton features" of degree zero. The relations are almost the same as in [WKO2, Section 4.2.1], which describes the relations for the associated graded of wTF^o: $\overrightarrow{4T}$ (the 4-Term relation), TC (Tails Commute), RI (Rotation Invariance), CP (the arrow Cap Pullout), and VI (Vertex Invariance). For $\widetilde{wTF}$ there is an additional relation TF (Tails Forbidden on strings). The TC and $\overrightarrow{4T}$ relations are shown in Figure 5. The Vertex Invariance relation is shown in Figure 6: here the $\pm$ signs depend on the strand orientations. Note that the type of the vertex and the types of each strand (thick black or thin red) are left ambiguous: the VI relation applies in all cases. Figure 7 shows the other relations: RI, CP and TF. Note that technically TF is not a relation: there were no generators with an arrow tail on a thin red strand, so saying that such an element vanishes is superfluous. However, without TF the VI relation would have to be stated for all the sub-cases of 0, 1 or 3 thin red strands, so we prefer this cleaner way, even if it is a slight abuse of notation.

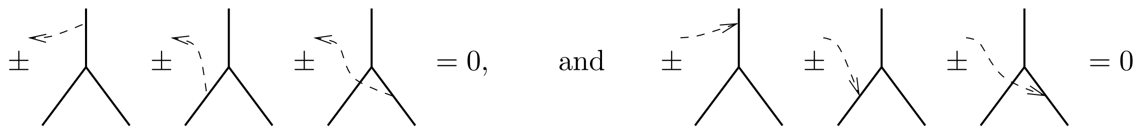


Figure 6. The VI relation: the vertices and strands could be of any type, but the same throughout the relation.

fig:VI

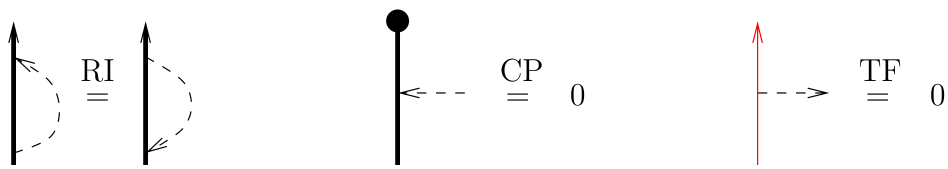
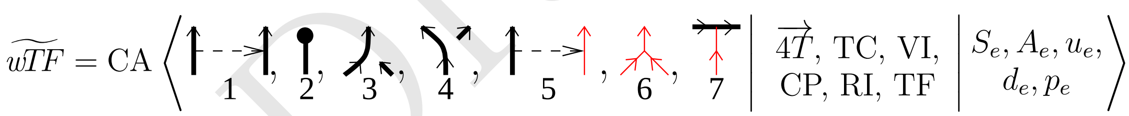


Figure 7. The RI and CP relations, and the TF relation (which is not really a relation).

fig:RICPTF

Each operation on $\widetilde{wTF}$ induces a corresponding operation on $\mathcal{A}^{sw}$. Orientation switch, adjoint, unzip, cap unzip, and long strand deletion act exactly the same way as they do for wTF^{oo}. We quickly recall these here, for details see [WKO2, Section 4.2.2]. The orientation switch S_e reverses the orientation of the skeleton strand e, and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow endings on } e\}}$. The adjoint operation also reverses the skeleton strand e and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow heads on } e\}}$. Given a skeleton S with a distinguished strand e, unzip (or disc unzip, if e is capped) is an operation $u_e : \mathcal{A}^{sw}(S) \to \mathcal{A}^{sw}(u_e(S))$ which maps each arrow ending on e to a sum of two arrows, one ending on each of the two new strands which replace e. Deleting a long strand e kills all arrow diagrams with any arrow ending on e. The operation induced by puncture, denoted p_e, turns the formerly thick black e into a thin red strand, and kills any arrow diagram with any arrow tails on e.

To summarise:



As in [WKO2, Definition 3.7], we define a "w-Jacobi diagram" (or just "arrow diagram") by also allowing trivalent chord vertices, each of which is equipped with a cyclic orientation, and modulo the $\overrightarrow{STU}$ relations of Figure 8. Denote the circuit algebra of formal linear combinations of these w-Jacobi diagrams by $\mathcal{A}^{swt}$. Then, as in [WKO2, Theorem 3.8], we have the following bracket-rise theorem:

Theorem 2.1. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 8 hold in $\mathcal{A}^{swt}$.*

The proof is identical to the proof of [WKO2, Theorem 3.8]. In light of this isomorphism, we will drop the extra "t" from the notation and use $\mathcal{A}^{sw}$ to denote either of these spaces. As in [WKO2], the primitive elements of $\mathcal{A}^{sw}$ are connected diagrams, denoted $\mathcal{P}^{sw}$, and $\mathcal{P}^{sw} = \{\text{trees}\} \oplus \{\text{wheels}\}$ as a vector space. Examples of trees and wheels are shown in Figure 9; for details see [WKO2, Section 3.1]. Note that the RI relation can now be rephrased (via $\overrightarrow{STU}_2$) as the vanishing of the wheel with a single spoke, or one-wheel.

We recall the following two crucial fact [WKO2, Lemmas 4.6 and 4.7]:

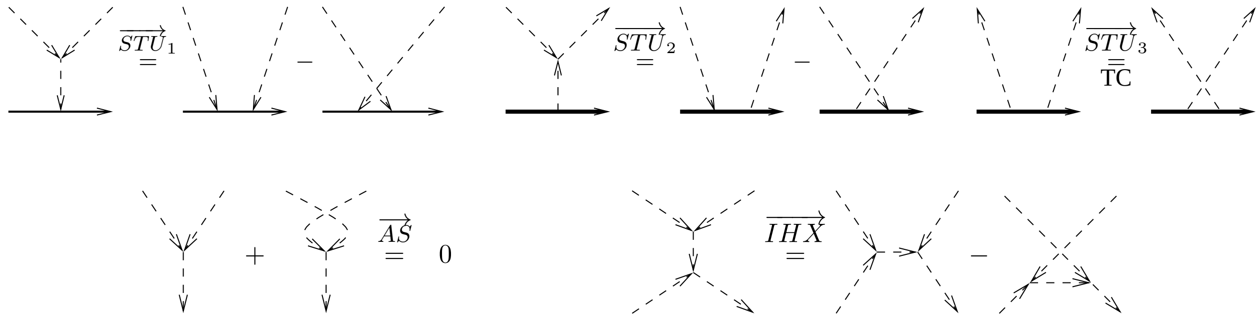


Figure 8. The $\overrightarrow{AS}$, $\overrightarrow{IHX}$ and the three $\overrightarrow{STU}$ rerations. Note that in $\overrightarrow{STU}_1$, the skeleton strand can be thin red or thick black, and that $\overrightarrow{STU}_3$ "degenerates" to the TC relation. fig:ASIHXS

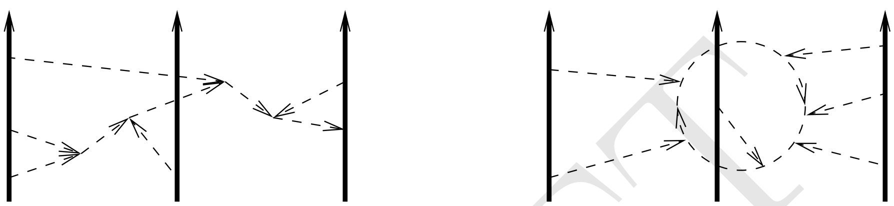


Figure 9. An example of a "tree", left, and a "wheel", right. fig:TreeAn

apIsWheels **Fact 2.2.** *$\mathcal{A}^{sw}(\uparrow)$, the part of $\mathcal{A}^{sw}$ with skeleton a single capped strand, is isomorphic as a vector space to the completed polynomial algebra freely generated by wheels w_k with $k \geq 2$.*

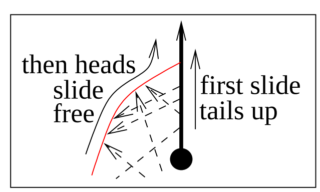
TwoStrands **Fact 2.3.** *$\mathcal{A}^{sw}(\nearrow\!\nwarrow) \cong \mathcal{A}^{sw}(\uparrow_2)$, where $\mathcal{A}^{sw}(\nearrow\!\nwarrow)$ stands for the space of arrow diagrams whose skeleton is a single vertex (the picture shows a positive vertex but the statement is true for all kinds of vertices with thick black strands), and $\mathcal{A}^{sw}(\uparrow_2)$ is the space of arrow diagrams on two (thick black) strands.*

The following Lemma will play an important role, in particular the second isomorphism stated is the map φ appearing in Theorem 1.1, part (2):

:CapString **Lemma 2.4.** *$\mathcal{A}^{sw}\left(\uparrow \cdots \uparrow\right) \cong \mathcal{A}^{sw}(\uparrow_n)$, and consequently $\mathcal{A}^{sw}\left(\nearrow\!\nwarrow\right) \cong \mathcal{A}^{sw}(\uparrow_2)$. The left hand side of the first isomorphism is the space of arrow diagrams on n strands whose bottom is capped and a thin red string is attached above the cap. Similarly, the left hand side of the second isomorphism involves a vertex whose two incoming strands are capped with a string attached.*

Proof. To prove the first isomorphism, we construct inverse maps between the two spaces. There is an obvious map $\mathcal{A}^{sw}(\uparrow_n) \to \mathcal{A}^{sw}\left(\uparrow \cdots \uparrow\right)$, namely, given an arrow diagram on two strands, place it on the capped/stringed strands above the attached string.

In the other direction, given an arrow diagram on capped/stringed strands, first note that in the same vein as Fact 2.2 we can assume that there are only arrow tails on the capped strand under the attached string. Also, arrow tails die on the thin red strand, hence it has only arrow heads. To produce an arrow diagram on regular strands, first push the arrow tails from the capped strand up and over the string/tube vertex.



This is done using the VI relation, but since tails die on the thin red strand they slide past the vertex without a cost. Once the capped side is cleared, continue by pushing the arrow tails up from the thin red string to the strand above the vertex. Note that now the cap kills any arrow heads and hence they also slide past the vertex at no cost. Once the capped and string strands are cleared of any arrow endings, the result can be regarded as an arrow diagram on n regular strands.

It is obvious that the two maps are inverses of each other, so we have proven the first isomorphism. For the second, do the same and combine with Fact 2.3. □

2.5. **The homomorphic expansion.** As discussed in [WKO2, Section 2.3], an expansion is a map $Z^w : \widetilde{wTF} \to \mathcal{A}^{sw}$ with the property that the associated graded map $\operatorname{gr} Z^w : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ is the identity map $\mathrm{id}_{\mathcal{A}^{sw}}$. A homomorphic expansion is an expansion which also intertwines each operation of $\widetilde{wTF}$ with its arrow diagrammatic counterpart, meaning that the appropriate squares commute. In [WKO2, Theorems 4.9 and 4.11] we proved that the existence of solutions for the Kashiwara–Vergne equations implies that there exsists a homomorphic expansion for wTF^o. In fact that homomorphic expansions for wTF^o (with the minor techical condition that the value of the vertex doesn't contain isolated arrows) are in one-to-one correspondence with solutions to the Kahiwara-Vergne problem with even Duflo function.

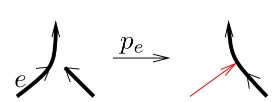
The point of this paper is to provide a topological construction for such a homomorphic expansion (and hence for the Kashiwara–Vergne conjecture), and this is easier to do for the slightly more general space $\widetilde{wTF}$. First we prove that finding a homomorphic expansion for wTF^o is equivalent to finding one for $\widetilde{wTF}$.

Theorem 2.5. *Homomorphic expansions for wTF^o are in one-to-one correspondence with homomorphic expansions for $\widetilde{wTF}$.*

Proof. For the duration of this proof we introduce the following notation: $\mathcal{A}^{sw}$ will denote the associated graded space of wTF^o and $Z^w : wTF^o \to \mathcal{A}^{sw}$ a homomorphic expansion. The associated graded space of $\widetilde{wTF}$ will be denoted by $\widetilde{\mathcal{A}}^{sw}$ and the corresponding homomorphic expansion by $\widetilde{Z}^w : \widetilde{wTF} \to \widetilde{\mathcal{A}}^{sw}$; see the diagram on the right.

$$\begin{array}{ccc} wTF^o & \hookrightarrow & \widetilde{wTF} \\ \downarrow{\scriptstyle Z^w} & & \downarrow{\scriptstyle \widetilde{Z}^w} \\ \mathcal{A}^{sw} & \hookrightarrow & \widetilde{\mathcal{A}}^{sw} \end{array}$$

The horizontal maps of the diagram are the obvious inclusions $wTF^o \hookrightarrow \widetilde{wTF}$ and $\mathcal{A}^{sw} \hookrightarrow \widetilde{\mathcal{A}}^{sw}$, namely the maps which send each generator of wTF^o or $\mathcal{A}^w$ to the same generator of the corresponding extended space. These maps are injective as none of the relations of $\widetilde{wTF}$ or $\widetilde{\mathcal{A}}^{sw}$ change the skeletons of the participating diagrams.



First assume that Z^w is a homomorphic expansion for wTF^o, we want to show that this extends uniquely to a homomorphic extension of $\widetilde{wTF}$. Z^w assigns values in $\mathcal{A}^{sw}$ to all the generators of wTF^o, namely the crossings, vertices and cap; hence we only need to define the value of the extension $\widetilde{Z}^w$ on the remaining generators of $\widetilde{wTF}$: those which involve thin red strands. Note that if such a *homomorphic* extension exists, then it is unique: each generator of $\widetilde{wTF}$ which involves thin red strand can be obtained via puncture operations from generators which do not, see the example in the figure on the right. Since we know the $\widetilde{Z}^w$-value of all the generators which do not involve thin red strands and $\widetilde{Z}^w$ should be homomorphic with respect to the puncture operation, this determines the value of $\widetilde{Z}^w$ on all generators.

It is straightforward to check that these values satisfy all equations arising from the relations of $\widetilde{wTF}$ and $\widetilde{Z}^w$ is indeed an expansion. Furthermore, $\widetilde{Z}^w$ is homomorphic by design, using the homomorphicity of Z^w.

Conversely, given a homomorphic expansion $\widetilde{Z}^w : \widetilde{wTF} \to \widetilde{A}^{sw}$, then the corresponding homomorphic expansion Z^w for wTF^o is the restriction (i.e., the composition of the inclusion of wTF^o into $\widetilde{wTF}$ with $\widetilde{Z}^w$). □

In light of the result above and because this paper focuses on the space $\widetilde{wTF}$, we are going to drop the tildes from $\widetilde{\mathcal{A}}^{sw}$ and $\widetilde{Z}^w$. For the rest of this paper $\mathcal{A}^{sw}$ denotes the associated graded space of $\widetilde{wTF}$ and Z^w a homomorphic expansion $\widetilde{wTF} \to \mathcal{A}^{sw}$.

In the proof above we argued that the values of the punctured vertices can be computed from the values of the vertices which already exist in wTF^o. In fact, more is true: the value of the completely thin red vertex is obviously trivial (thin red strands can only support arrow heads, not tails, hence no arrow diagram can live on a completely thin red skeleton). Moreover, the value of the left punctured vertex turns out to be trivial as well. This fact will be useful later, so we prove it here.

lem:pV **Lemma 2.6.** *For any homomorphic expansion Z^w, $Z^w\Big($*  *$\Big) = 1$, that is, the Z^w-value of a left punctured vertex is trivial, save for possibly short arrows supported on the right strand.*

Proof. Recall from [WKO2, Proof of Theorem 4.9] that the Z^w-value of the (positive, not punctured) vertex V can be written as $V = e^b e^t$, where b consists of wheels only and t (denoted uD in [WKO2]) consists of trees. Puncturing the left strand of V kills all arrow diagrams with tails on the left. This is almost everything: diagrams that survive are only wheels and short arrows supported entirely on the right strand, and arrows pointing from the right to the left. (Note: if all tails of a tree are supported on one strand, then the tree is a single arrow, otherwise it is killed by the anti-symmetry of the trivalent arrow vertices.) Observe that all of the surviving arrow diagrams commute with each other.

Let us then denote the value of the punctured vertex by $p_1V = e^{p_1(b)}e^{p_1(t)}$. Recall that V must satisfy the "unitarity" equation [WKO2, Equation (12)]: $V \cdot A_1A_2(V) = 1$, where A_i denotes the adjoint operation performed on strand i. Hence we have $p_1V \cdot A_1A_2(p_1V) = 1$. Since wheels have only tails, $A_1A_2(p_1(b)) = p_1(b)$. Each arrow has one head, so $A_1A_2(p_1(t)) = -p_1(t)$. Hence, using commutativity, $p_1V \cdot A_1A_2(p_1V) = e^{2p_1(b)} = 1$, which implies that $p_1(b) = 0$. As for $p_1(t)$, showing that there are no arrows pointing from the right to the left strand is a direct computation in degree 1.

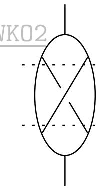
In [WKO2, Proof of Theorem 4.9] we showed that short arrows supported on one strand of V don't affect whether Z^w is a homomorphic expansion. In other words, if Z^w is a homomorphic expansion and a is a linear combination of such short arrows, then replacing V by e^aV gives rise to another homomorphic expansion. In this sense short arrows are unimportant in the value of V, and when they don't exist then $p_1V = 1$. □

3. Proof of Theorem 1.1

1TreeProof

3.1. **Tree level proof of Part (1): Z^u determines V^{tree}.** Let $\mathcal{A}^{tree}$ denote the quotient of $\mathcal{A}^{sw}$ by all wheels, and let $\pi : \mathcal{A}^{sw} \to \mathcal{A}^{tree}$ denote the quotient map (cf [WKO2, Section

3.2]). Part (1) of the main theorem states that if a "global" homomorphic expansion Z exists, then it is determined by Z^u, in other words, Z^w is determined by Z^u. Z^w on the other hand is given by the values V and C of the positive vertex and the cap [WKO2, Sections 4.3 and 4.5], so one only needs to show that V and C are determined by Z^u. Proving this "on the tree level" means showing only that $\pi(V)$ and $\pi(C)$ are determined by Z^u. We prove this partial result first since this contains the main idea without any clouding details. In particular, observe that since C consists entirely of wheels (Fact 2.2) we have $\pi(C) = 1$, so we only need to study $\pi(V)$.



Let B^u denote the "buckle" *sKTG*, as shown on the right (ignoring the dotted lines). All edges are oriented up, and by the drawing conventions of [WKO2, Section 4.6] all the vertices in the bottom half of the picture are negative and all the ones in the top half are positive. Let $B^w = a(B^u)$ be the corresponding element of $\widetilde{wTF}$. Let $\beta^u := Z^u(B^u)$, and note that β^u can be thought of as a chord diagram on four strands, as VI relations can be used to push all chord endings to the "middle" of the skeleton (between the dotted lines on the picture). We use the shorthand abusive notation $\beta^u \in \mathcal{A}^u(\uparrow_4)$, even though four vertical strands don't form an *sKTG*. Let $\beta^w = \alpha(\beta^u)$, and note that by the homomorphicity of Z we also have $\beta^w = Z^w(B^w)$. We are going to perform a series of operations on B^w and $\pi(\beta^w)$ which will allow us to recover $\pi(V)$ from it.

First, 'multiply' B^w by a positive vertex at the bottom, as shown in Figure 10. This is a circuit algebra operation, and correspondingly $\pi(\beta^w)$ is "circuit algebra multiplied" by $\pi(V)$, where V is the value of the vertex. Then unzip the edge marked by u, and puncture the edges marked e and e' (in that order). Then attach a cap to the thick black end at the bottom. This is a circuit algebra operation; keep in mind that image of the value of the cap is trivial in $\mathcal{A}^{trees}$. Finally, perform a disc unzip on the cap, followed by unzipping the red strand marked by u.

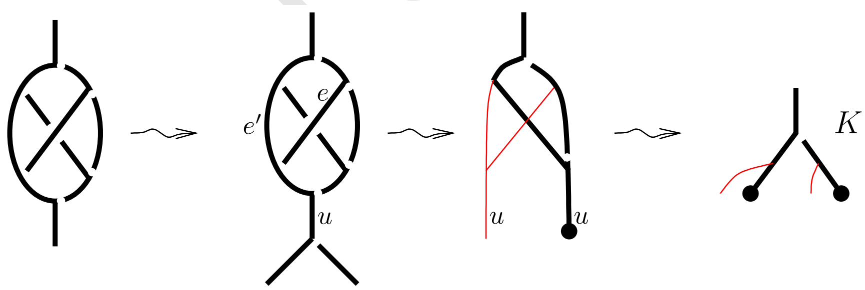


Figure 10. From the "buckle" to the vertex.

Let us call the resulting w-tangled foam K, as in Figure 10. What is $Z^w(K)$? Due to the homomorphicity of Z, it is obtained from β^w by performing the series of operations described above. Unfortunately, one of these operations was multiplication by $\pi(V)$, which is what we are trying to compute. However, notice that the left strand of that vertex later got punctured, and hence by Lemma 2.6 its value cancels, save for possible short arrows on its right strand. In turn, these cancel when the right strand is capped. Hence, $Z^w(K)$ can be computed from β^w by performing punctures and unzips, and since $\beta^w = \alpha(\beta^u)$, this means that $Z^w(K)$ is determined by Z^u.

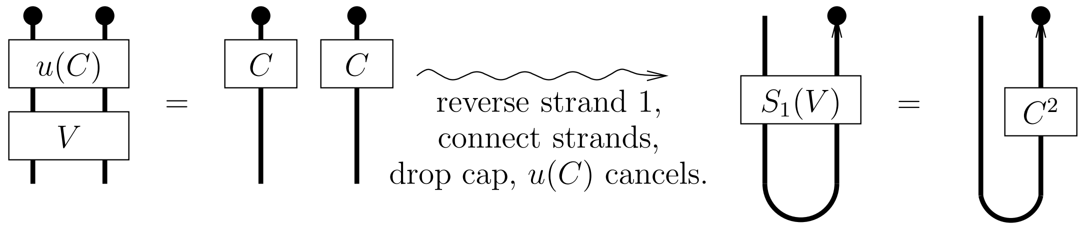


Figure 11. The Cap Equation and an implication. ***FIX: JUST CANCEL uC ON THE LEFT, NO REARRANGE NEEDED!***

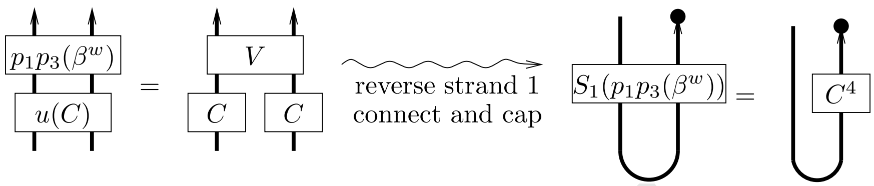


Figure 12. Proof of Part 1 of Theorem 1.1.

On the other hand, note that the space of chord diagrams on the skeleton of K is the space $\mathcal{A}(\uparrow_2)$ by Lemma 2.4. Also note that the Z^w-value of K in $\mathcal{A}^{tree}(\uparrow_2)$ is exactly $\pi(V)$. Hence, $\pi(V)$ is determined by Z^u as needed, so Z^u determines Z^w on the tree level. □

3.2. **Complete proof of Part (1): Z^u determines V and C.** The main idea of the complete proof is the same as in the tree level proof: we compute Z^w from β^u. This amounts to computing the Z^w-value of the foam K of Figure 10 two ways: from $\beta^w = \alpha(\beta^u)$ and from V and C. Then we use the resulting equation to solve for V and C. However, now with one equation and two unknowns, we need some additional information, and this is provided by the Cap Equation [WKO2, Equation (13)]. Recall that the Cap Equation is a relationship between V and C which Z^w must satisfy in order to be homomorphic with respect to disc unzip. The equation is shown on the left side of Figure 11.

Reverse the first strand of the cap equation and connect the two strands as shown in Figure 11. Note that Fact 2.2 implies that $\mathcal{A}^{sw}$ of a single strand capped on both ends is isomorphic to $\mathcal{A}^{sw}(\bullet\!\!\uparrow)$, hence we can ignore the cap at one end of the strand, as in Figure 11. Due to the commutativity of $\mathcal{A}^{sw}(\bullet\!\!\uparrow)$, we can rearrange the left side so that $S_1u(C)$ is at the bottom, as shown. Note that for any arrow diagram D, $S_1(u(D)) = 1$ when placed on a U-shaped strand as in Figure 11, by defitition of the unzip operation, so $S_1u(C)$ cancels. On the right side, recall from [WKO2, Section 4.5.3] that by the CW relation C is even (i.e. consists of even wheels only). Hence, $S_1(C) = C$.

Recall from the tree level proof (Section 3.1 that $Z^w(K) = u(C) \cdot p_1p_3(\beta^w) \in \mathcal{A}^{sw}(\uparrow_2)$, where p_1 and p_3 stand for punctures of the first and third strands of β^w. On the other hand, computing $Z^w(K)$ directly from V and C, its value is a vertex multiplied by two caps in $\mathcal{A}^{sw}(\uparrow_2)$, as shown in Figure 12, and these two results are equal by the compatibility of Z^w with Z^u.

Now reverse strand 1, connect the two strands and cap the top, as shown in Figure 12. As before, $u(C)$ cancels on the left hand side, and on the right we get C^4 since $S_1(V) = C^2$ in $\mathcal{A}^{sw}(\bullet\!\!\uparrow)$. Hence, we have computed C from β^w. Now plugging the value of C into the left side equation of Figure 12, we find V, hence we have computed V from β^w. The claim of

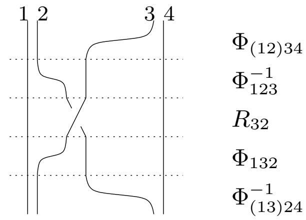


Figure 13. Computing β^b.

Theorem 1.1 was that we can compute V from associators: indeed, it is easy to write β^w in terms of associators, as we shall see shortly in Section 3.3. □

3.3. **Proof of Part (2): the explicit formula.** We compute V from Φ following the method of the proof of Part (1). We start with computing β^u. B^u is the closure of the parenthesized braid B^b shown in Figure 13, therefore it is enough to compute $\beta^b := Z^b(B^b)$. The result can be read off the picture in Figure 13:

$$\beta^b = \Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34}.$$

To interpret this formula, recall that the associator Φ is an element of $\mathcal{A}^{hor}(\uparrow_3)$, and the subscripts show which strands diagrams are placed on (this is often done in the literature using superscripts instead). For example, the notation $\Phi^{-1}_{(13)24}$ means doubling the first strand of Φ^{-1} and placing the resulting chord endings on starnds 1 and 3, as well as placing the chord endings from the other two strands of Φ^{-1} on strands 2 and 4. Also recall that $R = e^{c/2}$, where c is a single horizontal chord between two strands (and in this case R_{32} means that this chord runs between strands 3 and 2). Note that in figures we number the strands at the top and read multiplication bottom to top.

As β^u is the tree closure of β^b, it is given by the same formula interpreted as an element of $\mathcal{A}^u(\uparrow_4)$. In turn, $\beta^w = \alpha(\beta^u)$, and to compute $\pi(V)$, we puncture strands 1 and 3 of β^w, and cap strands 2 and 4. Note that $p_3\alpha(R_{32}) = e^{a_{23}/2}$, where a_{23} is a single arrow pointing from strand 2 to strand 3.

Recall that Φ_{123} is a horizontal chord associator which can be expressed as a power series in non-commuting variables c_{12} and c_{23} (i.e., chords between strands 1—2 and 2—3, respectively). The image of Φ in the quotient where c_{12} and c_{23} commute is 1. Hence, $p_1p_3(\alpha\Phi^{-1}_{123}) = 1$, as the punctured strands can only support heads, and tails on the middle strand commute by TC. Similarly, $p_1p_3(\alpha\Phi_{132}) = 1$ because the punctures kill the entire "left side" of the associator (that is, $p_1p_3\alpha(c_{13}) = 0$).

Thus $p_1p_3(\beta^w)$ can be expressed as $\alpha(\Phi^{-1}_{(13)24})e^{a_{23}/2}\alpha(\Phi_{(12)34})$. Since strands 1 and 3 are punctured, no arrows can be supported between these strands, hence $p_1p_3\alpha(\Phi_{(12)34}) = p_1p_3\alpha(\Phi_{234})$.

Observe that the punctured strand 3 can only support arrow heads. Expressing Φ as a power series in two variables, $\Phi_{234} = \Phi(c_{23}, c_{34})$, and $p_1p_3\alpha(\Phi(c_{23}, c_{34})) = \Phi(a_{23}, a_{43})$, where a_{23} is an arrow pointing from strand 2 to strand 3, and a_{43} is an arrow pointing from strand 4 to strand 3. Similarly, $\Phi^{-1}_{(13)24} = \Phi^{-1}(c_{(13)2}, c_{24})$, where $c_{(13)2} = c_{12} + c_{32}$. In turn, by a property of associators, $\Phi^{-1}(c_{(13)2}, c_{24}) = \Phi^{-1}(c_{(13)2}, -c_{(13)2} - c_{(13)4})$, and so $p_1p_3\alpha\Phi^{-1}_{(13)24} = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$.

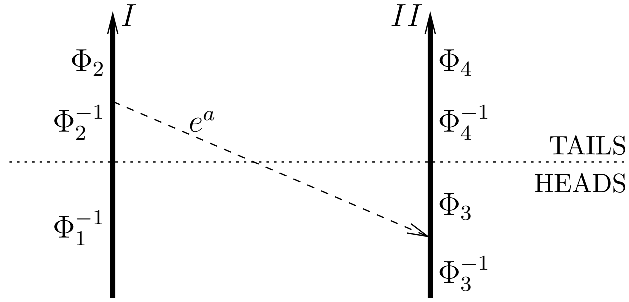


Figure 14. The value of $\pi(V)$ in terms of associators. Here Φ_1^{-1} stands for all the arrow heads on strand 1 in $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. Similarly Φ_2^{-1} denotes the arrow tails on strand 2 in the same formula, Φ_2 stands for the arrow tails on strand 2 of $\Phi(a_{23}, a_{43})$, etc. fig:ValueV

To summarise,

$$p_1p_3\beta^w = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}). \tag{2}$$

eq:wBuckle

Applying the isomorphism φ of Lemma 2.4 results in an arrow diagram on two strands. To obtain V, this result is multiplied by the appropriate cap values as in Figure 12, namely C^{-1} on each strand followed by $u(C)$. This gives the formula stated in Theorem 1.1. To make the formula truly explicit, the value of C could be computed at this point from the reasoning of Figure 12, but we skip this step here and will later see that it is in fact $\alpha(\nu^{1/4})$, where ν is the Kontsevich integral of the unknot.

To match this with the [AET] formulas, first note that those only concern the tree-level part $\pi(V)$. To be more precise, let $V = e^b e^{u(D)}$ as in Section 4.4 of [WKO2], where b is a "power series" of wheels in the sense of Fact 2.2, and $u(D)$ is a primitive element in $\mathcal{A}^{tree}$. Then $F = e^D \in TAut_2$ is a solution to the Kashiwara-Vergne problem in the sense of [AET]. For details see[4] Section 4 of [WKO2]. Thus, to match our formulas to those of [AET], we only need to worry about $\pi(V)$.

The main task is to compute the isomorphism φ of via Lemma 2.4 explicitly. The first strand of $\mathcal{A}^w(\uparrow_2)$ joins strands 1 and 2 in a vertex, and the second strand of $\mathcal{A}^w(\uparrow_2)$ joins strands 3 and 4. Let us call the two strands of $\mathcal{A}^w(\uparrow_2)$ strand I and strand II to avoid confusion. Recall from the construction of ϕ in Lemma 2.4 that one first slides arrow tails "up" through the vertices from the capped strands 2 and 4 (according to the forulas above there are no arrow heads on the capped strands), then one slides all the heads up from the punctured strands 1 and 3. Hence, one obtains an element of $\mathcal{A}^w(\uparrow_2)$ in which all arrow heads are below all tails on both strands. The result of this process is shown in Figure 14.

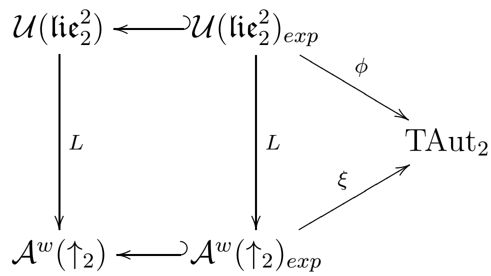
Next, we need to find the corresponding element $F \in \mathrm{TAut}_2$, and check that it agrees with the formula of [AET, Theorem 4]. According to this formula[5],

$$F = (\Phi^{-1}(x, -x-y), e^{-(x+y)/2}\Phi(-x-y, y)e^{y/2}), \tag{3}$$

eq:AET

[4] There are some notational differences between [AT] and [AET], hence we don't switch strands here as we did in [WKO2]. The value V and associator Φ are the same throughout this series of papers, in particular they agree with [WKO2] and [WKO4]. For computatinal implementation and verifications see [WKO4].

[5] There are sign differences from [AET] due to notational misalignment, for example our Φ is [AET]'s Φ^{-1}. Our notation is consistent with all other papers in this series and the formulas are computationally verified in [WKO4].



meaning that if x and y are the generators of the degree completed free Lie algebra $\mathfrak{lie}_2$, then F conjugates x by $\Phi^{-1}(x, -x-y)$, and conjugates y by $e^{-(x+y)/2}\Phi(-x-y, y)e^{y/2}$.

Section 3.2 of [WKO2] described the relationship between the primitives of $\mathcal{A}^w(\uparrow_n)$ and $\mathfrak{tder}_n$ in detail. Familiarity with those results will be helful, but we summarize the relevant facts about the relationship between $\mathcal{A}^w(\uparrow_2)$ and TAut$_2$ below.

The map ϕ is the interpretation map assumed in Formula (3): an element $(e^{\lambda_1}, e^{\lambda_2}) \in \mathcal{U}(\mathfrak{lie}_2^2)_{exp}$ is mapped to the automorphism of $\mathfrak{lie}_2$ which sends the generator x of $\mathfrak{lie}_2$ to $e^{-\lambda_1}xe^{\lambda_1}$, and the generator y to $e^{-\lambda_2}ye^{\lambda_2}$. Beware: this is not a group homomorphism! In other words, composition in TAut$_2$ is *not* given by piecewise multiplication of the conjugators.

The map L is defined as follows. A primitive element (λ_1, λ_2) of $\mathfrak{lie}_2^2$ can be represented by a linear combination of "floating trees" with symmetrized ends coloured by the symbols x and y. A "floating tree" is an oriented uni-trivalent graph with two-in-one-out vertices equipped with a cyclic orientation. The leaves (tails) are coloured by the symbols x and y, and hence modulo $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ each tree corresponds to an element in $\mathfrak{lie}_2$. The heads are also coloured by x or y, so there are two kinds of such trees, that is, two factors of $\mathfrak{lie}_2$. In other words, λ_1 represents a (possibly infinite) linear combination of trees whose head is coloured by x and λ_2 represents a linear combination of trees whose head is coloured by y. (Cf. primitive elements of $\mathcal{B}_2^w$ as in [WKO2][Theorem 3.16] and the discussion following it.)

Floating trees can be mapped into $\mathcal{A}^w(\uparrow_2)$ by "attaching its ends" to the appropriate strands, namely, the ends labelled x to first strand and the ends labelled y to the second strand. There is a choice involved in where to attach the head of the tree in relation to the tails on the same strand. In [WKO2][Proof of Proposition 3.19] we defined a map l which attaches the head of each tree below (*l*ower than) all of its tails; l is a Lie algebra map from $\mathfrak{lie}_2^2$ to the primitives of $\mathcal{A}^w(\uparrow_2)$.

The map L is an extension of l to $\mathcal{U}(\mathfrak{lie}_2^2)$: an element of $\mathcal{U}(\mathfrak{lie}_2^2)$ is a sum of products (disjoint unions) of coloured trees and L attaches the ends to the appropriate strands by attaching *all of the arrow heads* in a product below all of the arrow tails. Beware: $L \neq exp(l)$, and L is not a homomorphism of algebras or groups.

However, the maps L and ϕ fail to be group homomorphisms "in exactly the same way": there is a group homomorphism $\xi : \mathcal{A}^w(\uparrow_2)_{exp} \to$ TAut$_2$ which makes the triangle commute. The map ξ is the exponential of the action of the primitives of $\mathcal{A}^w(\uparrow_2)$ on $\mathfrak{lie}_2$ described in the "Conceptual argument" for [WKO2][Proposition 3.19]. Namely, an element $D \in \mathcal{A}^w(\uparrow_2)_{exp}$ acts on $\mathfrak{lie}_2$ the following way. Add an extra strand. Reresent an element $v \in \mathfrak{lie}_2$ by a tree whose head is on the extra strand (more precisely, a linear combination of such trees). Its conjugae $D^{-1}vD$ is once again a linear combination of such trees, defined to be the output of the action. The reader can check that this is indeed a hoomorphism and it makes the triangle commute.

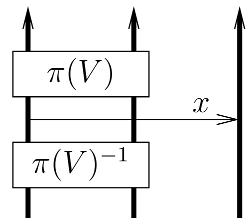


Figure 15. The action of $\pi(V)$ on the generator x of $\mathfrak{lie}_2$.

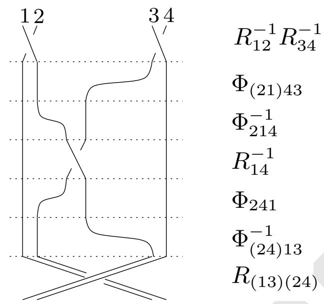


Figure 16. A different expression of β^b.

Now we are ready to compute how $\pi(V) \in \mathcal{A}^w(\uparrow_2)$ acts on the generator x of $\mathfrak{lie}_2$ and match this to the formula 3. Recall the value of $\pi(V)$ shown in Figure 14. The generator x is represented by an arrow from the first strand to the added third strand, and the result of the action is $\pi(V)^{-1}x\pi(V)$, as shown in Figure 15. To compute this, one commutes the tail of x up to the top of the strand across $\pi(V)$ using $\overrightarrow{STU}$ relations. At the end of this process, $\pi(V)$ and $\pi(V)^{-1}$ cancel, and the result of the action remains. Observe that due to the TC relation, only the heads on strand I act on x, in other words the action is by $\Phi_{124}^{-1} = \Phi^{-1}(a_{21}, -a_{21} - a_{41})$, arranged on strands I and II as shown in Figure 14. An L-preimage of this element is exactly $\Phi^{-1}(x, -x-y)$ in the first component, as in Formula (3).

As for the action of F on y, using the expression of Figure 13 results in a formula different from but equivalent to (3). To obtain (3), one needs to use a different expression of the "buckle braid" β^b, as shown in Figure 16.

Simplifying the formula as before, we have that after the cap and puncture operations $R_{(13)(24)} = e^{a_{(24)(13)}/2}$, $\Phi_{(24)13}^{-1} = 1$, $\Phi_{241} = \Phi(-a_{41} - a_{21}, a_{41})$, $R_{14}^{-1} = e^{-a_{41}/2}$, $\Phi_{214}^{-1} = \Phi^{-1}(a_{21}, a_{41})$, $\Phi_{(21)43} = \Phi(-a_{43} - a_{23}, a_{43})$, $R_{12}^{-1} = e^{-a_{21}/2}$, and $R_{34}^{-1} = e^{-a_{43}/2}$.

Using Lemma 2.4 slide arrow endings up to strands I and II, first from strands 2 and 4, then from strands 1 and 3. Since we are computing the action on y, the only part of the result which matters is arrow heads on strand II. These all come from strand 3, namely, read from bottom to top they are $e^{(a_{23}+a_{43})/2}$, followed by $\Phi(-a_{43} - a_{23}, a_{43})$, followed by $e^{-a_{43}/2}$. Taking the L-preimage we obtain $e^{(x+y)/2}\Phi(-x-y, y)e^{-y/2}$, as stated in Formula 3.

3.4. **Proof of part (3): the double tree construction.** What remains is to prove that the values of V and C above indeed give rise to a homomorphic expansion of $\widetilde{wTF}$. Unfortunately, checking directly that they satisfy all necessary equations is very difficult.

Note that most of the equations arise from relations that assert an equality between different planar algebra compositions of generators. For example, the crucial $R4$ relation is of

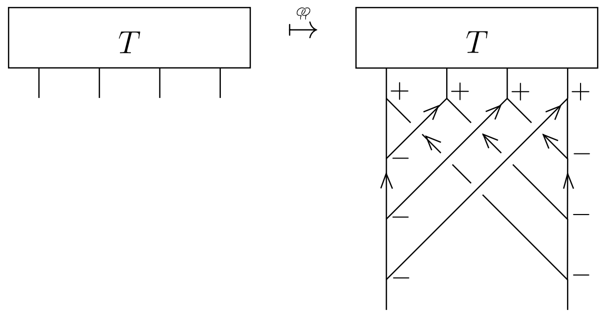
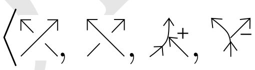


Figure 17. The *double tree map*: connect the ends of T by two binary trees (hence "double tree"), as shown. Note that the tree on the left always crosses over the tree on the right, and all edges of both trees are oriented towards T.

this kind. Hence, life would be much easier if Z^u were to be a planar algebra map. This unfortunately makes no sense, as *sKTG* is not a planar algebra but a different, more complicated algebraic structure. The reader might ask, why work with a space as inconvenient as *sKTG* instead of, say, the planar algebra of ordinary trivalent tangles? The answer is that the existence of a homomorphic expansion is a highly non-trivial property that is not guaranteed, and in particular ordinary trivalent tangles do not have one. Even without trivalent vertices, ordinary tangles, or u-tangles, do not have have a homomorphic expansion. Only *parenthesized tangles* (a.k.a. *q-tangles*) do, and in fact these are almost equivalent to *sKTG*.

Nonetheless, "wishing that Z^u were a planar algebra map" leads to the following strategy: we map ordinary trivalent tangles into *sKTG* via a *double tree* construction, and use this to define Z^w for the a-image of all usual trivalent tangles. Then we'll use the planar algebra structure to prove that this Z^w is a homomorphic expansion, and finally show that this Z^w is in fact the same as the one arising from part (1).

We start by defining usual trivalent tangles, to be denoted uTT:

$uTT :=$ PA $\langle$  $\mid$ R1^s, R2, R3, R4 $\rangle$

Here PA stands for *planar algebra*, which in our context is the algebraic structure similar to a circuit algebra, but with *planar* wiring diagrams. This is a slightly more simple-minded notion than the standard use of the term in the literature, in particular we do not use checkerboard shadings. The elements of uTT can be seen as usual (i.e., classical) trivalent tangles with signed vertices and no virtual crossings, modulo all of the relations which make sense in this context.

We define a *double tree* map $\varphi : uTT \to sKTG$, as in Figure 17. The map φ depends on two choices of binary trees: in Figure 17 we chose a particular example. It is important that, regardless of this choice, the "left side" tree always crosses over the "right side" tree. We will demonstrate that in fact the choice of trees becomes irrelevant after some post-compositions, as in Equation (4).

Working towards a construction of Z^w, we post-compose φ with the following sequence of maps:

$$T \in uTT \xrightarrow{\varphi} sKTG \xrightarrow{Z^u} \mathcal{A}^u(\varphi(T)) \xrightarrow{\alpha} \mathcal{A}^{sw}(\varphi(T)) \xrightarrow{C,u,p} \mathcal{A}^{sw}(\check{T}) \cong \mathcal{A}^{sw}(T). \qquad (4)$$

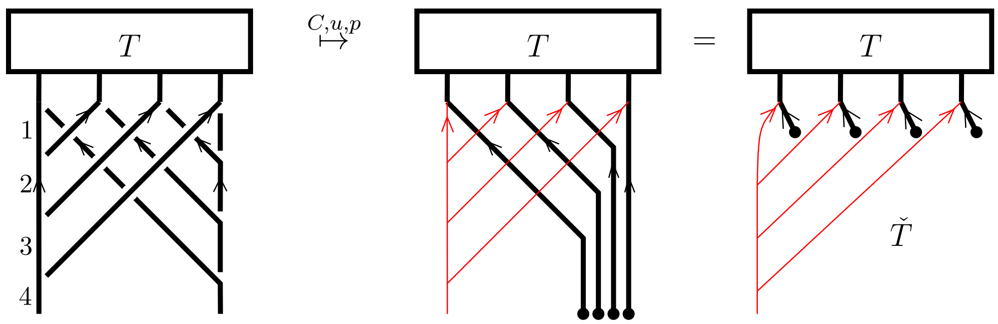


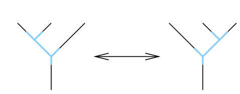
Figure 18. The punctures, caps and unzips. While these operations are applied in $\mathcal{A}^{sw}$, the picture only shows the effect on the skeleton.

Here T stands for an arbitrary tangle in uTT. The double tree construction maps T into sKTG, and by applying Z^u one obtains a value in $\mathcal{A}^u$, namely a chord diagram on the skeleton of $\wp(T)$. We denote the space of chord diagrams on this skeleton by $\mathcal{A}^u(\wp(T))$. Now α maps this to arrow diagrams on the skeleton of $\wp(T)$, that is, $\mathcal{A}^{sw}(\wp(T))$. In order to revert the skeleton back to that of T, we apply some operations in $\mathcal{A}^{sw}$: a cap attachment, unzips and punctures (as shown in Figure 18 and explained in the next paragraph), resulting in a slightly modified version of the desired skeleton, denoted $\check{T}$. Finally, we use that $\mathcal{A}^{sw}(\check{T}) \cong \mathcal{A}^{sw}(T)$ via the isomorphism of Lemma 2.4, and hence we obtain a value in $\mathcal{A}^{sw}(T)$, as needed. Note that the proof of Lemma 2.4 applies even though the punctured strands all connect in a binary tree: VI relations can be used as part of the isomorphism.

The puncture, cap and unzip operations are done in the following order: first attach a cap – a capped strand with no arrows on it – to the end of the right vertical strand: this is a circuit algebra operation in $\mathcal{A}^{sw}$. If T has n ends, perform $(n-1)$ consecutive disc unzips on the right vertical strand, as shown in Figure 18. Then puncture the strand marked with "1" on the left of Figure 18. Then puncture strands 2, 3, and 4 in that order, and recall that these punctures also spread to the connecting diagonal strands, as in Figure 4. Punctures could be done in any other "legal" order without changing the result. Note that since the punctured tree had originally always crossed over the capped tree, these crossings become virtual after puncturing, hence the equality in Figure 18.

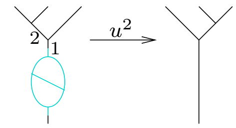
Let us denote the composition of the maps and operations shown in Equation (4) by ξ, that is, $\xi = \varphi \circ p \circ u \circ C \circ \alpha \circ Z^u \circ \wp$, where φ denotes the isomorphism of Lemma 2.4. To summarise, $\xi(T) \in \mathcal{A}^{sw}(T)$. The next task is to show that $\xi(T)$ is well-defined, namely, it doesn't depend on the choice of binary trees.

Lemma 3.1. *The choice of binary trees in the double tree construction does not affect $\xi(T)$.*

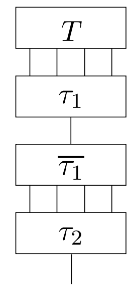


Proof. Any binary tree can be changed into any other binary tree via a sequence of "I to H" moves, as shown on the right. Hence, it is enough to analyze how an I to H move on one of the trees affects the value of $Z^u(\wp(T))$, and prove that the difference does not "survive" the cap and puncture operations.

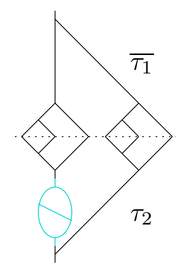
Suppose τ_1 and τ_2 are two binary trees which differ by a single I to H move, and let φ_{τ_1} and φ_{τ_2} denote the two resulting double-tree maps, assuming the tree on the other side is the same. The I to H move can be realised by inserting[6] an associator, followed by unzipping first the edge marked '1' on the right, then the edge marked '2'. Hence, by the homomorphicity of Z^u, the values $Z^u(\varphi_{\tau_2}(T))$ and $Z^u(\varphi_{\tau_1}(T))$, only differ in an inserted horizontal chord associator Φ on the three strands involved. In sloppy notation, $Z^u(\varphi_{\tau_2}(T)) = Z^u(\varphi_{\tau_1}(T)) * \Phi$. If the I to H move was done on the left side tree, then all the strands involved are later punctured, killing any arrow diagram that lived on them by the TF relation. As a result, the only surviving part of Φ is its constant term, 1, and the resulting values of ξ are equal.



If the I to H move is done on the right side tree, then the all participating strands are capped and disk unzipped. Arrow heads adjacent to a cap are zero by the CP relation, but there may be other arrow endings separating $\alpha(\Phi)$ from the caps, preventing it from being eliminated. To get around this roblem we use the trick shown on the right: an equivalent way to construct the right side tree of $\varphi_{\tau_2}(T)$ is to attach τ_1 under T, then τ_1 upside down (denoted $\overline{\tau_1}$), then τ_2. Unzip the edge connecting τ_1 and $\overline{\tau_1}$ several times until all vertices of τ_1 are gone, the result is just straight lines connecting T and τ_2, that is, $\varphi_{\tau_2}(T)$.



Now recall that τ_2 is obtained from τ_1 by inserting an associator (and unzipping). In the language of [WKO2, Section 5.2], $\overline{\tau_1}\tau_2$ consists of some "nested bubbles", with a single associator inserted, as shown on the right. This implies[7] that the only chords in $Z^u(\overline{\tau_1}\tau_2)$ are the ones in $Z^u(\Phi)$. Hence, after the cap attachment and disk unzips, $\alpha(\Phi)$ is directly adjacent to the caps, and since Φ is a horizontal chord associator, all positive degree parts of $\alpha(\Phi)$ canel due to the CP relation. This concludes the proof. □



There is an action of $\mathbb{Z}/n\mathbb{Z}$ on elements of $u\!\!\!/T$ with n ends, by cyclic permutations of the ends. The following lemma will be useful later in proving that Z^w is a planar algebra map; we present it now because its proof is similar to that of Lemma 3.1.

CyclicPerm **Lemma 3.2.** *The map ξ is invariant under cyclic permutation of the ends of T.*

Proof. To show that $\xi(T)$ is ivariat under cyclic permutations of ends of T, it is enough to show that $\xi(T)$ does not change when the rightmost end of T is moved to the far left (denote this by σT), as shown in the left versus middle pictures of Figure 19.

The rightmost picture of Figure 19 is equivalent as *sKTG*s to $\varphi(\sigma T)$. It differs from $\varphi(T)$ in three ways:

- the binary trees connecting the ends of T are different;
- two tree branches are connected to the trunk "on the wrong side", that is, these trivalent vertices have opposite cyclic orientation (marked by $*$ in Figure 19);
- one tree branch has a kink in it.

As before, we need to analyse how $Z^u(\varphi(\sigma T))$ differs from $Z^u(\varphi(T))$, and show that the difference doesn't survive the puncture, cap and unzip operations.

[6] See [WKO2, Section 4.6] for a detailed description of tangle insertion in *sKTG*.

[7]Since Z^u of the bubble is trivial and Z^u commmutes with insertions, see [WKO2, Section 5.2].

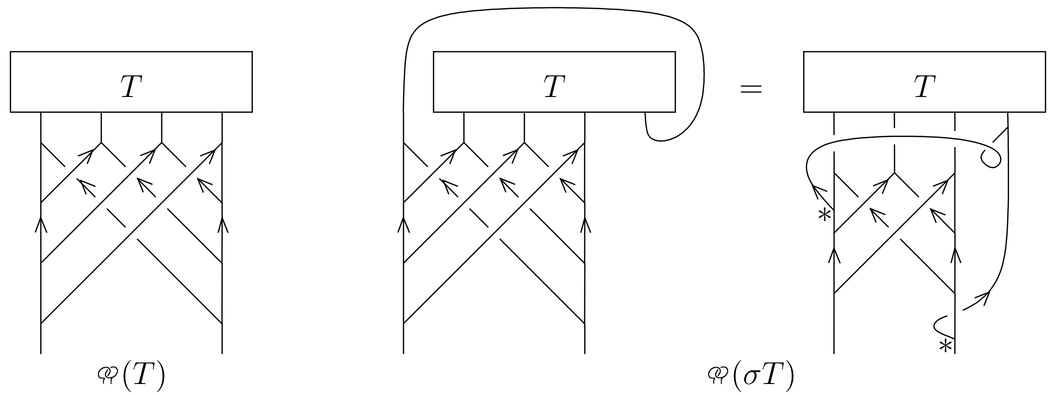


Figure 19. Double tree construction for cyclically permuted ends of T.

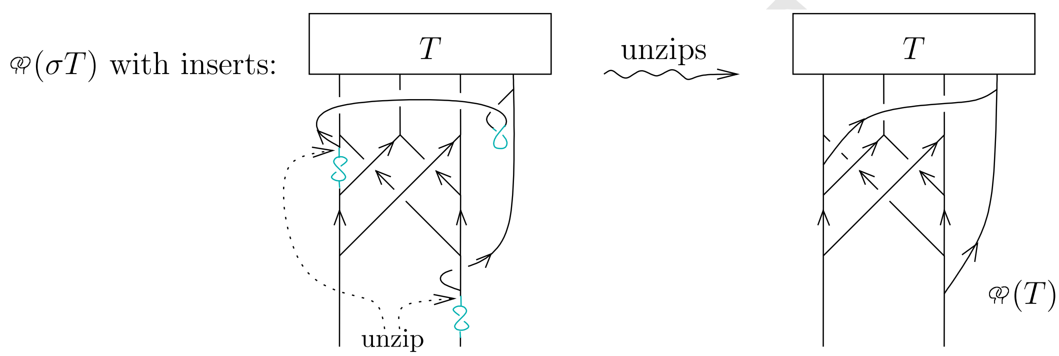


Figure 20. The difference between $\wp(T)$ and $\wp(\sigma T)$, understood via insertions.

To achieve this, we transform $\wp(\sigma T)$ into $\wp(T)$ using tangle insertions. First, cancel the kink by inserting the opposite kink on the same strand, as shown in Figure 20 in blue[8]. As Z^u is compatible with insertion, the Z^u values will differ by the value of a kink: a chord diagram on the one strand involved. Later in the process this strand is punctured, and no arrow diagram can live on a single punctured strand (TF), so the value of the kink cancels.

Similarly, switching the side that the tree branches are attached on amounts to inserting twists and unzipping the connecting edges, as also shown in Figure 20. Each of these operations changes the value of Z^u by inserting the value of a twist: $e^{c/2}$, where c denotes a single chord between the appropriate strands. Applying α maps this to $e^{(a_L+a_R)/2}$, where a_L and a_R denote horizontal left and right arrows, respectively. On the left side tree, this cancels after punctures, as before. On the right side tree, the strand directly underneath the twist is capped and uzipped, and hence the value of the twist cancels by the CP relation.

Now observe that the right side picture of Figure 20 only differs from $\wp(T)$ in the choices of binary trees, which do not change the value of ξ by Lemma 3.1. □

Lemma 3.3. *If there exists a homomorphic expansion Z^w for $\widetilde{wTF}$ compatible with Z^u, then for a tangle $T \in a(uTT)$, the value $Z^w(T)$ is equal to $\xi(T)$ multiplied by C^{-1} at each end of T, where C is the Z^w-value of the cap.*

[8]Or grey in black and white print.

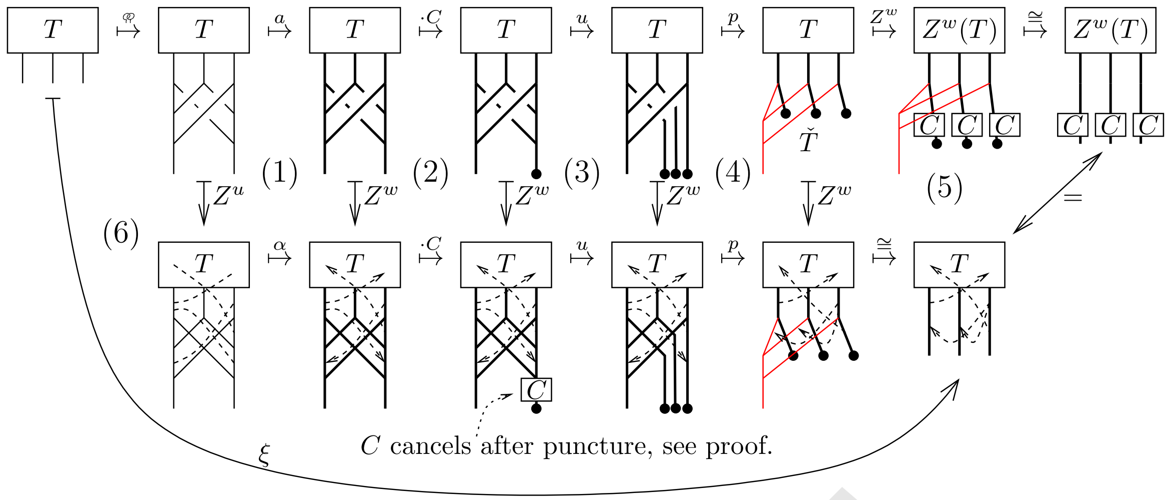
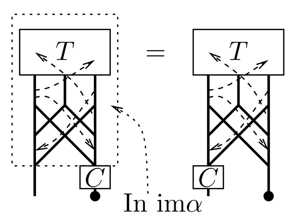


Figure 21. The compatibility of Z^w with Z^u and various operations. The chords and arrows are schematic indications of chord and arrow diagrams, not actual values.

Proof. Assume there exists a homomorphic expansion Z^w compatible with Z^u. We use, as in Figure 21, the homomorphicity of Z^w and its compatibility with Z^u to show that $\xi(T) = Z^w(\check{T})$.

The proof amounts to showing that the four squares, one pentagon and one octagon of the diagram in Figure 21 all commute. Square (1) commutes by the compatibility of Z^u and Z^w. In square (2) the top $\cdot C$ map denotes the circuit algebra operation of attaching a cap at the bottom right end of the w-foam. The bottom $\cdot C$ map denotes the circuit algebra operation of arrow diagrams, which attaches a cap with a value C on the strand, at the same place. The commutativity of this square is implied by the homomorphicity of Z^w. Square (3) is commutative because Z^w commutes with disc unzips. Square (4) commutes because Z^w commutes with punctures.

To explain the horizontal Z^w arrow of the pentagon (5), note that since Z^w is a circuit algebra homomorphism, $Z^w(\check{T})$ can be obtained from $Z^w(T)$ by attaching the Z^w-value of a left-punctured right-capped vertex at each tangle end. By Lemma 2.6 the value of the left-punctured vertex is 1, so the only arrows outside of $Z^w(T)$ are C values at each end, as shown. The horizontal isomorphisms of the pentagon are the map $\mathcal{A}^{sw}(T) \cong \mathcal{A}^{sw}(\check{T})$ of Lemma 2.4. The commutativity of the pentagon is obvious.



The commutativity of the octagon (6) would be true by definition, if not for the insertion of the value C. However, this cancels after punctures, by a property of arrow diagrams in the image of α, called *tail-invariance*. In short, as long as a w-foam is in the image of a, one can slide a strand under it. The arrow diagram implication of this in the current situation is that the value C in the diagram, which has only tails on the skeleton, can be moved from one tangle end to the other, as shown on the right. Consequently, C cancels when the left strand is punctured. For details on tail-invariance, see [WKO2], Remark 3.11 and early in Section 3.3. □

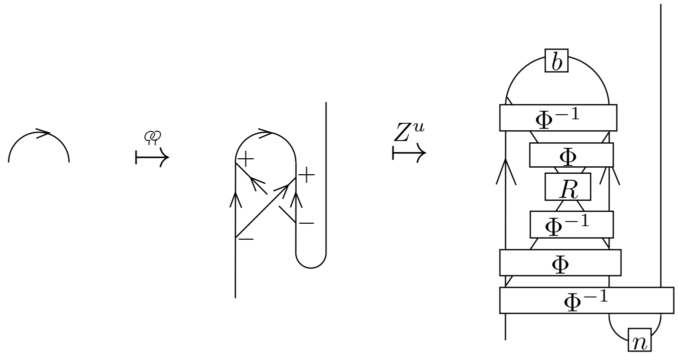


Figure 22. Double tree composed with Z^u, applied to a single strand.

Strategic preview: By Lemma 3.3, if Z^w exists, then one can compute C by evaluating ξ on a single un-knotted strand (denote this by $\uparrow$). Since $Z^w(\uparrow) = 1$ for any homomorphic expansion, we have $\xi(\uparrow) = C^2$, determining the value of C.

To compute the value V of the vertex, which is an element of $\mathit{u}\!\!\!/TT$, evaluate ξ on it, then multiply by C^{-1} at each end. We'll need to show that these agree with the values constructed in Part (1), Section 3.2. More significantly, we'll have to show that Z^w is indeed a homomorphic expansion of $\widetilde{wTF}$. In order to do this, we'll first show that it is a planar algebra homomorphism, which implies that it satisfies all and deal with the cap equation separately. The compatibility of Z^w with Z^u is automatic by design.

We start by computing $\xi(\uparrow)$ and C. Recall Fact 2.2, which implies that C involves wheels only, so – as a reality check – $\xi(\uparrow)$ must consist only of wheels if it is to equal C^2.

Lemma 3.4. *For the un-knotted strand, $\xi(\uparrow) = \alpha(\nu^{1/2})$. Here $\nu \in \mathcal{A}^u(\uparrow)$ denotes the Kontsevich integral of the un-knot, which indeed involves wheels only*[9].

Proof. We apply $\wp$ to $\uparrow$, as shown in Figure 22. Our first goal is to comupte $Z^u(\wp(\uparrow))$. In [WKO2, Section 5.2] we give an algorithm for writing any *sKTG* as a "porduct" of generators, and hence expressing its Z^u value in terms of the Z^u-values of the generators. To feed $\wp(\uparrow)$ into this algorithm, one needs to "curve up" one strand, in this case the strand on the right (this choice doesn't affect the outcome).

The result after applying $\wp$ and Z^u are shown in Figure 22. The value of Z^u is expressed in terms of Φ, a horizontal-chord Drinfel'd associator which is the value of the *associator graph*, the value of the *twist* $R = e^{c/2}$ where c is a single chord, and the values n and b of the *noose* and *balloon* graphs, respectively. See [WKO2, Section 4.6] for details.

In $\xi(\uparrow)$, Z^u is followed by α, a cap attachment, unzips and punctures. As explained in [WKO2, Section 4.6], there is a one-parameter uncertainty in the exact values of n and b, but we do know that and $\alpha(b) = e^{a/2}\alpha(\nu)^{1/2}$, and $\alpha(n) = e^{a/2}\alpha(\nu)^{1/2}$. Note that the exponential part of n cancels by the CP relation once the cap is attached. We "push" most of the arrow diagrams to the middle four strands using the VI relation. The result after α, cap attachment, unzip and puncture is shown in Figure 23 and explained below.

Recall that α maps a chord to the sum of its two possible orientations. However, when one supporting strand is punctured, only one of these orientations survive. Hence, for example, $p_2(\alpha(R_{23}^{-1})) = (e^{-a_{32}/2})$. Figure 23 shows a schematic picture of $puC\alpha Z^u\wp(\uparrow)$ with

[9]The value of ν was conjectured in [BGRT] and proven in [BLT].

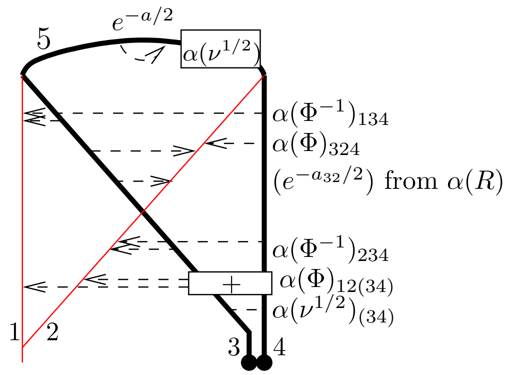


Figure 23. The result after applying α, cap, unzips and puncture.

exponentials and associators indicated by single arrows. To explain the notation for associators, recall that $\Phi \in \mathcal{A}^{hor}(\uparrow_3)$ can be written as a power series in any two of the three generators of $\mathcal{A}^{hor}(\uparrow_3)$: c_{12}, c_{23} and c_{13}. For each associator above, we chose the presentation in which $\alpha(\Phi)$ is of the simplest form. For example, we write the top associator $\Phi^{-1}_{13(24)}$ of Figure 22 in terms of c_{13} and $c_{1(24)} = c_{12} + c_{14}$, since after the puctures $p_1\alpha(c_{13}) = a_{31}$ and $p_1p_2\alpha(c_{1(24)}) = a_{41}$. This is reflected in Figure 23 in drawing only these two arrows for this associator.

Now observe that the two arguments of $\alpha(\Phi)_{12(34)}$ commute by the TC relation: strands 3 and 4 support only tails in this associator. As mentioned before, the value of an associator in the quotient where its two arguments commute is 1, hence $p\alpha(\Phi)_{12(34)} = 1$. Next, study $p\alpha(\Phi^{-1})_{134}$: the tail of an arrow a_{41} can be "pulled over the top on strand 5" using the VI relation and the fact that $e^{-a/2}\alpha(\nu)$ is a local arrow diagram on one strand and hence it is central. So $a_{41} = a_{31}$, meaning in particular that a_{41} commutes with a_{31}, and so $p\alpha(\Phi^{-1})_{134} = 1$. Applying the same trick also cancels $\alpha(\Phi)_{324}$, and $\alpha(\Phi^{-1})_{234}$. In the latter case note that the tail of a_{42} also commutes with the tails of $\alpha(R^{-1})$. In summary, all associators cancel.

Next we show that $\alpha(\nu^{1/2})_{(34)}$, which remains from n, cancels as well. Since ν is an exponential of wheels, so is $\alpha(\nu^{1/2}) \in \mathcal{A}^{sw}(\uparrow)$. Recall from [WKO1, Section 3.8] that wheels in $\mathcal{A}^{sw}$ have two possible orientations. For odd wheels these are negatives of each other by the AS relation, for even wheels they are equal. Hence, α kills odd wheels and multiplies even wheels by 2, as well as orienting them. Let us write $\alpha(\nu^{1/2})$ as $e^{w(x)}$, where $w(x)$ is an (even) power series in x with constant term 0: to interpret this as an element of $\mathcal{A}^{sw}(\uparrow)$, expand it and interpret each monomial x^k as a k-wheel on the single strand. Then by the action of unzip, $\alpha(\nu^{1/2})_{(34)} = e^{w(x_3+x_4)}$, where each monomial is interpreted as a cyclic word which is in turn interpreted as a wheel on strands 3 and 4. Now slide this arrow diagram up on strands 3 and 4 to strand 5. Since all associators have canceled, and $\alpha(R^{-1})$ has only tails on strand 3, there is no obstruction to doing this. At the junction of strands 3, 1 and 5, we need to apply the VI relation. Tails on the punctured strand 1 are zero (TF relation), so each tail slides onto strand 5, whose orientation is compatible with strand 3. In other words we replace x_3 by x_5 in the expression. On the other side, tails again slide onto strand 5 but now the orientations are opposite, and hence x_4 is replaced by $-x_5$. In summary, $\alpha(\nu^{1/2})_{(34)} = e^{w(x_5-x_5)} = 1$.

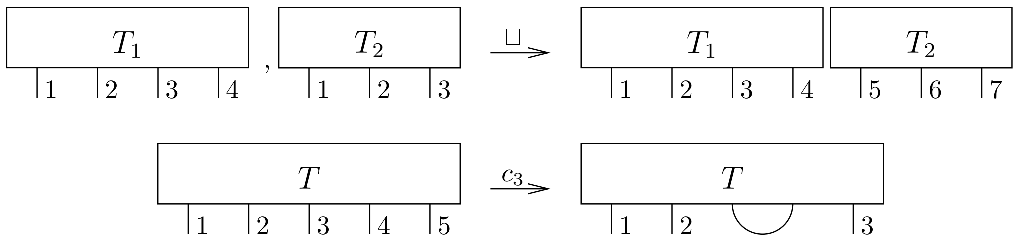


Figure 24. Basic planar algebra operations: disjoint union and contraction.

Finally, move the top exponential $e^{-a/2}$ to strands 3 and 2, using the VI relation at both vertices. The tail of the arrows moves freely from strand 5 to strand 3. The heads commute with $\alpha(\nu)$, they are killed on strand 4 due to the CP relation, so they slide onto strand 2 but acquire a negative sign. Hence, $(e^{-a_{55}/2}) = (e^{a_{32}/2})$, and this cancels $\alpha(R^{-1})$.

To summarize, $\xi(\uparrow) = \alpha(\nu^{1/2})$, as claimed. □

Corollary 3.5. *The value of the cap is* $C = \alpha(\nu^{1/4})$. □

Recall that this lets us define $Z^w(T)$ for any $T \in uTT$, in particular the value V of the vertex: namely, $Z^w(T)$ is defined to be $\xi(T)$ multiplied by the value C^{-1} at each end of T. Hence, we have defined Z^w from Z^u, and it is compatible with Z^u by design. The next step is to prove that Z^w is a homomorphic expansion of $\widetilde{wTF}$. The bulk of the argument relies on the following proposition:

Theorem 3.6. *The restriction of* Z^w *to* $a(uTT)$ *is a planar algebra map.*

Proof. Planar algebra operations can be written as compositions of two simpler, basic operations: disjoint unions and contractions. For two tangles T_1 and T_2, the disjoint union $T_1 \sqcup T_2$ is the disjoint union of the two tangles where the ends are ordered by declaring that the ordered ends of T_1 come first, followed by the ordered ends of T_2. The contraction operation c_i can be applied to any tangle with at least $i+1$ ends, and acts by joining the i-th and $(i+1)$-st ends of T and re-numbering the rest, resulting in a tangle with two less ends. Both operations are shown in Figure 24.

All we need to show then is that Z^w commutes with these two operations, that is, $Z^w(T_1 \sqcup T_2) = Z^w(T_1) \sqcup Z^w(T_2)$, and $Z^w(c_i(T)) = c_i(Z^w(T))$. Note that the right sides of these equalities make sense: arrow diagrams on uTT skeleta, where Z^w takes its values, also form a planar algebra.

Disjoint unions. We need to compute $\xi(T_1 \sqcup T_2)$, where T_1 and T_2 are two uTT-s. The $\wp$ map applied to a disjoint union of T_1 and T_2 is shown in Figure 25. Recall that the trees can be chosen arbitrarily by Lemma 3.1, we chose the most convenient trees for the proof. Observe that $\wp(T_1 \sqcap T_2)$ is the same *sKTG* as $\wp(T_1)$ and $\wp(T_2)$ inserted into a simpler *sKTG*, denoted H, as shown in the same figure (up to some orientation switches which don't impact what follows and hence will be ignored for simplicity). Hence, $Z^u(\wp(T_1 \sqcup T_2))$ is given by inserting $Z^u(\wp(T_1))$ and $Z^u(\wp(T_2))$ into $Z^u(H)$. One could compute $Z^u(H)$ using the same algorithm as before, but we can save ourselves the work, as follows. All chords in $Z^u(H)$ can be assumed to be located in the rectangle shown in Figure 25. After applying α, both strands on which $\alpha(Z^u(H))$ lives is punctured, so after punctures $p\alpha(Z^u(H)) = 1$ in $\mathcal{A}^{sw}$. After the subsequent caps and unzips, T_1 and T_2 are once again separated, hence we have shown that $\xi(T_1 \sqcup T_2) = \xi(T_1) \sqcup \xi(T_2)$, and it follows immediately that $Z^w(T_1 \sqcup T_2) = Z^w(T_1) \sqcup Z^w(T_2)$.

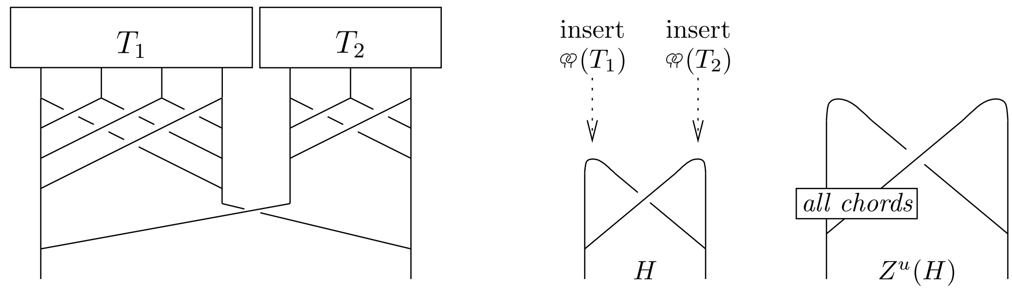


Figure 25. The double tree map applied to a disjoint union of $w\mathcal{TT}$-s is the same as inserting the double tree of each individual $w\mathcal{TT}$ into the $s\mathcal{KTG}$ H. In $Z^u(H)$ all chords can be pushed into the rectangle shown, using VI relations when necessary.

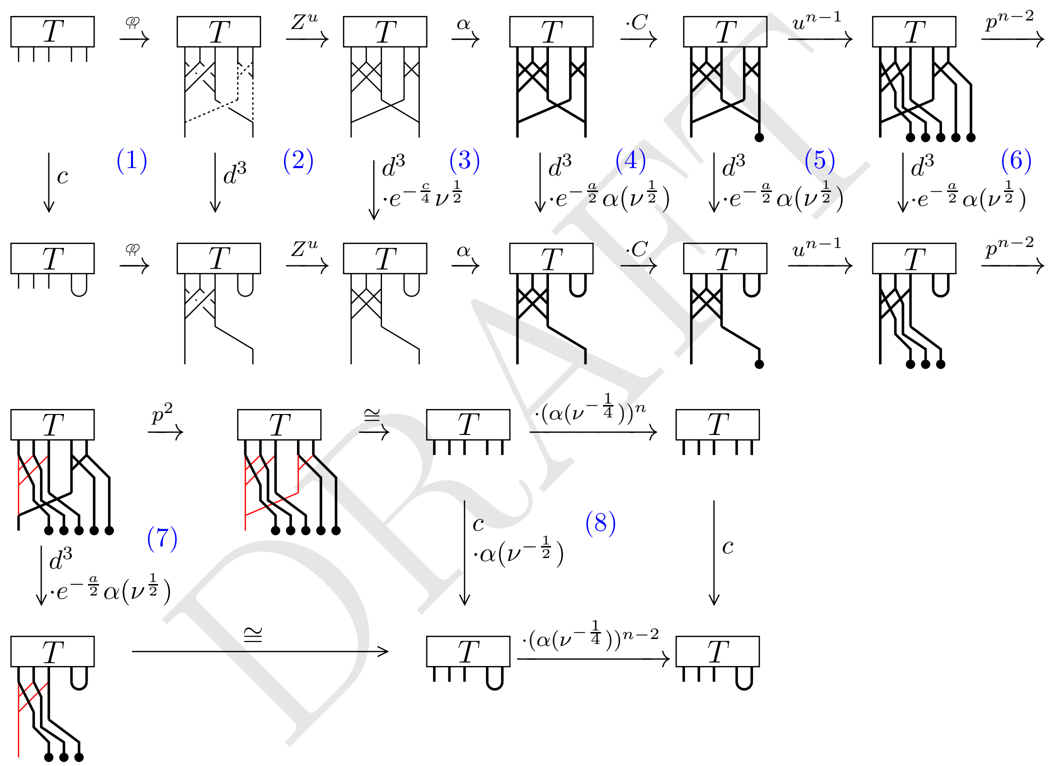


Figure 26. Proving that Z^w commutes with contractions.

Contractions. Proving that Z^w commutes with contractions is somewhat more involved. Firs of all, by Lemma 3.2, we can assume without loss of generality that the ends contracted are the last (rightmost) two ends of T. Hence we will drop the subscript from c_i and denote this operation simply by c. We need to show that $Z^w(cT) = cZ^w(T)$, for any $T \in w\mathcal{TT}$. The commutativity with contractions is the outer perimeter of the diagram shown in Figure 26. Note that in the diagram is schematic: we indicate skeleta but omit chords and arrows for simplicity, and we suppress orientations and orientation switches.

We're going to show that each numbered square (and in one case, pentagon) of this diagram commutes.

(1) This square plays out in *sKTG*, and the picture is the proof: contraction followed by the double tree map has the same effect as the double tree map followed by deleting three strands. The strands to delete are indicated by broken lines. Deletion in the future always refers to these same strands, so we no longer break up the lines.

(2) There is an "edge delete" operation of *sKTG*-s which we have not mentioned before. When one deletes an edge (which ends in a vertex at both ends), the vertices at each end disappear (cease being vertices). The associated graded operation on chord diagrams deletes the skeleton edge, and a chord diagram with any chord endings on the deleted edge is sent to 0.

The invariant Z^u is not entirely homomorphic with respect to edge deletions. Recall that in [WKO2, Section 4.6.1] Z^u was constructed from an invariant Z^{old}. In fact, Z^{old} does commute with edge deletions [Da, Proposition 6.7], so the behaviour of Z^u depends on the vertex normalisations implemented in [WKO2]. Namely, each vertex of an *sKTG* has one incident edge marked "distinguished", in drawings this is usually the vertical edge. If an edge is "not distinguished" at either of its ending vertices, then deleting it commutes with Z^u. When deleting an edge is distinguished at a vertex, such as the right vertical edge of square (2), then the square only commutes after a correction term of $e^{-c/4}\nu^{1/2}$ is inserted at the place of the vertex. (That is, where the two ends of T are contracted at the bottom right picture of square (2).) Here c denotes a single chord.

(3) There is a corresponding "edge delete" operation of $\widetilde{wTF}$, which works the same way: when deleting a tube or a string, the vertices at either end cease being vertices. (It is also possible to delete a capped edge.) The associated graded operation deletes the appropriate skeleton strand and sends any arrow diagram with arrow endings on the deleted strand to zero. The edge delete operations for chord and arrow diagrams clearly make a commutative square with α, hence square (3) commutes.

(4) This square plays out in $\mathcal{A}^{sw}$ and it is obviously commutative.

(5) Once again clearly commutes, note that on the right now two of the three deleted edges are capped tubes.

(6) Deletions commute with punctures in $\mathcal{A}^{sw}$. This follows from the definitions, the only thing to note is that when a tube strand is deleted at a "tube-and-string" vertex, the other tube strand deflates to a string (just like it does in the case of a puncture, as shown in Figure 4).

(7) For the pentagon (7), we will show only that it commutes up to a possible small error (on the u-shaped strand), and later prove that this error is necessarily zero.

To show commutativity up to an error, a better understanding of the arrow diagram in the top left corner is necessary. This arrow diagram is the result of a sequence of operations $(p^{n-2} \circ u^{n-1} \circ \cdot C \circ \alpha \circ Z^u \circ \Phi)$. All of these operations with the exception of Z^u are "easy" in the sense that we have a complete understanding of their effect. Z^u is "hard", but the proof of [WKO2, Proposition 4.13] presents a technique for computing the relevant part of its value, see Figures 27 and 28 and their captions.

The value at the top left corner of the pentagon (7) is the chord diagram D of Figure 28, with $Z^u(A)$ inserted, and then the appropriate sequence of α, cap, unzips and punctures performed. We need to analyze which parts of D survive these operations, and the operations of the pentagon (7): this is an exercise very similar to what

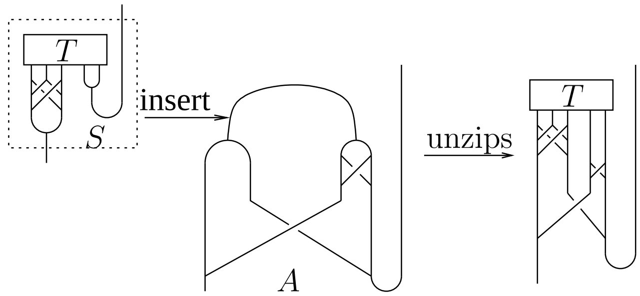


Figure 27. Computing the top left corner of Square 7. Step 1: $\wp(T)$ can be expressed as the *sKTG* denoted S inserted into the *sKTG* denoted A, followed by some unzips, as shown. Z^u respects insertions, hence computing $Z^u(A)$ determines the value of $Z^u(\wp(T))$ outside of S.

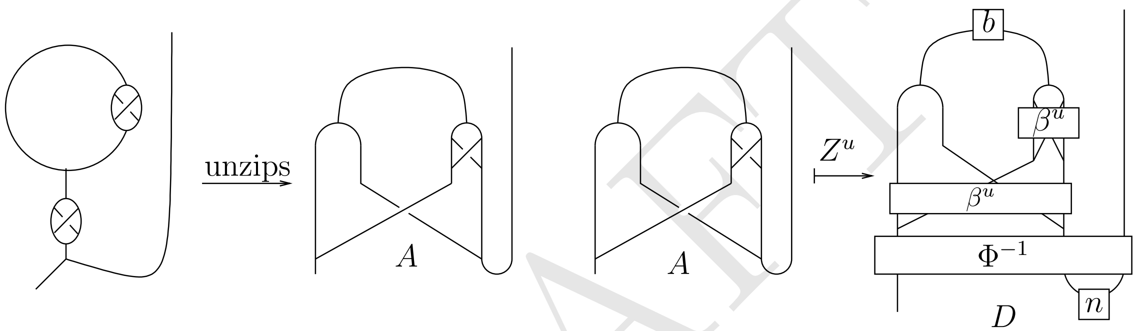


Figure 28. Computing the top left corner of Square 7, Step 2: computing $Z^{u(A)}$. The *sKTG* A can be obtained by inserting the *buckle sKTG* twice into a simpler *sKTG*, and unzipping, as shown on the left. The value of the buckle was computed in Figure 13. Using this value—denoted β^u—and the algorithm in [WKO2, Section 5.2], one computes $Z^u(A)$. The result is denoted D and shown on the right.

has been done for Lemma 3.4 for example. The result is shown in Figure 29, and explained below. Note that the n value in D cancels after punctures, see the last paragraph of Lemma 3.3 for the reasoning; the bottom Φ^{-1} also cancels, as in the proof of Lemma 3.4, hence we don't show these values in Figure 29.

Going down from the top left corner, the three edge deletions kill both buckle (β^w) values. The value arrow diagram $S\alpha u(b)$ cancels for the following simple reason, also illustrated in Figure 30:

Fact 3.7. *Suppose that an arrow a ends on a strand e. Unzipping the strand e produces a diagram a_1+a_2, one summand ending on each daughter strand. Reversing the orientation of the first strand gives $-a_1+a_2$. Now connecting the two daughter strands to form a u-shaped strand identifies a_1 and a_2, sending their difference to 0.*

This is essentially what happens to the arrow diagram $\alpha u(b)$, canceling it in both directions of the pentagon (7). The second down-going arrow on the left is applying the isomorphism φ of Lemma 2.4.

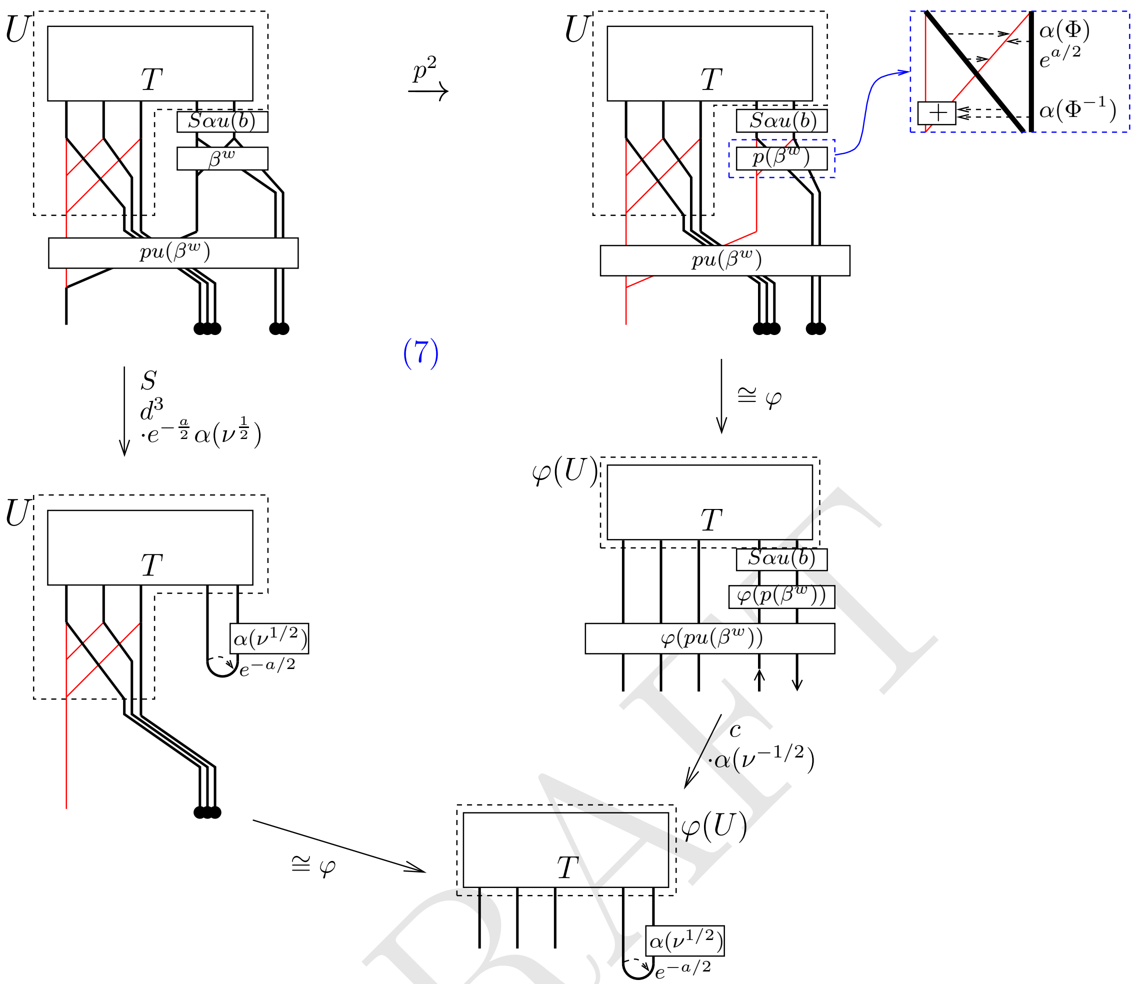


Figure 29. The more detailed picture of the Pentagon 7. fig:Square

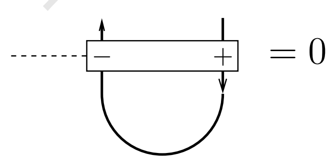


Figure 30. Unzip, switch orientation and connect kills arrows on the strand. fig:KillAn

On to the right side of the pentagon: from Equation (2) and with the strand numbering of Figure 13, we have that

$$p_1p_3\beta^w = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}).$$

This is shown in the enlarged rectangle at the to right corner of Figure 29.

Once again, the down-going arrow applies the isomorphism φ of Lemma 2.4, which is followed by the contraction. Understanding the result requires a similar analysis to that of the Proof of Part 2 of the main theorem, in particular Figure 14. We'll briefly outline the argument. At the bottom, $\varphi(pu(\beta^w))$ cancels altogether after contraction, essentially by Fact 3.7. This is straightforward except for the Φ^{-1} component,

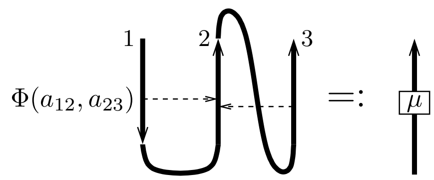


Figure 31. The Φ component of $\varphi(p(\beta^w))$ after contraction.

which requires a minor additional commutativity argument. Of the other "buckle", $\varphi(p(\beta^w))$, the Φ^{-1} component cancels, again by Fact 3.7; only the exponential and the Φ component remains. The arrow in the exponent of $e^{a/2}$ switches sign due to the reversed orientation, and becomes the $e^{-a/2}$ component at the bottom of the pentagon in Figure **??**. After contraction, the Φ component gives rise to a local arrow diagram on a single strand, shown in Figure 31 and denoted μ from here on.

In summary, we see that the pentagon (7) commutes if and only if $\mu = \alpha(\nu)$, and otherwise has an error, which is a local arrow diagram on the contracted strand of the value $\alpha(\nu)^{-1}\mu$.

(8) Finally, the square (8) clearly commutes.

We have therefore shown that Z^w commutes with contraction up to an error $\alpha(\nu)^{-1}\mu$ on the contracted strand. It remains to show that this error is 1. This follows from the facts that $Z^w(\uparrow) = 1$, and that Z^w commutes with disjoint unions:

$$1 = c(Z^w(\downarrow\uparrow)) = \alpha(\nu)^{-1}\mu Z^w(c(\downarrow\uparrow)) = \alpha(\nu)^{-1}\mu \cdot 1.$$

This completes the proof. □

Note that as a side result we have proven the following curious fact about associators:

Proposition 3.8. *For any Φ horizontal chord associator, μ defined from Φ as in Figure 31, and ν the Kontsevich integral of the unknot, $\mu = \alpha(\nu)$.* □

To summarise

Theorem 3.9.

Proof.

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Recycling

This square requires some effort, and first of all a definition of the "inflate" operation of the bottom arrow. This operation is well-defined when a 1-dimensional (red) strand connects two capped tubes, as illustrated in Figure **??**. Inflating replaces the 1-dimensional string with a narrow tube, creating two smooth "pair of pants" vertices where the string joined the capped strands. Now by isotopy the capped strands can be retracted and just one tube is left, as shown in Figure **??**. Alternatively, the 1D string can first be moved by isotopy to join near the caps, then inflating it produces a single tube. This is well-defined. The square (7) involves the associated graded (arrow diagram) version of this operation: use the VI relation to move all arrow endings off the capped strands, then turn the thin red strand thick black, and dispose of the capped strands, as seen in square (7).

To Do

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