FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS II: THE DOUBLE TREE CONSTRUCTION

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Abstract. In this paper we provide a topological interpretation and independent topological proof of the formula by Alkeseev-Enriquez-Torossian AlekseevEnriquezTorossian:ExplicitSolutions [AET] for solutions of the Kashiwara-Vergne problem in terms of associators.

We study a class of w-knotted objects: knottings of "2-dimensional foams" and various associated features in four-dimensioanl space. We utilize a "double tree construction" to show that every "expansion" (also called "universal finite type invariant" or "UFTI") of parenthesized braids extends uniquely first to an expansion/UFTI of knotted trivalent graphs (a well known result), and then on to an expansion/UFTI of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is the same as a "Drinfel'd associator" Φ, and an expansion for the aforementioned w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem Kashiwaravergne:Conjecture [KV] as reformulated by Alekseev and Torossian AlekseevTorossian:Kashiwaravergne [AT]. Hence our result amounts to a topological framework for the result of AlekseevEnriquezTorossian:ExplicitSolutions [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof that the said formula works — namely that the equations satisfied by V follow from the equations satisfied by Φ.

Contents

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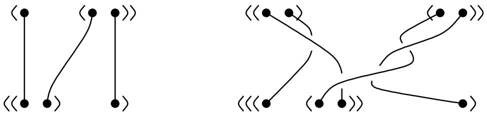


Figure 1. Two examples of parenthesized braids. Note that by convention the parenthetization can be read from the distance scales between the endpoints of the braid, and so we are going to omit the parentheses in the future.

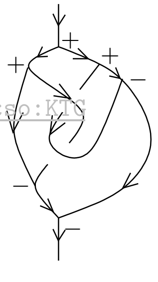
1. Introduction

1.1. **Topology.** We begin by describing a chain of maps from "parenthesized braids" to "(signed) knotted trivalent graphs" to "w-tangled foams":

$$\mathcal{K} := \{\mathit{uPB} \xrightarrow{\mathrm{cl}} \mathit{sKTG} \xrightarrow{a} \widetilde{\mathit{wTF}}\}.$$

Let us first briefly elaborate on each of these spaces and maps.

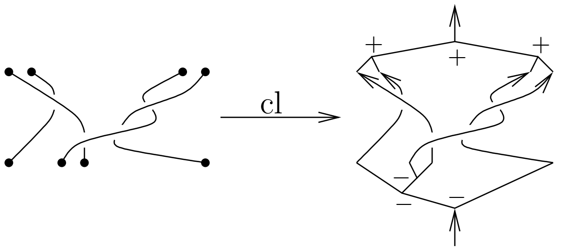
Parenthesized braids are braids are braids whose ends are ordered along two lines, the "bottom and the "top", along with parenthetizations of the endpoints on the bottom and on the top. Two examples are shown in Figure 1. Parentehesized braids form a category whose objects are parenthetizations, morphisms are the parenthesized braids themselves, and composition is given by stacking. In addition to stacking, there are several operations defined on parenthesized braids: strand addition, removal and doubling. A detailed introduction to parenthesized braids is in [BN1].



Trivalent graphs are oriented graphs with three edges meeting at each vertex and whose vertices are equipped with a cyclic orientation of the edges. A knotted trivalent graph (KTG) is a framed embedding of a trivalent graph into $\mathbb{R}^3$. KTGs are studied from a finite type invariant point of view in [BND1]. In this paper we use a version of KTGs that was introduced and studied in Section 6.6 of [BND2], namely trivalent $(1,1)$-tangles with some extra combinatorial information: signs assigned to the trivalent vertices. We call this space *sKTG*, as in [BND2]. An example is shown on the right. The space *sKTG* is also equipped with several operations: tangle insertion, edge unzip, and edge orientation switch.

The space $\widetilde{\mathit{wTF}}$ is a minor extension of wTF^o studied in Section 6 of [BND2], and will be introduced in detail in Section 2. It can be described as a planar algebra generated by certain features (various kinds of crossings and vertices, as well as "caps") modulo certain relations ("Reidemeister moves") and equipped with a number of auxiliary operations beyond planar algebra composition. This Reidemeister theory conjecturally represents knotted tubes in $\mathbb{R}^4$ with singular "foam vertices", caps, and attached one-dimensional strings.

The map $\mathrm{cl} : \mathit{uPB} \to \mathit{sKTG}$ is the "closure map". Given a parenthesized braid, close up its top and bottom each by a tree according to the parentetization; this produces a *sKTG* with the convention that all strands are oriented upwards, bottom vertices are negative, and top vertices are positive. An example is shown below.



The map $a : sKTG \to \widetilde{wTF}$ arises combinatorially from the fact that all *sKTG* diagrams can be interpreted as elements of $\widetilde{wTF}$, and all $sKTG$ Reidemeister moves remain true in $\widetilde{wTF}$. Topologically, it is an extended version of Satoh's tubing map, described in Remark 3.1.1 of [BND2].

1.2. **Algebra.** The chain of maps $\mathcal{K}$ is an example of a general "algebraic structure", as defined in Section 4.1 of [BND2]. An algebraic structure consists of a collection of objects belonging to a number of "spaces" or "different kinds", and operations that may be unary, binary, multinary or zeronary, between these spaces. In this case there are many spaces (or kinds of objects): for example, parenthesized braids with specified bottom and top parenthetizations form one space, so do knottings of a given trvalent graph. There is also a large collection of operations, consisting of all the internal operations of *uPB*, *sKTG* and $\widetilde{wTF}$, as well as the maps a and cl.

In Section 4.2 and 4.3 of [BND2]we discuss projectivizations and expansions for general algebraic structures. A projectivization is the associated graded space taken with respect to the filtration by powers of the "augmentation ideal". For the spaces *uPB*, *sKTG* and $\widetilde{wTF}$, the projectivizations $\mathcal{A}^{hor}$, $\mathcal{A}^u$ and $\mathcal{A}^{sw}$ are the spaces of "horizontal chord diagrams on parenthesized strands", "chord diagrams on trivalent skeleta", and "arrow diagrams", as described in [BN1], Section 6.6 of [BND2], and Section 2 of this paper, respectively. As a result, the projectivization of $\mathcal{K}$ is the structure

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{\text{cl}} \mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and a, respectively. More specifically, cl is the "closure of chord diagrams", and α is "sending each chord to the sum of its two possible oriantations".

An expansion is a filtration-respecting map from an algebraic structure (where linear combinations of objects of the same kind are allowed) to its projectivization, satisfying a certain non-degeneracy property. Expansions are also called universal finite type invariants in knot theory. A homomorphic extension also behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the projectivization. Hence, a homomorphic expansion $Z : \mathcal{K} \to \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u, and Z^w for *uPB*, *sKTG* and $\widetilde{wTF}$, respectively, so that the following diagram commutes:

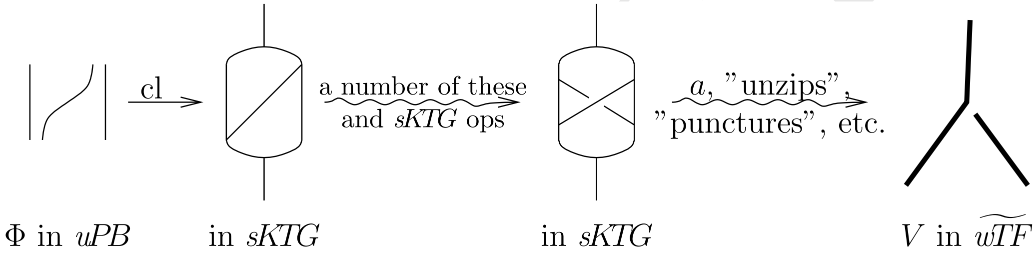
$$\begin{tikzcd} uPB \arrow[r, "\text{cl}"] \arrow[d, "Z^b"] & sKTG \arrow[r, "a"] \arrow[d, "Z^u"] & \widetilde{wTF} \arrow[d, "Z^w"] \\ \mathcal{A}^{hor} \arrow[r, "\text{cl}"] & \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^w \end{tikzcd} \tag{1}$$

We recall (see [BN1]) that a homomorphic expansion Z^b for parenthesized braids is determined by a "horizontal chord associator" $\Phi = Z^b(|\diagup|)$. A homomorphic expansion Z^u of *sKTG* is also determined by an associator (horizontal chords or not; see Section 6.6 of [BND2]), so the significance of left commutative square is to force Z^u to come from a horizontal chord associator. In turn, Z^w (roughly speaking) is determined by a solution $V = Z^w(\nearrow\!_{\nwarrow})$ to the Kashiwara-Vergne problem (see Section 6 of [BND2]), and the goal of this paper is to prove the following theorem:

thm:main

Theorem 1.1. (1) *Assuming that* $Z : \mathcal{K} \to \mathcal{A}$ *exists,* Z *is determined by* Φ*. In other words, there is a formula for* V *in terms of* Φ*, assuming that* V *exists.*
(2) *Said formula is the formula proven in* [AET].
(3) *Every* Z^b *extends to a* Z*.*

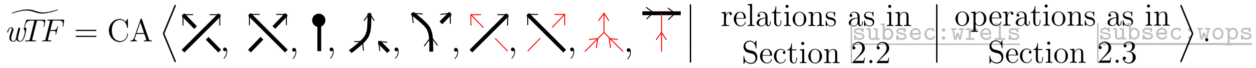
The key to the proof of the theorem is to show that the generator $\nearrow\!_{\nwarrow}$ of $\widetilde{wTF}$ can be expressed in terms of the generator $|\diagup|$ of *uPB* and the operations of $\mathcal{K}$. Assuming that Z exists, this yields a formula for V in terms of Φ. The expression of $\nearrow\!_{\nwarrow}$ in terms of $|\diagup|$ uses a "Double Tree Construction", which will be discussed in Section ??. For now, let us display a picture with no explanation:



sec:wTFe

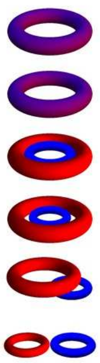
2. THE SPACES $\widetilde{wTF}$ AND $\mathcal{A}^{sw}$ IN MORE DETAIL

As we mentioned in the introduction, $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in Section 6 of [BND2]. It can be introduced as a planar algebra or a circuit algebra, we will do the latter as it is simpler and more concise. Circuit algebras are defined in Section 4.4 of [BND2]; in short, they are similar to a planar algebras but the "connecting strands" are allowed to cross. As in [BND2], each generator and relation of $\widetilde{wTF}$ has a local topological interpretation. In [BND2], wTF^o represented knotted tubes in $\mathbb{R}^4$ with "foam vertices" and "capped ends". Two-dimensional tubes will be denoted by thick lines and one dimensional ones by thin red lines. The space $\widetilde{wTF}$ extends wTF^o by adding one-dimensional strands to the picture. Note that one dimensional strands cannot be knotted in $\mathbb{R}^4$, however, they can be knotted *with* two-dimensional tubes.



bsec:wgens

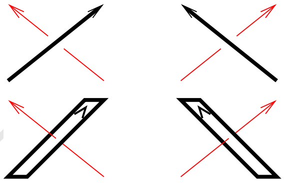
2.1. The generators of $\widetilde{wTF}$. We begin by discussing the local topological meaning of each generator shown above.



Bar-NatanDancso:WKO

The first five generators are as described in Section 6.1.1 of [BND2]. Recall that knotted (more precisely, braided) tubes in $\mathbb{R}^4$ can equivalently be thought of as movies of flying rings in $\mathbb{R}^3$. The two crossings stand for movies where two rings trade places while the one corresponding to the under strand flies through the one corresponding to the over strand. The dotted end represents a tube "capped off" by a disk. The fourth and fifth generators stand for singular "foam vertices", and will be referred to as the positive and negative vertex, respectively. The positive vertex represents the movie shown on the left: the right ring approaches the left ring from below, flies inside it and merges with it. To be completely precise, $\widetilde{wTF}$ as a circuit algebra has more generators than shown above: the vertices appear with all possible orientations of the strands. However, all other versions can be obtained from the ones shown above using "orientation switch" operations (to be discussed in Section 2.3).

subsec:wops



The red (thin) strands denote one dimensional strings in $\mathbb{R}^4$, or "flying points in $\mathbb{R}^3$". The crossings between the two types of strands (sixth and seventh generators) denote "points flying through rings". They are both shown on the right in band notation (see section 5.4 of [BND2]for an explanation of band notation). For example, the bottom left picture means "the point on the approaches the ring on the left from below, flies through the ring and out to the left above it." This explains why there are no generators with a thick strand crossing under a thin red strand: a ring cannot fly through a point.

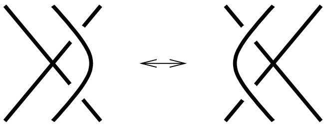
Bar-NatanDancso:WKO

Next is a trivalent vertex of 1-dimensional strings in $\mathbb{R}^4$. Once again, this generator should be shown in all possible strand orientation combinations. Finally, tha last generator is a "mixed vertex", in other words a one-dimesional string attached to the wall of a 2-dimensional tube.

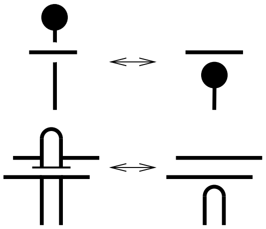
bsec:wrels

2.2. **The relations.** As a list, the relations for $\widetilde{wTF}$ look the same as the relations for wTF^o: $\{\mathrm{R1}^s, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}, \mathrm{OC}, \mathrm{CP}\}$. Recall that $\mathrm{R1}^s$is the weak (framed) version of the Reidemeister 1 move; R2 and R3 are the usual Reidemeister moves; R4 allowes moving a strand over or under a vertex. OC is the "Overcrossings Commute" relation, and CP (Cap Pullout) allows for pulling a capped strand out from under a crossing, as shown below:

OC:

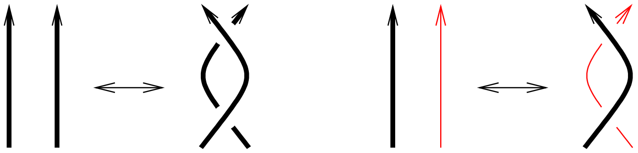


CP:



However, all relations should be interpreted in all possible combinations of strand types, for example the lower strand of the Reidemeister two relation can either be thick black or thin red:

R2:



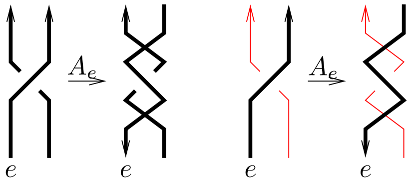
Similarly, any of the bottom strands of the R3, R4, or OC relations may be thin red.

As in wTF^o, the relations all have local topological meaning and conjecturally $\widetilde{wTF}$ is a Reidemeister theory for ribbon knotted tubes in $\mathbb{R}^4$ with caps, singular foam vertices and attached strings. For example, Reidemeister 2 with a thin red bottom strand is imposed because a point flying in through a ring and then immediately flying back out is isotopic to not having any interaction between the point and ring at all.

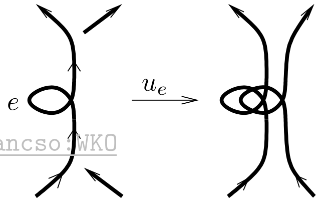
2.3. **The operations.** Like $wTFo$, $wTFe$ is equipped with a set of auxiliary operations in addition to the circuit algebra structure.

The first of these is orientation reversal. For the thin (red) strands, this simply means reversing the direction of the strand. For the thick (tube) strands, orientation switch comes in two flavours. Recall from [BND2] that in the topological interpretation of wTF^o, each tube is oriented as a 2-dimensional surface, and also has a distinguished "core": a line along the tube which is oriented as a 1-dimensional manifold and determines the "direction" or "1-dimensional orientation" of the tube. Both of these are determined by the direction of the strand in the circuit algebra, via Satoh's tubing map.

Topologically, the operation "orientation switch", denoted S_e for a given strand e, acts by reversing both the (1-dimensional) direction and the (2-dimensional) orientation of the tube e. Diagrammatically, this corresponds to simply reversing the direction of the corresponding strand e.

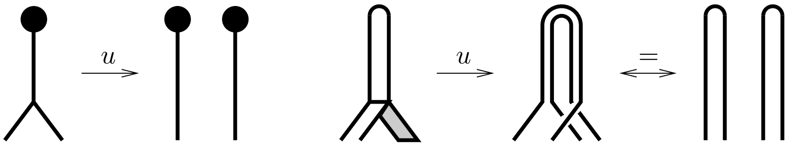


The "adjoint" operation, denoted A_e, on the other hand only reverses the (1-dimensional) direction of the tube e, not the orientation as a surface. Diagrammatically, this manifests as reversing the strand direction and adding two virtual crossings on either side of each crossing where e crosses *over* another strand, as shown on the right (note that the strand below e may be thick or thin). For more details on orientations and orientation switches, see [BND2].



The unzip operation u_e doubles the strand e using the blackboard framing, and then attaches the ends of the doubled strand to the connecting ones, as shown on the right. We restrict unzip to strands whose two ending vertices are of different signs. (For the definition of crossing and vertex signs, see Sections 5.4 and 6.1 of [BND2].) Topologically, the blackboard framing of the diagram induces a framing of the corresponding tube in $\mathbb{R}^4$ via Satoh's tubing map, and unzip is the act of "pushing the tube off of itself slightly in the framing direction". Note that unzips preserve the ribbon property.

A related operation, *disk unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagram world, in the direction of the blackboard framing), as before. An example in the line and band notations is shown below.



We also allow the deletion of "long linear" strands, meaning strands that do not end in a vertex on either side.

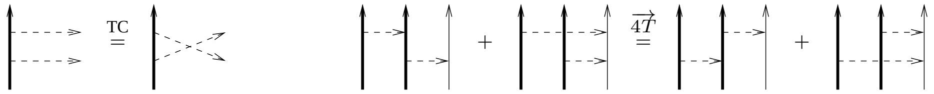
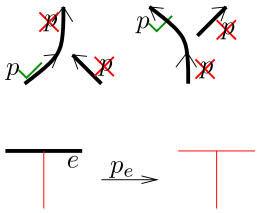


Figure 2. The TC and $\overrightarrow{4T}$ relations

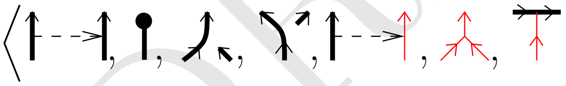
fig:TCand4

So far all the operations we have introduced had already existed in wTF^o. The first new operation is called "puncture", denoted p_e, which diagrammatically simply turns the thick black strand e into a thin red one. The corresponding topological picture is "puncturing a tube", i.e., removing a small disk from it and retracting the rest to its core. Any crossings where e passes under another strand are not affected, while crossings in which e is the over strand turn into virtual crossings.



For simplicity, we place a restriction on which strands can be punctured, namely at each (fully thick black) vertex puctures are only allowed for one of the three meeting strands, as shown in the top row of the figure on the right. More general puctures could be allowed in a theory complete with "wens", as in Section 6.5 of [BND2]. The bottom row of the same figure shows what happens when puncturing one of the thick strands of a mixed vertex. Topologically, this is because the mixed vertex represents a string attached to the ouside of a tube, so when puncturing e, the entire tube retracts to its core. Finally, puncturing a capped tube makes it disappear.

2.4. **The projectivization $\mathcal{A}^{sw}$.** As in [BND2], the space $\widetilde{wTF}$ is filtered by powers of the augmentation ideal and its associated graded space or projectivization, denoted $\mathcal{A}^{sw}$, is a "space of arrow diagrams on foam skeletons with strings". As a circuit algebra, $\mathcal{A}^{sw}$ is presented as follows:



$$\widetilde{wTF} = \mathrm{CA}\left\langle \quad \middle| \begin{array}{c}\text{relations}\\ \text{as below}\end{array} \middle| \begin{array}{c}\text{operations}\\ \text{as below}\end{array} \right\rangle .$$

The first and fifth generators are called single arrows and they are of degree one, while all others are "skeleton features" of degree zero. The relations are almost the same as those for the projectivization of wTF^o: $\overrightarrow{4T}$ (the 4-Term relation), TC (Tails Commute), RI (Rotation Invariance), CP (Cap Pullout), VI (Vertex Invariance), with the additional new relation TF (Tails Forbidden on strings). The TC and $\overrightarrow{4T}$ relations are shown in Figure 2, note that the 3rd strand in each term of the $\overrightarrow{4T}$ relation is ambiguous: it can be either thick black or thin red, the relation applies in either case. VI is picturd in Figure 3: here the ± signs depend on the strand orientations and the type of the vertex and the types of each strand (thick black or thin red) is left ambiguous: the VI relation applies in all cases. Figure 4 shows the other three relations: RI, CP and TF. Note that technically TF is not even a relation: there were no generators with an arrow tail on a thin red strand, so saying that such an element vanishes is superfluous. However, without TF the VI relation would have to be stated for all the sub-cases of 0, 1 or 3 thin red strands meeting at the vertex, instead of simply saying that arrow tails on these strands vanish. We prefer stating them this way as it is cleaner, even if it is a slight abuse of notation.

As in [BND2](see Definition 3.13), we define a "w-Jacobi diagram" (or just "arrow diagram") to be similar to by also allowing tricalent chord vertices, each of which is equipped

Figure 3. The VI relation: the vertices and strands could be of any type, but the same throughout the relation.

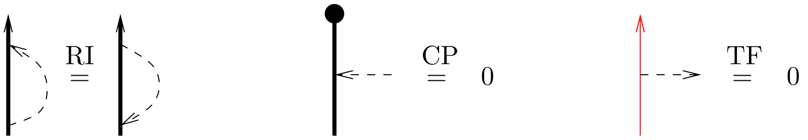


Figure 4. The RI relation, CP relation and the TF relation (which is not really a relation).

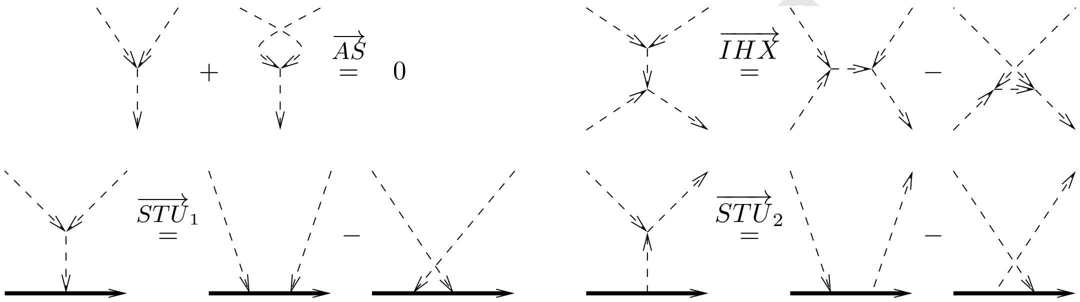


Figure 5. The $\overrightarrow{AS}$, $\overrightarrow{IHX}$ and the two $\overrightarrow{STU}$ rerations.

with a cyclic orientation. Denote the circuit algebra of formal linear combinations of these w-Jacobi diagrams by, $\mathcal{A}^{swt}$. Then, as in Theorem 6.5 in [BND2], we have the following bracket-rise theorem:

Theorem 2.1. *The obvious inclusion of diagrams induces a circuit algebra isomorphism* $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$. *Furthermore, the* $\overrightarrow{AS}$, $\overrightarrow{IHX}$ *and* $\overrightarrow{STU}$ *relations (see Figure 5) hold in* $\mathcal{A}^{swt}$.

The proof is identical to the proof of Theorem 3.15 in [BND2]. In light of this isomorphism, we will drop the extra "t" from the notation and use $\mathcal{A}^{sw}$ to denote either of these spaces.

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RECYCLING

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