All upcaps are downcaps.

We need an index of notation?

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: THE DOUBLE TREE CONSTRUCTION

DROR BAR-NATAN AND ZSUZSANNA DANCSO

Abstract. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological solution to the Kashiwara-Vergne problem. In particular we recover via a topological argument the Alkeseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of *2-dimensional foams* and various associated features in four-dimensioanl space. We use a topological construction which we name the double tree construction to show that every *expansion* (also known as *universal finite type invariant*) of parenthesized braids extends first to an expansion of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is the same as a *Drinfel'd associator* Φ, and an expansion for w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem [KV] as reformulated by Alekseev and Torossian [AT]. Hence our result provides a topological framework for the result of [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof that the said formula works — namely that the equations satisfied by V follow from the equations satisfied by Φ.

Contents

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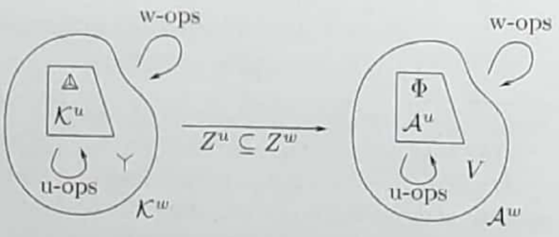
1. Introduction

1.1. Executive Summary. This section is a large-scale overview of the main result of this paper and the idea behind its proof; it is followed by a more detailed introduction.

A *homomorphic expansion* for a class of topological objects $\mathcal{K}$ is an invariant $Z\colon \mathcal{K} \to \mathcal{A}$ whose target space $\mathcal{A}$ is canonically associated with $\mathcal{K}$ (its *associated graded*). Homomorphic expansions satisfy a certain universality property, and respect operations which exist on $\mathcal{K}$ and therefore also on $\mathcal{A}$. Such invariants are often hard to find, and when they are found, they are often intimately connected with deep mathematics:

- For many classes of knotted objects in 3-dimensional spaces homomorphic expansions don't exist — for example, one would have loved ordinary tangles to have homomorphic expansions, but they don't.
- Yet a certain class $\mathcal{K}^u$ of knotted objects in 3-space, *parenthesized tangles*, or nearly-equivalently, *knotted trivalent graphs* – which we adopt in this paper and denote by *sKTG* – do have homomorphic expansions. A homomorphic expansion $Z^u\colon \mathcal{K}^u \to \mathcal{A}^u$ is defined by its values on a couple of elements of $\mathcal{K}^u$ which generate $\mathcal{K}^u$ using the operations $\mathcal{K}^u$ is equipped with. The most interesting of these generators is the tetrahedron $\triangle$, and $\Phi = Z^u(\triangle)$ turns out to be equivalent to a *Drinfel'd associator.*
- A certain class $\mathcal{K}^w$ of graphs, called *w-foams* and denoted *wTF*^{o} in the paper – the name is based on a conjectured equivalence to a class of 2-dimensional *welded* knotted tubes in 4-dimensional space – also has homomorphic expansions. The most interesting generator of $\mathcal{K}^w$ is the *vertex* $\curlywedge$, and if $Z^w\colon \mathcal{K}^w \to \mathcal{A}^w$ is a homomorphic expansion, then it turns out that $V = Z^w(\curlywedge)$ is equivalent to a solution of the *Kashiwara-Vergne problem* in Lie theory.

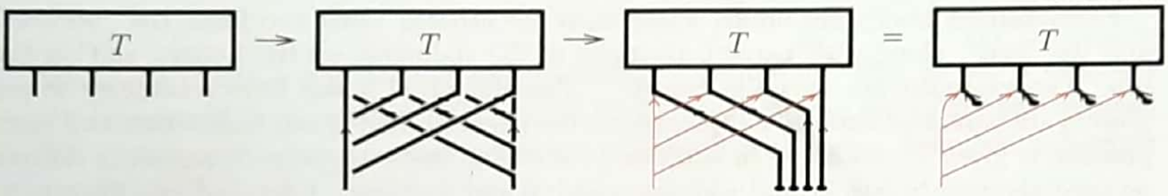
Roughly speaking, $\mathcal{K}^u$ is a part of $\mathcal{K}^w$ and $\mathcal{A}^u$ is a part of $\mathcal{A}^w$, as in the figure on the right (more precisely, there are natural maps $a\colon \mathcal{K}^u \to \mathcal{K}^w$ and $\alpha\colon \mathcal{A}^u \to \mathcal{A}^w$). The main purpose of this paper is to prove the following theorem, whose precise version is stated later as Theorem 1.1:



Theorem. Any homomorphic expansion Z^u for $\mathcal{K}^u$ extends uniquely to a homomorphic expansion Z^w for $\mathcal{K}^w$, and therefore, any Drinfel'd associator Φ yields a solution V of the Kashiwara-Vergne problem.

The proof of this theorem is conceptually simple: we show that the generators of $\mathcal{K}^w$ can be explicitly expressed using the generators of $\mathcal{K}^u$ and the operations of $\mathcal{K}^w$, and that the resulting explicit formulas for $Z^w(\curlywedge)$ (and for Z^w of the other generators) satisfies all the required relations.

The devil is in the details. It is in fact impossible to express the generators of $\mathcal{K}^w$ in terms of the generators of $\mathcal{K}^u$ — to do that, one first has to pass to a larger space $\tilde{\mathcal{K}}^w$ (in the paper $\widetilde{uTF}$) that has more objects and more operations, and in which the desired explicit expressions do exist. But even in $\tilde{\mathcal{K}}^w$ these expressions are complicated, and are best described within a certain "double tree construction" which also provides the framework for the verification of relations. Here's an unexplained summary; the explanations make the bulk of this paper:



1.2. **Detailed Introduction.** This paper is the third in a sequence [WKO1, WKO2, WKO3] studying finite type invariants of w-knotted objects, and contains the strongest result: a topological construction for a homomorphic expansion of *w-foams* starting from the Kontsevich integral. This in particular implies the Kashiwara-Vergne Theorem, more precisely, the [AET] formula for solutions of the Kashiwara-Vergne equations in terms of Drinfel'd associators. Readers familiar with finite type invariants in general should have no trouble reading [WKO2] and this paper without having read [WKO1]. The setup and main results of [WKO2] are used heavily in this paper. Reproducing all necessary details would be lengthy, so we only include concise summaries for readers already familiar with the content, and otherwise refer to specific sections of [WKO2] throughout.

The Kashiwara-Vergne conjecture (KV for short) — proposed in 1978 [KV] and proven in 2006 by Alekseev and Meinrenken [AM] — asserts that solutions exist for a certain set of equations in the space of "tangential automorphisms" of the free lie algebra on two generators. For a precise statement we refer the reader to [WKO2] or [AT]. The existence of such solutions has strong implications in Lie theory and harmonic analysis, in particular it implies the multiplicative property of Duflo isomorphism, which was shown to be knot-theoretic in [BLT, BDS].

In [AT] Alekseev and Torossian give another proof of the KV conjecture based on a deep connection with Drinfel'd associators. In turn, Drinfel'd's theory of associators [Dr] can be interpreted as a theory of well-behaved universal finite type invariants of parenthesized tangles[1] [LM, BN2], or of knotted trivalent graphs [Da]. In [AET] Alekseev, Enriquez and Torossian gave an explicit formula for solutions of the Kashiwara-Vergne equations in terms of Drinfel'd associators.

In [WKO2] we re-interpreted the Kashiwara-Vergne conjecture as the problem of finding a "homomorphic" universal finite type invariant of a class of knotted trivalent tubes

[1] "q-tangles" in [LM], "non-associative tangles" in [BN2].

in 4-dimansional space (called w-tangled foams), and explained the connection to Drilfeld associators in terms of a relationship between 3-dimensional and 4-dimensional topology. Another topological interpretation for the KV problem has recently emerged in [AKKN].

In this paper we present a topological construction for a homomorphic universal finite type invariant of w-tangled foams, thereby giving a new, topological proof for the KV conjecture. This construction also leads to an explicit formula for solutions, which we prove agrees with [AET].

1.2.1. *Topology.* We begin by describing a chain of maps from "parenthesized braids" to "(signed) knotted trivalent graphs" to "w-tangled foams":

$$\mathcal{K} := \{uPaB \xrightarrow{\text{cl}} sKTG \xrightarrow{a} \widetilde{wTF}\}.$$

Let us first briefly elaborate on each of these spaces and maps.

Parenthesized braids are braids whose ends are ordered along two lines, the "bottom" and the "top", along with parenthetizations of the endpoints on the bottom and on the top. Two examples are shown in Figure 1. Parentehesized braids form a category whose objects are parenthetizations, morphisms are the parenthesized braids themselves, and composition is given by stacking. In addition to stacking, there are several operations defined on parenthesized braids: strand addition, removal and doubling. A detailed introduction to parenthesized braids is in [BN1].

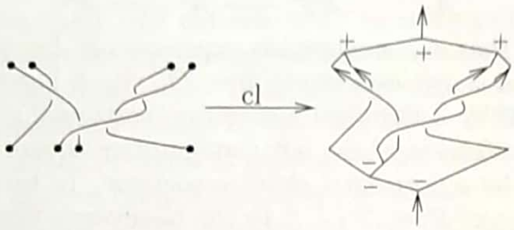
Trivalent graphs are oriented graphs with three edges meeting at each vertex and whose vertices are equipped with a cyclic orientation of the incident edges. A knotted trivalent graph (KTG) is a framed embedding of a trivalent graph into $\mathbb{R}^3$. KTGs are studied from a finite type invariant point of view in [BND1]. In this paper we use a version of KTGs that was introduced and studied in [WKO2, Section 4.6], namely trivalent tangles with one or two ends and with some extra combinatorial information: trivalent vertices are equipped with a marked "distinguished edge" and signs. We call this space *sKTG* (for signed KTGs), as in [WKO2]. An example is shown on the right. The space *sKTG* is also equipped with several operations: tangle insertion, sticking a 1-tangle onto an edge of another tangle, disjoint union of 1-tangles, edge unzip, and edge orientation switch (see [WKO2, Section 4.6] for details).

The space $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4], and will be introduced in detail in Section 2. It can be described as a circuit algebra (similar to a planar algebra but with non-planar connections allowed, see [WKO2, Section 2.4]) generated by certain features (various kinds of crossings and vertices, as well as "caps") modulo certain relations ("Reidemeister moves") and equipped with a number of auxiliary

Figure 1. Two examples of parenthesized braids. Note that by convention the parenthetization can be read from the distance scales between the endpoints of the braid, and so we omit the parentheses in parts of this paper.

operations beyond circuit algebra composition. This Reidemeister theory conjecturally represents knotted tubes in 4-dimensional space with singular *foam vertices*, caps, and attached one-dimensional strings.

The map cl : *uPaB* $\to$ *sKTG* is the "closure map". Given a parenthesized braid, close up its top and bottom each by gluing a binary tree according to the parentetization; this produces a *sKTG* with the convention that all strands are oriented upwards, bottom vertices are negative, and top vertices are positive. An example is shown below.



The map a : *sKTG* $\to$ $\widetilde{uTF}$ arises combinatorially from the fact that all *sKTG* diagrams can be interpreted as elements of $\widetilde{uTF}$, and all $sKTG$ Reidemeister moves are also imposed in $\widetilde{uTF}$. Topologically, it is an extended version of Satoh's tubing map, described in Remark 3.1.1 of [WKO2].

1.2.2. *Algebra.* The chain of maps $\mathcal{K}$ is an example of a general "algebraic structure", as discussed in [WKO2, Section 2.1]. An algebraic structure consists of a collection of objects belonging to a number of "spaces" or "different kinds", and operations that may be unary, binary, multinary or nullary, between these spaces. In this case there are many spaces (or kinds of objects): for example, parenthesized braids with specified bottom and top parenthetizations form one space, so do knottings of a given trivalent graph (skeleton). There is also a large collection of operations, consisting of all the internal operations of *uPaB*, *sKTG* and $\widetilde{uTF}$, as well as the maps a and cl.

In Sections 2.1 to 2.3 of [WKO2] we discuss associated graded structures and expansions for general algebraic structures. For any algebraic structure (think braids, or tangles with tangle composition), one allows formal linear compositions of elements of the same *kind* (think, same skeleton). Associated graded structures are taken with respect to the filtration by powers of the *augmentation ideal*. For the spaces *uPaB*, *sKTG* and $\widetilde{uTF}$, the associated graded spaces $\mathcal{A}^{hor}$, $\mathcal{A}^{u}$ and $\mathcal{A}^{sw}$ are the spaces of "horizontal chord diagrams on parenthesized strands", "chord diagrams on trivalent skeleta", and "arrow diagrams", as described in [BN1], [WKO2, Section 4.6], and Section 2 of this paper, respectively. As a result, the associated graded structure of $\mathcal{K}$ is

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{\text{cl}} \mathcal{A}^{u} \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and a, respectively. More specifically, cl is the "closure of chord diagrams", and α is "replacing each chord with the sum of its two possible orientations", see [WKO2, Section 3.3].

An expansion [WKO2, Section 2.3] is a filtration-respecting map from an algebraic structure to its associated graded structure, whose associated graded map is the identity. In knot theory, expansions are also called universal finite type invariants. A homomorphic expansion is an expansion which behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the associated graded structure; for a detailed definition and introduction see [WKO2, Section 2.3]. Hence, a homomorphic expansion $Z : \mathcal{K} \to \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u, and Z^w for $\mathcal{K}^b := uPaB$, $\mathcal{K}^u := sKTG$ and $\mathcal{K}^w := \widetilde{wTF}$, respectively, so that the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{K}: & & \mathcal{K}^b & \xrightarrow{\text{cl}} & \mathcal{K}^u & \xrightarrow{a} & \mathcal{K}^w \\ \downarrow Z & & \downarrow Z^b & & \downarrow Z^u & & \downarrow Z^w \\ \mathcal{A}: & & \mathcal{A}^{hor} & \xrightarrow{\text{cl}} & \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array} \tag{1}$$

We recall (see [BN1]) that a homomorphic expansion Z^b for parenthesized braids is determined by a "horizontal chord associator" $\Phi = Z^b(|\diagup|)$. A homomorphic expansion Z^u of *sKTG* is also determined[2] by a Drinfel'd associator (horizontal chords or not; see [WKO2, Section 4.6]), so the significance of the left commutative square is to force the associator corresponding to Z^u to be a horizontal chord associator. In turn, Z^w is determined by a solution F (a close cousin of $V = Z^w(\curvearrowright)$) to the Kashiwara-Vergne problem (see [WKO2, Section 4.4 – 4.5]). The goal of this paper is to prove the following theorem, which, via the correspondence above, implies the KV conjecture:

Theorem 1.1. (1) *Assuming that $Z : \mathcal{K} \to \mathcal{A}$ exists, it is determined*[3] *by Z^u.*

(2) *There is a formula for V in terms of the Drinfel'd associator Φ associated to Z^u:*

$$V = C_1^{-1} C_2^{-1} C_{(12)} \varphi \left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \Phi(a_{23}, a_{43}) \right),$$

the notation will be explained later. This agrees[4] *with the formula proven in* [AET].

(3) *Every Z^b extends to a Z.*

The key to the proof of the theorem is to show that the generator $\curvearrowright$ of $\widetilde{wTF}$ can be expressed in terms of the generator $|\diagup|$ of *uPaB* and the operations of $\mathcal{K}$. Assuming that Z exists, this yields a formula for V in terms of Φ.

1.3. **Paper Structure.** In Section 2 we provide an overview of the space wTF^o of (oriented) w-foams and its extension with strings $\widetilde{wTF}$. We provide a brief review of definitions and crucial facts from [WKO2], and details of the extension. We prove that homomorphic expansions for wTF^o extend uniquely to homomorphic expansions for $\widetilde{wTF}$.

Section 3 makes up the bulk of the paper and is devoted to the proof of Theorem 1.1. In Section 3.1 we prove part (1). In Section 3.2 we deduce the formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators, proving part (2). In Section 3.3 we prove statement (3), the hardest part. Section 4 . . . .

2. The spaces $\widetilde{wTF}$ and $\mathcal{A}^{sw}$ in more detail

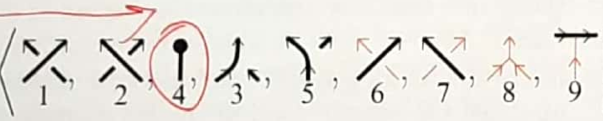
As we mentioned in the introduction, $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4]. It can be introduced as a planar algebra or as a circuit

[2] With the exception of some minor normalization, see [WKO2, Section 4.6], in particular Lemma 4.14 and the paragraph following it.

[3] In fact, almost entirely determined by Z^b, with the exception of some minor normalization of Z^u which is not determined by an associator.

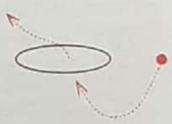
[4] Although the two formulas are written in different languages, and checking that they agree is not obvious. See Section 3.2.

algebra; we will do the latter as it is simpler and more concise. Circuit algebras are defined in [WKO2, Section 2.4]; in short, they are similar to planar algebras but without the planarity requirement for "connecting strands". As in [WKO2], each generator and relation of $\widetilde{w\mathit{TF}}$ has a local topological interpretation. Recall [WKO2, Sections 1.2, 3.4, 4.1] that $w\mathit{TF}^o$ diagrams represent certain ribbon knotted tubes with foam vertices in $\mathbb{R}^4$, and the circuit algebra $w\mathit{TF}^o$ is conjecturally a Reidemeister theory for this space (i.e., there is a surjection δ from the circuit algebra $w\mathit{TF}^o$ to ribbon knotted tubes with foam vertices, and δ is conjectured to be an isomorphism). The space $\widetilde{w\mathit{TF}}$ extends $w\mathit{TF}^o$ by adding one-dimensional strands to the picture. Note that ~~one dimensional strands cannot be knotted in $\mathbb{R}^4$~~, however, they can be knotted *with* the two-dimensional tubes. In figures two-dimensional tubes will be denoted by thick lines and one dimensional strings by thin red lines. With this in mind, we define $\widetilde{w\mathit{TF}}$ as a circuit algebra defined in terms of generators and relations, and with some extra operations beyond circuit algebra composition. Each generator, relation and operation has a local topological interpretation which provides much of the intuition behind the proofs. However, the corresponding Reidemeister theorem is only conjectural.

$\widetilde{w\mathit{TF}} = \mathrm{CA}\Big\langle$  $\Big|$ relations as in Section 2.2 $\Big|$ auxiliary operations as in Section 2.3 $\Big\rangle$

2.1. **The generators of $\widetilde{w\mathit{TF}}$.** We begin by discussing the local topological meaning of each generator shown above.

The first five generators are as described in [WKO2, Sections 4.1.1], we briefly recall their descriptions here. Knotted (more precisely, braided) tubes in $\mathbb{R}^4$ can equivalently be thought of as movies of flying rings in $\mathbb{R}^3$. The two crossings stand for movies where two rings trade places by the ring of the under strand flying through the ring of the over strand. The dotted end represents a tube "capped off" by a disk. Generators 4 and 5 stand for singular "foam vertices", and will be referred to as the positive and negative vertex, respectively. The positive vertex represents the movie shown on the left: the right ring approaches the left ring from below, flies inside it and merges with it. The negative vertex represents a ring splitting and the inner ring flying out below and to the right. To be completely precise, $\widetilde{w\mathit{TF}}$ as a circuit algebra has more vertex generators than shown above: the vertices appear with all possible orientations of the strands. However, all other versions can be obtained from the ones shown above using "orientation switch" operations (to be discussed in Section 2.3).



The thin red strands denote one dimensional strings in $\mathbb{R}^4$, or "flying points in $\mathbb{R}^3$". The crossings between the two types of strands (generators 6 and 7) represent "points flying through rings". For example, the picture on the left shows generator 6, where "the point on the right approaches the ring on the left from below, flies through the ring and out to the left above it". This explains why there are no generators with a thick strand crossing under a thin red strand: a ring cannot fly through a point.

Generator 9 is a trivalent vertex of 1-dimensional strings in $\mathbb{R}^4$. Finally, the last generator is a *mixed vertex:* a one-dimensional string attached to the wall of a

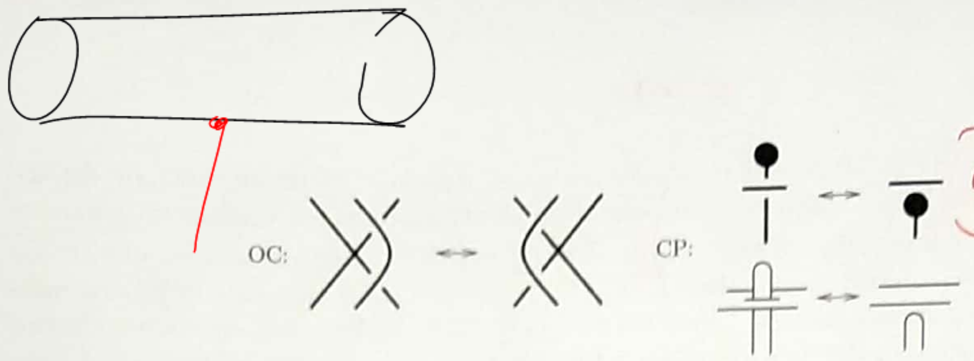
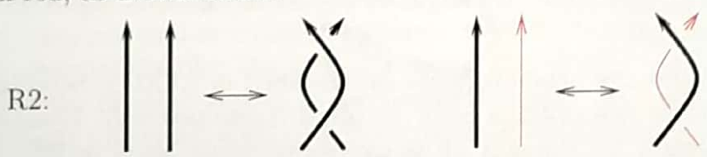


Figure 2. The OC and CP relations.

2-dimensional tube. All generators should be shown in all possible strand orientation combinations; we are suppressing this to save space.

2.2. **The relations.** As a list, the relations for $\widetilde{wTF}$ are the same as the relations for wTF^o [WKO2, Section 4.5]: {R1^s, R2, R3, R4, OC, CP}. Recall that R1^s is the weak (framed) version of the Reidemeister 1 move; R2 and R3 are the usual second and third Reidemeister moves; R4 allowes moving a strand over or under a vertex. OC stands for *Overcorssings Commute*, CP for *Cap Pullout*: these two relations are shown in Figure 2, for a detailed explanation see [WKO2, Section 4.1.2].

In $\widetilde{wTF}$ all relations should be interpreted in all possible combinations of strand types and orientations (tube or string), for example the lower strand of the R2 relation can either be thick black or thin red, as shown below:



Similarly, any of the lower strands of the R3, R4, and OC relations may be thin red.

As in wTF^o, the relations all have local topological meaning and conjecturally $\widetilde{wTF}$ is a Reidemeister theory for ribbon knotted tubes in $\mathbb{R}^4$ with caps, singular foam vertices and attached strings. For example, Reidemeister 2 with a thin red bottom strand is imposed because a point flying in through a ring and then immediately flying back out is isotopic to not having any interaction between the point and ring at all.

It is easy to verify that all relations represent local isotopies of welded (ribbon knotted) tubes in $\mathbb{R}^4$ with singular vertices and attached strings. What is not clear at this stage is that this is a complete Reidemeister theory, that is, whether this is a complete set of relations. For more detail on this see [WKO2, Section 1.2].

2.3. **The operations.** Like wTF^o, $\widetilde{wTF}$ is equipped with a set of auxiliary operations in addition to the circuit algebra structure.

The first of these is orientation reversal. For the thin (red) strands, this simply means reversing the direction of the strand. For the thick strands (tubes), orientation switch comes in two versions. Recall from [WKO2, Section 3.4] that in the topological interpretation of wTF^o, each tube is oriented as a 2-dimensional surface, and also has a distinguished "core": a line along the tube which is oriented as a 1-dimensional manifold and determines the "direction" or "1-dimensional orientation" of the tube. Both of these are determined by the direction of the strand in the circuit algebra, via Satoh's tubing map.

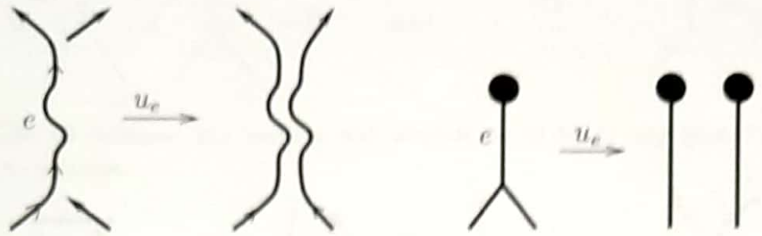
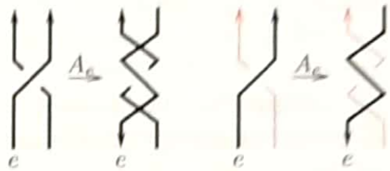


Figure 3. Unzip and disc unzip.

Topologically, the operation "orientation switch", denoted S_e for a given strand e, acts by reversing both the (1-dimensional) direction and the (2-dimensional) orientation of the tube e. Diagrammatically, this corresponds to simply reversing the direction of the corresponding strand e.

The "adjoint" operation, denoted A_e, on the other hand only reverses the (1-dimensional) direction of the tube e, not the orientation as a surface. Diagrammatically, this manifests itself as reversing the strand direction and adding two virtual crossings on either side of each crossing where e crosses *over* another strand, as shown on the right (note that the strand below e may be thick or thin). Note that virtual crossings don't appear when e crosses under another strand. For more details on orientations and orientation switches, see [WKO2, Sections 3.4 and 4.1.3].



The unzip operation u_e doubles the strand e using the blackboard framing, and then attaches the ends of the doubled strand to the connecting ones, as shown in Figure 3. We restrict unzip to strands whose two ending vertices are of different signs. (For the definition of crossing and vertex signs, see [WKO2, Sections 3.4 and 4.1].) Topologically, the blackboard framing of the diagram induces a framing of the corresponding tube in $\mathbb{R}^4$ via Satoh's tubing map, and unzip is the act of "pushing the tube off of itself slightly in the framing direction". Note that unzips preserve the ribbon property.

A related operation, *disc unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagram world, in the direction of the blackboard framing), as before. An example is shown in Figure 3; see [WKO2, Section 4.1.3] for details on framings and unzips.

So far all the operations we have introduced had already existed in wTF^o. There is also a new operation is called "puncture", denoted p_e, which diagrammatically simply turns the thick black strand e into a thin red one. The corresponding topological picture is "puncturing a tube", i.e., removing a small disk from it and retracting the rest to its core. Any crossings where e passes under another strand are not affected, while crossings in which e is the over strand turn into virtual crossings.

For simplicity, we place a restriction on which strands can be punctured, namely at each (fully thick black) vertex puctures are only allowed for one of the three meeting strands, as shown on the left of Figure 4. More general puctures could be allowed in a theory with more than one kind of "string to tube" vertex. The right of the same figure shows that when puncturing one of the thick strands of a mixed vertex, the puncture "spreads". Topologically, this is because the mixed vertex represents a string attached to a tube, so when puncturing e, the entire tube retracts to its core. Finally, a capped tube disappears when punctured.

p

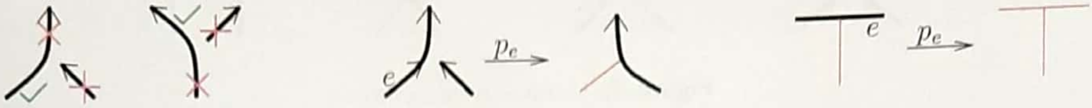


Figure 4. Puncture operations: the picture on the left shows which edges can be punctured at each vertex. The middle and right pictures show the effect of puncture operations.

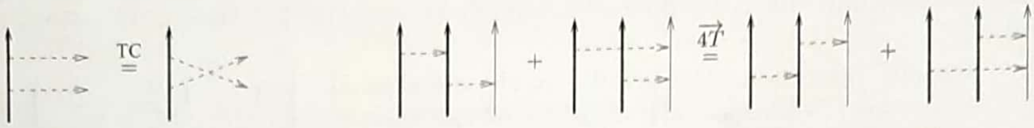
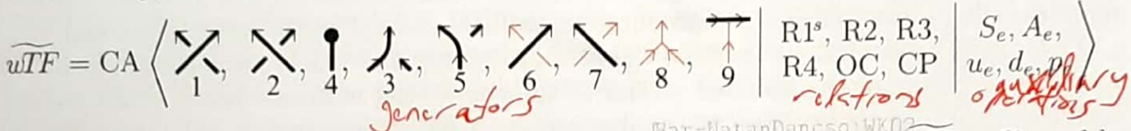
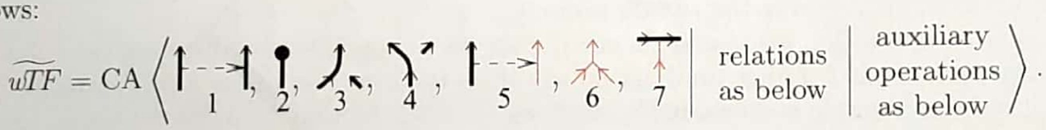


Figure 5. The TC and $\overrightarrow{4T}$ relations. Note that the 3rd strand in each term of the $\overrightarrow{4T}$ relation can be either thick black or thin red, the relation applies in either case.

In summary,



2.4. **The associated graded structure $\mathcal{A}^{sw}$.** As in [WKO2], the space $\widetilde{wTF}$ is filtered by powers of the augmentation ideal and its associated graded space, denoted $\mathcal{A}^{sw}$, is a "space of arrow diagrams on foam skeletons with strings". As a circuit algebra, $\mathcal{A}^{sw}$ is presented as follows:



Generators 1 and 5 are called single arrows and they are of degree one, while all others are "skeleton features" of degree zero. The relations are almost the same as in [WKO2, Section 4.2.1], which describes the relations for the associated graded of wTF^o: $\overrightarrow{4T}$ (the 4-Term relation), TC (Tails Commute), RI (Rotation Invariance), CP (the arrow Cap Pullout), and VI (Vertex Invariance). For $\widetilde{wTF}$ there is an additional relation TF (Tails Forbidden on strings). The TC and $\overrightarrow{4T}$ relations are shown in Figure 5. The Vertex Invariance relation is shown in Figure 6: here the $\pm$ signs depend on the strand orientations. Note that the type of the vertex and the types of each strand (thick black or thin red) are left undetermined: the VI relation applies in all cases. Figure 7 shows the other relations: RI, CP and TF. Note that technically TF is not a relation: there were no generators with an arrow tail on a thin red strand, so saying that such an element vanishes is meaningless. However, without TF the VI relation would have to be stated for all the sub-cases of 0, 1 or 3 thin red strands, so we prefer this cleaner way, even if it is a slight abuse of notation.

Figure 6. The VI relation: the vertices and strands could be of any type, but the same throughout the relation.

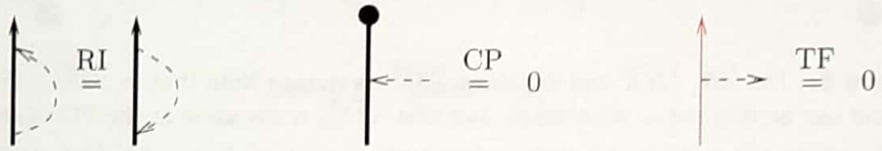
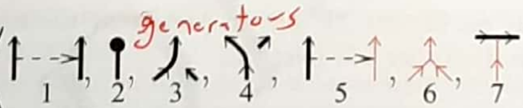


Figure 7. The RI and CP relations, and the TF relation (which is not really a relation).

Each operation on $\widetilde{uTF}$ induces a corresponding operation on $\mathcal{A}^{sw}$. Orientation switch, adjoint, unzip, cap unzip, and long strand deletion act exactly the same way as they do for uTF^{oo}. We quickly recall these here, for details see [WKO2, Section 4.2.2]. The orientation switch S_e reverses the orientation of the skeleton strand e, and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow heads and tails on } e\}}$. The adjoint operation also reverses the skeleton strand e and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow heads on } e\}}$. Given a skeleton S with a distinguished strand e, unzip (or disc unzip, if e is capped) is an operation $u_e : \mathcal{A}^{sw}(S) \to \mathcal{A}^{sw}(u_e(S))$ which maps each arrow ending on e to a sum of two arrows, one ending on each of the two new strands which replace e. Deleting a long strand e kills all arrow diagrams with any arrow ending on e. The operation induced by puncture, denoted p_e, turns the formerly thick black e into a thin red strand, and kills any arrow diagram with any arrow tails on e.

To summarise:

$\widetilde{uTF} = CA \Big\langle$  $\Big|$ $\overrightarrow{4T}$, TC, VI, CP, RI, TF $\Big|$ S_e, A_e, u_e, d_e, p_e $\Big\rangle$

As in [WKO2, Definition 3.7], we define a "w-Jacobi diagram" (or just "arrow diagram") by also allowing trivalent chord vertices, each of which is equipped with a cyclic orientation, and modulo the $\overrightarrow{STU}$ relations of Figure 8. Denote the circuit algebra of formal linear combinations of these w-Jacobi diagrams by $\mathcal{A}^{swt}$. Then, as in [WKO2, Theorem 3.8], we have the following bracket-rise theorem:

Theorem 2.1. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 8 hold in $\mathcal{A}^{swt}$.*

The proof is identical to the proof of [WKO2, Theorem 3.8]. In light of this isomorphism, we will drop the extra "t" from the notation and use $\mathcal{A}^{sw}$ to denote either of these spaces. As in [WKO2], the primitive elements of $\mathcal{A}^{sw}$ are connected diagrams, denoted $\mathcal{P}^{sw}$, and $\mathcal{P}^{sw} = \{\text{trees}\} \oplus \{\text{wheels}\}$ as a vector space. Examples of trees and wheels are shown in Figure 9; for details see [WKO2, Section 3.1]. Note that the RI relation can now be rephrased (via $\overrightarrow{STU}_2$) as the vanishing of the wheel with a single spoke, or one-wheel.

We recall the following two crucial facts [WKO2, Lemmas 4.6 and 4.7]:

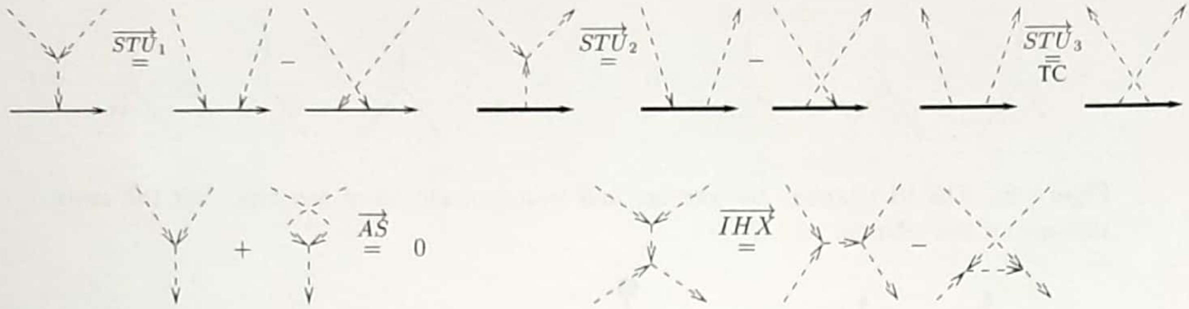


Figure 8. The $\overrightarrow{AS}$, $\overrightarrow{IHX}$ and the three $\overrightarrow{STU}$ rerations. Note that in $\overrightarrow{STU}_1$, the skeleton strand can be thin red or thick black, and that $\overrightarrow{STU}_3$ is the same as the TC relation.

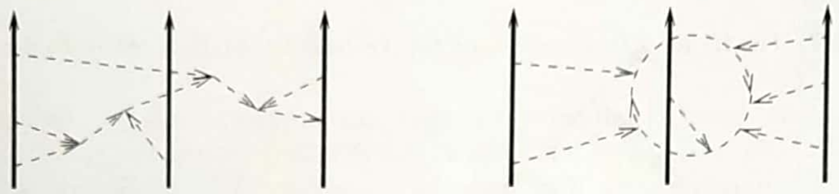


Figure 9. An example of a *tree*, left, and a *wheel*, right.

Fact 2.2. *$\mathcal{A}^{sw}(\uparrow)$, the part of $\mathcal{A}^{sw}$ with skeleton a single capped strand, is isomorphic as a vector space to the completed polynomial algebra freely generated by wheels w_k with $k \geq 2$.*

Fact 2.3. *$\mathcal{A}^{sw}(\nearrow\!\!\searrow) \cong \mathcal{A}^{sw}(\uparrow_2)$, where $\mathcal{A}^{sw}(\nearrow\!\!\searrow)$ stands for the space of arrow diagrams whose skeleton is a single vertex (the picture shows a positive vertex but the statement is true for all kinds of vertices with thick black strands), and $\mathcal{A}^{sw}(\uparrow_2)$ is the space of arrow diagrams on two (thick black) strands.*

The following Lemma will play an important role, in particular the second isomorphism stated is the map φ appearing in Theorem 1.1, part (2):

Lemma 2.4. *$\mathcal{A}^{sw}\left(\uparrow\right) \overset{\varphi}{\cong} \mathcal{A}^{sw}(\uparrow)$, where each side of the isomorphism is a space of arrow diagrams on the skeleta shown in parentheses. On the left, the thin red string is a tangle end. The thick black strand may continue past the arrow, and there may be additional skeleton components: the same on both sides. Applying the isomorphism φ twice, one also obtains $\mathcal{A}^{sw}\left(\nearrow\!\!\searrow\right) \cong \mathcal{A}^{sw}(\uparrow_2)$.*

Proof. We construct inverse maps between the two spaces. There is an obvious map $\mathcal{A}^{sw}(\uparrow) \xrightarrow{\psi} \mathcal{A}^{sw}\left(\uparrow\right)$, shown in Figure 10: given an arrow diagram on a single thick black strand, place all arrow endings (denoted "x") on the strand above the tube/string vertex.

In the other direction, consider an arrow diagram on the capped/stringed vertex. One may assume that there are only arrow tails on the capped strand under the vertex: any arrow head may be commuted using $\overrightarrow{STU}$ relations towards the cap, where it is killed by the CP relation[5]. On the thin red strand there are only arrow heads. To construct φ, first

[5]This argument also appears in [WKO2], for example as the basic idea for the proof of Fact 2.2.

Flip the order to match the order they appear in the proof.

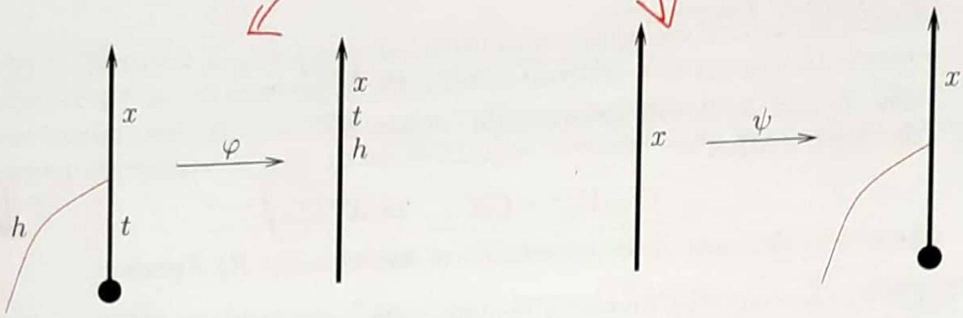


Figure 10. Inverse maps.

"push" the arrow tails (denoted "t") from the capped strand up across the vertex using the VI relation. Since tails vanish on the thin red strand, they simply slide past the vertex. Once the capped side is cleared, continue by sliding the arrow heads "h" up from the thin red string to the strand above the vertex. Now the cap relation kills any arrow heads on the capped strand, so once again they simply slide past the vertex. The result placed on a single thick black strand is shown in Figure 10.

It is clear that ψ is well-defined, we leave it to the reader to check that so is φ as a short exercise. Given that both maps are well-defined, it is clear that they are inverses of each other. □

2.5. **The homomorphic expansion.** As discussed in [WKO2, Section 2.3], an expansion for $\widetilde{wTF}$ is a map $Z^w : \widetilde{wTF} \to \mathcal{A}^{sw}$ with the property that the associated graded map $\operatorname{gr} Z^w : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ is the identity map $\mathrm{id}_{\mathcal{A}^{sw}}$. A homomorphic expansion is an expansion which also intertwines each operation of $\widetilde{wTF}$ with its arrow diagrammatic counterpart. In [WKO2, Theorems 4.9 and 4.11] we proved that the existence of solutions for the Kashiwara–Vergne equations implies that there exsists a homomorphic expansion for wTF^o. In fact that homomorphic expansions[6] for wTF^o are in one-to-one correspondence with solutions to the Kahiwara-Vergne problem.

The point of this paper is to provide a topological construction for such a homomorphic expansion (and hence for a solution of the Kashiwara–Vergne conjecture), and this is easier to do for the slightly more general space $\widetilde{wTF}$.

Let $\mathcal{A}^{osw} \subset \mathcal{A}^{sw}$ denote arrow diagrams on wTF^o skeleta, the associated graded space of wTF^o. In [WKO2, Section 4.3] we prove the following crucial fact:

Fact 2.5. *$Z^{ow} : wTF^o \to \mathcal{A}^{osw}$ is a (group-like) homomorphic expansion for wTF^o, if and only if the (group-like) values V and C for the positive vertex and the cap, respectively, satisfy the following equations:*

(1) *The* R4 Equation:

$$V_{12}R_{(12)3} = R_{23}R_{13}V_{12} \quad in\ \mathcal{A}^{sw}(\uparrow_3).$$

Here $R = e^{a_{12}}$ is the value of the crossing, where a denotes a single arrow from the over strand 1 to the under strand 2. In the equation V_{12} denotes the value V placed on strands 1 and 2; $R_{(12)3}$ means the first strand of R is doubled and placed on strands 1 and 2, and the second strand of R is placed on strand 3; ~~*and so on.*~~

and similarly for R_{23} and R_{13}, and $R_{(12)3}$ ....

[6]Subject to the minor technical condition that the value of the vertex doesn't contain isolated arrows.

(2) *The* Unitarity Equation:

$$V \cdot A_1A_2(V) = 1 \qquad in\ \mathcal{A}^{sw}(\uparrow_2),$$

where A_1 and A_2 denote the antipode operations.

(3) *The* Cap Equation[7]:

$$C_{(12)}V_{12}^{-1} = C_1C_2 \qquad in\ \mathcal{A}^{sw}(\uparrow_2),$$

where the subscripts mean strand placements as in the R4 Equation.

We begin by showing that finding a homomorphic expansion for $\widetilde{uTF}$ is no harder than finding one for uTF^o:

Theorem 2.6. *Homomorphic expansions for uTF^o are in one-to-one correspondence with homomorphic expansions for $\widetilde{uTF}$ via unique extension and restriction.*

Proof. Every element of uTF^o is also in $\widetilde{uTF}$, hence any Z^w restricts to a homomorphic expansion Z^{ow} of uTF^o. Every element of $\widetilde{uTF}$ is the result of puncturing – possibly on multiple strands – an element of uTF^o, and Z^w is required to commute with punctures. Hence any Z^{ow} uniquely extends to a Z^w. □

$$\begin{array}{ccc} uTF^o & \hookrightarrow & \widetilde{uTF} \\ \downarrow Z^{ow} & & \downarrow Z^w \\ \mathcal{A}^{osw} & \hookrightarrow & \mathcal{A}^{sw} \end{array}$$

As it turns out, the value of the left-punctured vertex is trivial under any homomorphic expansion. This fact will be useful later, so we prove it here.

Lemma 2.7. *For any homomorphic expansion Z^w, $Z^w($$) = 1$, that is, the Z^w-value of a left punctured vertex is trivial.*[8]

Proof. Recall from [WKO2, Proof of Theorem 4.9] that the Z^w-value V of the positive (not punctured) vertex can be written as $V = e^be^t$, where b is a linear combination of wheels only and t (denoted uD in [WKO2]) is a linear combination of trees. Puncturing the left strand of V kills all arrow diagrams with tails on the left. Diagrams that survive are: wheels and short arrows (whose head and tail are not separated by any other arrow ending) supported entirely on the right strand, and arrows pointing from the right to the left[9]. Observe that all of the surviving arrow diagrams commute with each other. *when regarded in $\mathcal{A}(\uparrow\uparrow)$? otherwise, what does "commute" mean?*

Denote the value of the punctured vertex by $p_1V = e^{p_1(b)}e^{p_1(t)}$. Recall that V must satisfy the Unitarity Equation of Fact 2.5, so $p_1V \cdot A_1A_2(p_1V) = 1$. Since wheels have only tails, $A_1A_2(p_1(b)) = p_1(b)$. Each arrow has one head, so $A_1A_2(p_1(t)) = -p_1(t)$. Hence, using commutativity, $p_1V \cdot A_1A_2(p_1V) = e^{2p_1(b)} = 1$, which implies that $p_1(b) = 0$. As for $p_1(t)$, showing that there are no arrows pointing from the right to the left strand is a direct computation in degree 1.

[7] For convenience we state the Cap Equation phrased for caps at the bottom of strands, hence the difference from the equivalent formulation in [WKO2].

[8] Assuming that V is free of short arrows on the right strand: arrows whose tail and head are not separated by any other arrow endings. Adding or removing such arrows does not affect whether Z^w is a homomorphic expansion, hence the assumption. For details see the last paragraph of the proof of this Lemma, or [WKO2, Section 4.4].

[9] Using that if all tails of a tree are supported on one strand, then the tree is a single arrow, due to TC and the anti-symmetry of the trivalent arrow vertices.

I don't like Lemma 2.7. As stated, it is a lie that is corrected by Footnote 8, and by the last paragraph of the proof. Either the assumptions for the lemma should be strengthened or the conclusions weakened.

In [WKO2, Section 4.4] we showed that short arrows supported on either strand of V don't affect whether Z^w is a homomorphic expansion: if Z^w is a homomorphic expansion and a is a linear combination of such short arrows, then replacing V by e^aV gives rise to another homomorphic expansion. Hence, as in [WKO2] we assume there are no short arrows in V, so $p_1V = 1$. □

3. Proof of Theorem 1.1

Let us now return to the proof of our main theorem, Theorem 1.1. For convenience we recall that it consists of three parts, with $\mathcal{K}^b := uPaB$, $\mathcal{K}^u := sKTG$ and $\mathcal{K}^w := \widetilde{wTF}$:

$$\begin{array}{ccccccc} \mathcal{K}: & & \mathcal{K}^b & \xrightarrow{\text{cl}} & \mathcal{K}^u & \xrightarrow{a} & \mathcal{K}^w \\ \downarrow Z & & \downarrow Z^b & & \downarrow Z^u & & \downarrow Z^w \\ \mathcal{A}: & & \mathcal{A}^{hor} & \xrightarrow{\text{cl}} & \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array}$$

(1) Assuming that $Z: \mathcal{K} \to \mathcal{A}$ exists, it is determined[10] by Z^u.

(2) Given a Drinfel'd associator Φ and associated Z^u, there is an explicit formula for V in terms of Φ:

$$V = C_1^{-1}C_2^{-1}\varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2}\Phi(a_{23}, a_{43})\right) C_{(12)}, \qquad (2)$$

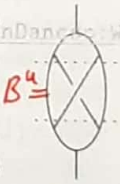
where a denotes a single arrow. This agrees with the formula proven in [AET].

(3) Every Z^b extends to a Z.

3.1. Proof of Part (1).

We prove Part 1 in two steps: first verifying the easier "tree level" case, which nonetheless contains the main idea, then in general.

3.1.1. *Tree level proof of Part (1).* Let $\mathcal{A}^{tree}$ denote the quotient of $\mathcal{A}^{sw}$ by all wheels, and let $\pi : \mathcal{A}^{sw} \to \mathcal{A}^{tree}$ denote the quotient map (cf [WKO2, Section 3.2]). Part (1) of the main theorem is the same as stating that Z^w is determined by Z^u. Z^w, in turn is determined by the values V and C of the positive vertex and the cap [WKO2, Sections 4.3 and 4.5], so one only needs to show that V and C are determined by Z^u. Proving this "on the tree level" means showing only that $\pi(V)$ and $\pi(C)$ are determined by Z^u, a partial result which makes use of the main idea without the technical details. In particular, observe that since C is a linear combination of products of wheels (Fact 2.2), we have $\pi(C) = 1$, so we only need to show that $\pi(V)$ is determined by Z^u.



Let B^u denote the "buckle" *sKTG*, as shown on the right (ignore the dotted lines for now). All edges are oriented up, and by the drawing conventions of [WKO2, Section 4.6] all the vertices in the bottom half of the picture are negative and all the ones in the top half are positive. Let $B^w = a(B^u) \in \widetilde{wTF}$, and $\beta^u := Z^u(B^u)$. Note that β^u can be thought of as a chord diagram on four strands: use VI relations to move all chord endings to the "middle" of the skeleton, between the dotted lines on the picture. Hence, we use the abusive notation $\beta^u \in \mathcal{A}^u(\uparrow_4)$. Let $\beta^w = \alpha(\beta^u)$, and note that by the compatibility of Z^u and Z^w we have $\beta^w = Z^w(B^w)$. We will perform a series of operations on B^w and $\pi(\beta^w)$ to recover $\pi(V)$ from it.

[10] In fact, determined by Z^b, aside from a minor normalization of Z^u: for details see the discussion of the "noose and balloon" KTGs in [WKO2, Section 4.6].

First, connect (a circuit algebra operation in $\widetilde{uTF}$) positive vertex to the bottom of B^w, as shown in Figure 11. Then unzip the edge marked by u, and puncture the edges marked e and e', in that order. Then attach a cap (once again a circuit algebra operation) to the thick black end at the bottom. Finally, unzip the capped strand.

This is a circuit algebra operation, and correspondingly $\pi(\beta^w)$ is circuit algebra multiplied by $\pi(V)$, where V is the value of the vertex. This is a circuit algebra operation; keep in mind that image of the value of the cap is trivial in $\mathcal{A}^{trees}$.

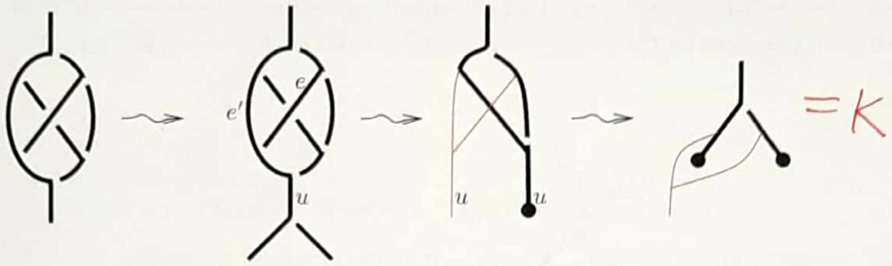


Figure 11. From the "buckle" β^w to the (modified) vertex.

Let us call the resulting w-tangled foam K, as shown at the right in Figure 11. What is $Z^w(K)$? Due to the homomorphicity of Z, it is obtained from β^w by performing the same series of operations in the associated graded: a circuit algebra composition with V, unzip, punctures, circuit algebra composition with C, and disc unzip. Notice that the left strand of that attached vertex got punctured, and hence by Lemma 2.7 the attached value V cancels.[11] $Z^w(K)$ still depends on the value C. At the tree level, since $\pi(C) = 1$, $\pi(Z^w(K))$ can be computed from β^w by performing punctures and unzips. Since $\beta^w = \alpha(\beta^u)$, this means that $\pi(Z^w(K))$ is determined by Z^u.

On the other hand, note that the space of chord diagrams on the skeleton of K is the space $\mathcal{A}(\uparrow_2)$ by Lemma 2.4 and VI. Note also that K is a circuit algebra combination of a vertex, two left-punctured right-capped vertices and an all-red-strings vertex, and the Z^w-values of the latter three are trivial. So $\pi(Z^w(K)) = \pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$. Hence, $\pi(V)$ is determined by Z^u as needed. □

3.1.2. *Complete proof of Part (1).* In the previous subsection we showed that Z^u determines $\pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$. Now we show that in turn, $\pi(V)$ determines both V and C uniquely, using a perturbative argument.

By contradiction, assume this is not the case, in particular, first assume that there exist $V \neq V'$, both of which are vertex values of Z^u-compatible homomorphic expansions, such that $\pi(V) = \pi(V')$. Let v denote the lowest degree term of $V - V'$. Note that v is primitive and $v \in \ker \pi$, so v is a homogeneous linear combination of wheels. By the Unitarity Equation of Fact 2.5, we have $A_1A_2(v) = -v$. Recall that A_i reverses the direction of the strand i and multiplies each arrow diagram by (-1) to the number of heads on that strand. Since v has only tails, $A_1A_2(v) = v$, so $v = -v$, so $v = 0$, a contradiction. Therefore, $\pi(V)$ determines V uniquely.

[11] Any short arrows would also cancel when the right strand is capped.

Now we show that V determines C uniquely. Assume there are different values C and C' in $\mathcal{A}^{sw}(\uparrow)$ so that (V, C) and (V, C') are both vertex-cap value pairs of Z^u-compatible homomorphic expansions. Let c denote the lowest degree term of $C - C'$, then c is a scalar multiple of a single wheel. The Cap Equation of Fact 2.5 implies $c_{(12)} = c_1 + c_2$ in $\mathcal{A}^{sw}(\uparrow_2)$.

There is a well-defined linear map $\omega : \mathcal{A}^{sw}(\uparrow_2) \to \mathbb{Q}[x, y]$ sending an arrow diagram – which has arrow tails only on each strand – to "x to the power of the number of tails on strand 1, times y to the power of the number of tails on strand 2". Assume $c = \alpha w_r$, where w_r denotes the r-wheel, and $\alpha \in \mathbb{Q}$. Then $0 = \omega(c_{(12)} - c_1 - c_2) = \alpha((x+y)^r - x^r - y^r)$, so either $r = 1$ or $\alpha = 0$. But $w_1 = 0$ in $\mathcal{A}^{sw}$ by the RI relation, hence $\alpha = 0$ and thus $c = 0$, a contradiction. □

3.2. **Proof of Part (2).** In this section we compute V, the value of the vertex, from Φ, the Drinfel'd associator determining Z^b, using the construction of Part (1). We then show that the result translates to the [AET] formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators.

3.2.1. *From Φ to V.* To compute V, consider once again the w-tangled foam K on the right of Figure 11. On one hand, $Z^w(K)$ can be computed directly from the generators: $Z^w(K) = C_1C_2V_{12} \in \mathcal{A}^{sw}(\uparrow_2)$, since the values of the left-punctured vertices are trivial. On the other hand, one can compute $Z^w(K)$, using the compatibility with Z^u, from $\beta^u = Z^u(B^u)$, where B^u is the closure of the parenthesised braid B^b shown in Figure 12. In particular, $Z^w(K) = C_{12}\varphi(p_1p_3\alpha(\beta^u))$, where φ is the isomorphism of Lemma 2.4. In summary, $V = C_1^{-1}C_2^{-1} \cdot \varphi(p_1p_3\alpha(\beta^u))C_{12}$. To obtain the formula (2) of Theorem 1.1, one needs to compute $p_1p_3\alpha(\beta^u)$ in terms of Φ.

Too big a leap! needs explanation.

By the compatibility of Z^u and Z^b, it is enough to compute $\beta^b := Z^b(B^b)$. The result can be read from the picture in Figure 12:

$$\beta^b = \Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34}.$$

To interpret this formula, recall that the associator Φ is an element of $\mathcal{A}^{hor}(\uparrow_3)$, and the subscripts show which strands diagrams are placed on. For example, the notation $\Phi^{-1}_{(13)24}$ means doubling the first strand of Φ^{-1} and placing the resulting chord endings on strands 1 and 3, as well as placing the chord endings from the other two strands of Φ^{-1} on strands 2 and 4. Also recall that $R = e^{c/2}$, where c is a single horizontal chord between two strands (and in this case R_{32} means that this chord runs between strands 3 and 2).

As β^u is the tree closure of β^b, it is given by the same formula interpreted as an element of $\mathcal{A}^u(\uparrow_4)$. One then applies α to obtain $\beta^w = \alpha(\beta^u)$, followed by puncturing strands 1 and 3 and caping strands 2 and 4.

To begin understanding the effect of these operations, we note that $p_3\alpha(R_{32}) = e^{a_{23}/2}$, where a_{ij} is a single arrow pointing from strand i to strand j.

Recall that Φ_{123} is a horizontal chord associator which can be expressed as a power series in non-commuting variables c_{12} and c_{23} (i.e., chords between strands 1—2 and 2—3, respectively). The image of Φ in the quotient where c_{12} and c_{23} commute is 1. Hence, $p_1p_3(\alpha\Phi^{-1}_{123}) = 1$, as the punctured strands only support arrow heads, and tails on the middle strand commute by TC. Similarly, $p_1p_3(\alpha\Phi_{132}) = 1$ because the punctures kill the entire "left side" of the associator (that is, $p_1p_3\alpha(c_{13}) = 0$).

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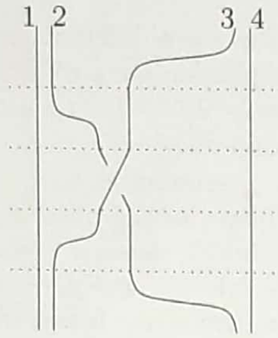

$$
\begin{array}{ll}
\Phi_{(12)34} & \longrightarrow \Phi(a_{23}, a_{43}) \\
\Phi^{-1}_{123} & \longrightarrow 1 \\
R_{32} & \longrightarrow e^{a_{23}/2} \\
\Phi_{132} & \longrightarrow 1 \\
\Phi^{-1}_{(13)24} & \longrightarrow \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})
\end{array}
$$

Figure 12. Computing β^b. Strands are numbered at the top and multiplication is read from bottom to top; the rightmost column lists the images of the factors under $p_1p_2\alpha$.

Thus $p_1p_3(\beta^w)$ can be expressed as $p_1p_3(\alpha(\Phi^{-1}_{(13)24}))e^{a_{23}/2}p_1p_3(\alpha(\Phi_{(12)34}))$. Since strands 1 and 3 are both punctured, no arrows can be supported between these strands, hence $p_1p_3\alpha(\Phi_{(12)34}) = p_3\alpha(\Phi_{234})$.

Expressing Φ as a power series in two variables (abusively also denoted by Φ), $\Phi_{234} = \Phi(c_{23}, c_{34})$, and $p_1p_3\alpha(\Phi(c_{23}, c_{34})) = \Phi(a_{23}, a_{43})$. Similarly, $\Phi^{-1}_{(13)24} = \Phi^{-1}(c_{(13)2}, c_{24})$, where $c_{(13)2} = c_{12} + c_{32}$.

A well-known property of associators is $\Phi(c_{ij}, c_{jk}) = \Phi(c_{ij}, -c_{ij} - c_{ik})$. Hence, $\Phi^{-1}(c_{(13)2}, c_{24}) = \Phi^{-1}(c_{(13)2}, -c_{(13)2} - c_{(13)4})$, so $p_1p_3\alpha\Phi^{-1}_{(13)24} = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. To summarise,

$$p_1p_3\beta^w = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}). \tag{3}$$

Combining this with the computation of V at the beginning of this section, we obtain the formula of Theorem 1.1 part (2). Later in Lemma 3.5 we'll also compute the even part of the value of the cap explicitly, and find that it is $\alpha(\nu^{1/4})$, where ν is the Kontsevich integral of the unknot.

3.2.2. *From V to Kashiwara–Vergne.* This technical section is mainly interesting for readers familiar with the Alekseev–Enriquez–Torossian work on Kashiwara–Vergne solutions. Results of this section are not used afterwards.

The value $\varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}))$ can be computed more explicitly, which is necessary in order to compare it with the [AET] formulas. The first strand of $\mathcal{A}^{sw}(\uparrow_2)$ joins strands 1 and 2 in a vertex, and the second strand of $\mathcal{A}^w(\uparrow_2)$ joins strands 3 and 4. Strands 1 and 3 are punctured and strands 2 and 4 are capped. Let us call the two strands of $\mathcal{A}^w(\uparrow_2)$ strand I and strand II to avoid confusion. Recall from the construction of φ that one first slides arrow tails from the capped strands "up" through the vertices, then slides all the heads up from the punctured strands 1 and 3. Thus one obtains an element of $\mathcal{A}^w(\uparrow_2)$ in which all arrow heads are below all tails on both strands. The result is shown in Figure 13, and explained in the caption.

For a quick re-cap of [AET] notions, let $\mathfrak{lie}_2$ denote the free Lie algebra on two generators x and y. Let $\mathfrak{tder}_2$ denote *tangential derivations* of this Lie algebra, that is, derivations d with the property that $d(x) = [x, a_1]$ and $d(y) = [y, a_2]$, where $a_1, a_2 \in \mathfrak{lie}_2$. Let $\mathrm{TAut}_2 := \exp(\mathfrak{tder}_2)$ denote the group of *tangential automorphisms* of $\mathfrak{lie}_2$. There is a map $\theta : \mathfrak{lie}_2^2 \to \mathfrak{tder}_2$, sending a pair (a_1, a_2) to the derivation d given by $d(x) = [x, a_1], d(y) = [y, a_2]$. The

$P_1 P_3 \alpha(\beta^u) =$

Interface formulas w/ explanations!!

$P_1 P_3 \alpha(\beta^b)$

$= P_1 P_3 \alpha(\Phi^{-1}_{(13)24} - - - -)$

$= P_1 P_3 \alpha(\Phi^{-1}(c_{12} + c_{32}, c_{24})$

$= P_1 P_3 \alpha \, \Phi^{-1} \quad P_1 P_3 \alpha \, \Phi \quad P_1 P_3 \alpha \, R$

$a \overset{(1)}{=} b \overset{(2)}{=} c \qquad = Z$ Option I

We want to compute $\Upsilon := P_1 P_3 \alpha \, Z^u(\beta^u)$

$\Upsilon = a$ Option II

Par of explaining, so

$\Upsilon = b$

Par of nonsense

$\Upsilon = c$