

We need an index of notation! (Do-)

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: W-FOAMS, THE KASHIWARA-VERGNE THEOREM AND DRINFEL'D ASSOCIATORS

DROR BAR-NATAN AND ZSUZSANNA DANCZO

ABSTRACT. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological solution to the Kashiwara-Vergne problem. In particular we recover via a topological argument the Alekseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of *2-dimensional foams* and various associated features in four-dimensional space. We use a topological construction which we name the double tree construction to show that every *expansion* (also known as *universal finite type invariant*) of parenthesized braids extends first to an expansion of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is uniquely determined by *Drinfel'd associator* Φ , and an expansion for w-knotted objects is uniquely determined by a solution V of the Kashiwara-Vergne problem [KV], as reformulated by Alekseev and Torossian [AT]. Hence our result provides a topological framework for the result of [AET] that "there is a formula for V in terms of Φ ", along with an independent topological proof of the Kashiwara-Vergne Theorem and the Alekseev-Enriquez-Torossian formula.

CONTENTS

1. Introduction	2
1.1. Executive Summary	2
1.2. Detailed Introduction	3
1.3. Computations	7
1.4. Paper Structure	7
2. The spaces \widetilde{uTF} and \mathcal{A}^{sw} in more detail	7
2.1. The generators of \widetilde{uTF}	8
2.2. The relations	8

Date: first edition in future, this edition Jun. 6, 2023. The arXiv:???????? edition may be older.

1991 *Mathematics Subject Classification.* 57M25.

Key words and phrases. virtual knots, w-braids, w-knots, w-tangles, knotted graphs, finite type invariants, Kashiwara-Vergne, associators, double tree, free Lie algebras.

The first author's work was partially supported by NSERC grants RGPIN-264374 and RGPIN-2018-04350, and by the Chu Family Foundation (NYC), and wishes to thank the Sydney Mathematics Research Institute for their hospitality and support. The second author was partially supported by NSF grant no. 0932078 000 while in residence at the Mathematical Sciences Research Institute, and by the Australian Research Council DECRA DE170101128 Fellowship. Electronic version and related files at [WKO3], <http://www.math.toronto.edu/~drorbn/papers/WK03/>.

In Sections 2.1 to 2.3 of [WKO2] we discuss associated graded structures and expansions for general algebraic structures. For any algebraic structure (think braids, or tangles with tangle composition), one allows formal linear compositions of elements of the same *kind* (think, same skeleton). Associated graded structures are taken with respect to the filtration by powers of the *augmentation ideal*. For the spaces $uPaB$, $sKTG$ and \widetilde{wTF} , the associated graded spaces \mathcal{A}^{hor} , \mathcal{A}^u and \mathcal{A}^{sw} are the spaces of “horizontal chord diagrams on parenthesized strands”, “chord diagrams on trivalent skeleta”, and “arrow diagrams”, as described in [BNI], [WKO2, Section 4.6], and Section 2 of this paper, respectively. As a result, the associated graded structure of \mathcal{K} is

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{cl} \mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and α , respectively. More specifically, cl is the “closure of chord diagrams” and α is “replacing each chord with the sum of its two possible orientations”, see [WKO2, Section 3.3].

An expansion [WKO2, Section 2.3] is a filtration-respecting map from an algebraic structure to its associated graded structure, whose associated graded map is the identity. In knot theory, expansions are also called universal finite type invariants. A homomorphic expansion is an expansion which behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the associated graded structure; for a detailed definition and introduction see [WKO2, Section 2.3]. Hence, a homomorphic expansion $Z : \mathcal{K} \rightarrow \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u , and Z^w for $\mathcal{K}^b := uPaB$, $\mathcal{K}^u := sKTG$ and $\mathcal{K}^w := \widetilde{wTF}$, respectively, so that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{K} : & \mathcal{K}^b & \xrightarrow{cl} & \mathcal{K}^u & \xrightarrow{\alpha} & \mathcal{K}^w \\ \downarrow Z & \downarrow Z^b & & \downarrow Z^u & & \downarrow Z^w \\ \mathcal{A} : & \mathcal{A}^{hor} & \xrightarrow{cl} & \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array} \quad (2)$$

We recall (see [BNI]) that a homomorphic expansion Z^b for parenthesized braids is determined by a “horizontal chord associator” $\Phi = Z^b(|\prime|)$. A homomorphic expansion Z^u of $sKTG$ is also determined² by a Drinfel’d associator (horizontal chords or not; see [WKO2, Section 4.6]), so the significance of the left commutative square is to force the associator corresponding to Z^u to be a horizontal chord associator. In turn, Z^w is determined by a solution F (a close cousin of $V = Z^w(\uparrow_*)$) to the Kashiwara-Vergne problem (see [WKO2, Section 4.4 – 4.5]). The goal of this paper is to prove the following theorem, which, via the correspondence above, implies the Kashiwara-Vergne Theorem:

Theorem 1.1. (1) Assuming that $Z : \mathcal{K} \rightarrow \mathcal{A}$ exists, it is determined³ by Z^u .
 (2) There is a formula for $V = Z^w(\uparrow_*)$ in terms of the Drinfel’d associator Φ associated to Z^u and $C = Z^w(\downarrow)$:

$$V = C_1^{-1} C_2^{-1} \varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \Phi(a_{23}, a_{43})) C_{(12)}, \quad (3)$$

²With the exception of some minor normalization, see [WKO2], Lemma 4.14 and the paragraph after.

³In fact, almost entirely determined by Z^b , with the exception of some minor normalization of Z^u which is not determined by an associator.

thm:main
 can we be
 explicit
 about what
 is C_2

eq:MainDia

eq:AEI

where a denotes a single arrow⁴. This agrees⁵ with the formula proven in [AET].
 (3) Every Z^b extends to a Z .

Remark 1.2. The formula in part (2) of the Theorem, expresses $V = Z^w(\downarrow, \cdot)$ in terms of the Drinfel'd associator Φ , and $C = Z^w(\downarrow)$. One might wonder if there are separate formulas for both V and C in terms of Φ . In fact, in Corollary 3.13 we compute the “even part” of C explicitly, and show that it is fixed, in other words, does not depend on Φ . Part (1) of the Theorem, proven in Section 3.1.2, shows that the complete value of C (even and odd part) is uniquely determined by Φ . In Theorem 3.6 we also prove that a suitable value exists for any choice Φ .

The key to the proof of the theorem is to show that the generator \downarrow of \widetilde{wTF} can be expressed in terms of the generator \downarrow of $wPaB$ and the operations of \mathcal{K} . Assuming that Z exists, this yields a formula for V in terms of Φ .

1.3. Computations. We note that this paper is “abstract”, yet everything difficult in it occurs within graded spaces, and can be computed explicitly up to a certain degree. The highest degree to which computations can be completed depends on the specific object. In a follow-up paper [WKO4] many of these computations are carried out.

1.4. Paper Structure. In Section 2 we provide an overview of the space wTF^o of (oriented) w-foams and its extension with strings \widetilde{wTF} . We provide a brief review of definitions and crucial facts from [WKO2], and details of the extension. We prove that homomorphic expansions for wTF^o extend uniquely to homomorphic expansions for \widetilde{wTF} .

Section 3 makes up the bulk of the paper and is devoted to the proof of Theorem 1.1. In Section 3.1 we prove part (1). In Section 3.2 we deduce the formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators, proving part (2). In Section 3.3 we prove statement (3), the hardest part of the proof.

Section ?? is a short section of closing remarks, and in Appendix A we give an explicit comparison and equivalence between our formula in Part (2) and the Alekseev–Enriquez–Torossian [AET] formula.

2. THE SPACES \widetilde{wTF} AND \mathcal{A}^{sw} IN MORE DETAIL

As mentioned in the introduction, \widetilde{wTF} is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4]. It can be introduced as a planar algebra or as a circuit algebra; we will do the latter as it is simpler and more concise. Circuit algebras are defined in [WKO2, Section 2.4]; in short, they are similar to planar algebras but without the planarity requirement for “connecting strands”. As in [WKO2], each generator and relation of \widetilde{wTF} has a local topological interpretation. Recall from [WKO2, Sections 1.2, 3.4, 4.1] that wTF^o diagrams represent certain ribbon knotted tubes with foam vertices in \mathbb{R}^4 , and the circuit algebra wTF^o is conjecturally a Reidemeister theory for this space (i.e., there is a surjection δ from the circuit algebra wTF^o to ribbon knotted tubes with foam vertices, and δ is conjectured to be an isomorphism). The space \widetilde{wTF} extends wTF^o by adding one-dimensional strands

⁴The notation is explained in detail in Section 3.2

⁵The two formulas are written in different languages, and checking that they agree takes effort. See Section 3.2 and Appendix A.

between the dotted lines on the picture. Hence, we write $\beta^u \in \mathcal{A}^u(\uparrow_4)$. Let $\beta^w = \alpha(\beta^u)$, and note that by the compatibility of Z^u and Z^w we have $\beta^w = Z^w(B^w)$. We will perform a series of operations on B^w and $\pi(\beta^w)$ to recover $\pi(V)$ from it.

First, connect (a circuit algebra operation in \widetilde{wTF}) a positive vertex to the bottom of B^w , as shown in Figure 12. Then unzip the edge marked by u , and puncture the edges marked e and e' . Then attach a cap (once again a circuit algebra operation) to the thick black end at the bottom. Finally, unzip the capped strand.

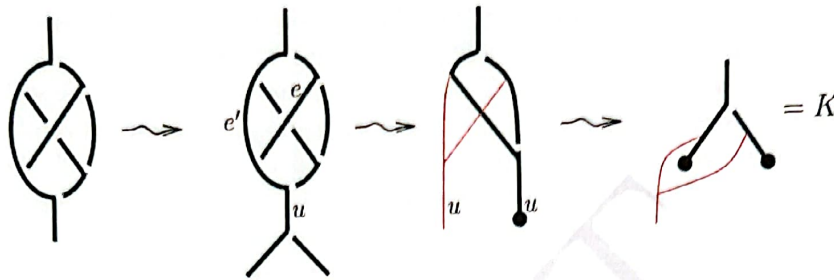


Figure 12. From the "buckle" β^w to the (modified) vertex.

fig:Buckle

Call the resulting w-foam K , as shown at the right in Figure 12. What is $Z^w(K)$? Due to the homomorphicity of Z , it is obtained from β^w by performing the same series of operations in the associated graded: a circuit algebra composition with V , unzip, punctures, circuit algebra composition with C , and disc unzip. Notice that the left strand of that attached vertex got punctured, and hence by Lemma 2.9 the attached value V cancels.¹¹ $Z^w(K)$ still depends on the value C . At the tree level, since $\pi(C) = 1$, $\pi(Z^w(K))$ can be computed from β^w by performing punctures and unzips. Since $\beta^w = \alpha(\beta^u)$, this means that $\pi(Z^w(K))$ is determined by Z^u .

On the other hand, note that the space of chord diagrams on the skeleton of K is the space $\mathcal{A}(\uparrow_2)$ by the Sorting Lemma (Lemma 2.5) and VI. Note also that K is a circuit algebra combination of a vertex, two left-punctured right-capped vertices and an all-red-strings vertex, and the Z^w -values of the latter three are trivial. So $\pi(Z^w(K)) = \pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$. Hence, $\pi(V)$ is determined by Z^u as needed. \square

3.1.2. Complete proof of Part (1). In the previous subsection we showed that Z^u determines $\pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$. The proof of Part (1) is completed by the following Lemma:

Lemma 3.1. For any homomorphic expansion Z^w of \widetilde{wTF} set $V = Z^w(\uparrow_*)$ and $C = Z^w(\downarrow)$. Then $\pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$ determines both V and C uniquely.

Proof. We use a perturbative argument. By contradiction, assume this is not the case, in particular, first assume that there exist $V \neq V'$, both of which are vertex values of Z^u -compatible homomorphic expansions, such that $\pi(V) = \pi(V')$. Let v denote the lowest degree term of $V - V'$. Note that v is primitive and $v \in \ker \pi$, so v is a homogeneous linear combination of wheels. By the Unitarity Equation of Fact 2.6, we have $A_1 A_2(v) = -v$. Recall that A_i reverses the direction of the strand i and multiplies each arrow diagram by

¹¹Any short arrows would also cancel when the right strand is capped.

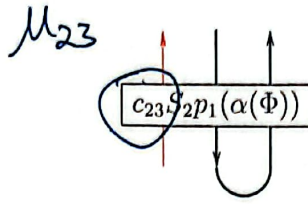


Figure 13. A concatenated associator.

fig:Concat

(-1) to the number of heads on that strand. Since v has only tails, $A_1 A_2(v) = v$, so $v = -v$, so $v = 0$, a contradiction. Therefore, $\pi(V)$ determines V uniquely.

Now we show that V determines C uniquely. Assume there are different values C and C' in $\mathcal{A}^{sw}(\downarrow)$ so that (V, C) and (V, C') are both vertex-cap value pairs of Z^u -compatible homomorphic expansions. Let c denote the lowest degree term of $C - C'$, then c is a scalar multiple of a single wheel. The Cap Equation of Fact 2.6 implies $c_{(12)} = c_1 + c_2$ in $\mathcal{A}^{sw}(\downarrow_2)$.

There is a well-defined linear map $\omega : \mathcal{A}^{sw}(\downarrow_2) \rightarrow \mathbb{Q}[x, y]$ sending an arrow diagram - which has arrow tails only on each strand - to "x to the power of the number of tails on strand 1, times y to the power of the number of tails on strand 2". Assume $c = \alpha w_r$, where w_r denotes the r-wheel, and $\alpha \in \mathbb{Q}$. Then $0 = \omega(c_{(12)} - c_1 - c_2) = \alpha((x+y)^r - x^r - y^r)$, so either $r = 1$ or $\alpha = 0$. But $w_1 = 0$ in \mathcal{A}^{sw} by the RI relation, hence $\alpha = 0$ and thus $c = 0$, a contradiction. \square

AEIFormula

3.2. Proof of Part (2). In this section we compute V , the value of the vertex, from Φ , the Drinfel'd associator determining Z^b , using the construction of Part (1). In Appendix A we also show that this result translates to the [AET] formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators.

In the computation of V from Φ , as well as later in the paper, we use two facts about Drinfel'd associators. We summarise these in the following Lemma:

m:PhiFacts

Lemma 3.2. Let $\Phi = \Phi(c_{12}, c_{23}) \in \mathcal{A}^u(\uparrow_3)$ be a Drinfel'd associator, where c_{ij} denotes a chord between strands i and j . Let $\alpha(\Phi) \in \mathcal{A}^w(\uparrow_3)$ denote the image of Φ in arrow diagrams. Then, the following facts hold:

it:Central
oPunctures

- (1) $\Phi(x, y) = \Phi(x, -x - z)$, whenever $(x + y + z)$ is central.
- (2) $p_i p_j \alpha(\Phi) = 1$, whenever $i, j \in \{1, 2, 3\}$, $i \neq j$, and p_i denotes puncture of the i -th strand.
- (3) $\mu_{23}(S_2 p_1(\alpha(\Phi))) = 1$, where S_2 stands for orientation switch of strand 2, and μ_{23} is concatenation (multiplication) of strands 2 and 3 - a circuit algebra operation - as shown in Figure 13.

it:Concat

Proof. Property (1) follows from the fact that the logarithm of Φ is a Lie series in x and y with no constant term.

To show Property (2), recall that if $\Phi(x, y)$ is a Drinfel'd associator, then $\Phi(0, y) = \Phi(x, 0) = 1$. Therefore $p_1 p_2 \alpha(\Phi) = 1$, because $p_1 p_2 \alpha(c_{12}) = p_1 p_2 (a_{12} + a_{21}) = 0$. Similar reasoning shows that $p_2 p_3 \alpha(\Phi) = 1$. Finally, $p_1 p_3 \alpha(\Phi(c_{12}, c_{23})) = \Phi(a_{21}, a_{23})$, and since $[a_{21}, a_{23}] = 0$ by the TC relation, $\Phi(a_{21}, a_{23}) = 1$.

For Property 3, note that $p_1(\alpha(\Phi)) = \Phi(a_{21}, -a_{21} - a_{31})$. Thus, strands 2 and 3 carry only arrow tails, and these commute by the TC relation, and $S_2 p_1(\alpha(\Phi)) = \Phi(-a_{21}, a_{21} - a_{31})$.

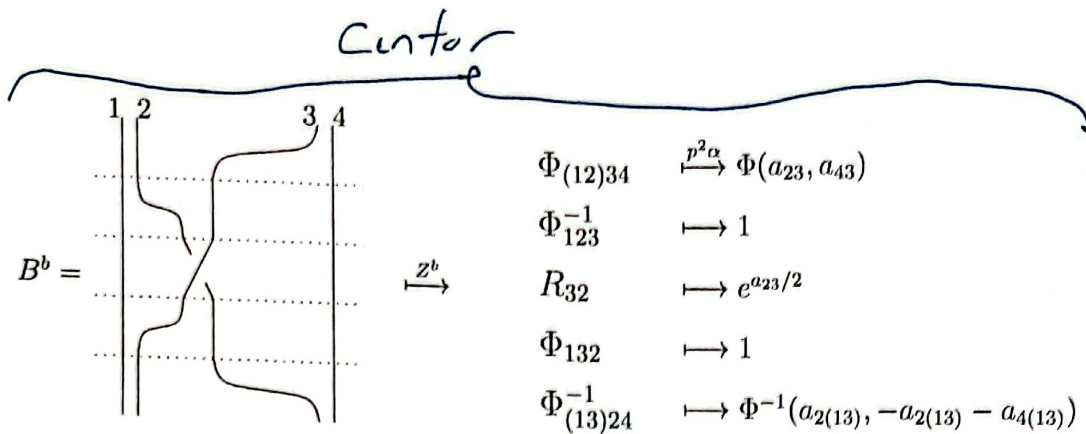


Figure 14. Computing β^b . Strands are numbered at the top and multiplication is read from bottom to top; the rightmost column lists the images of the factors under $p_1 p_3 \alpha$.

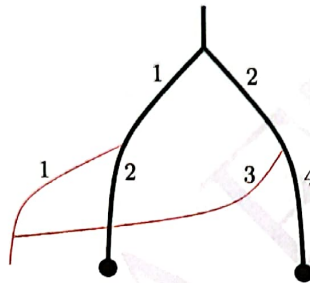


Figure 15. Strand numbering convention for K and V : arrow endings from strand 1 and 2 of K are "pushed" to strand 1 of V when applying φ , and arrow endings from strands 3 and 4 are pushed to strand 2.

By Property (1) of Lemma 3.2, $\Phi_{(13)24}^{-1} = \Phi^{-1}(c_{(13)2}, c_{24}) = \Phi^{-1}(c_{(13)2}, -c_{(13)2} - c_{(13)4})$, so $p_1 p_3 \alpha \Phi_{(13)24}^{-1} = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. To summarise,

$$\Upsilon = \varphi(C_{24} \cdot \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})).$$

Note that the expression $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})$ has only arrow tails on strands 2 and 4, and therefore commutes with C_{24} by the TC relation. Hence, by the definition of φ ,

$$\begin{aligned} \Upsilon &= \varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}) \cdot C_{24}) \\ &= \varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})) \cdot \varphi(C_{24}). \end{aligned}$$

Furthermore, by the strand numbering convention shown in Figure 15, we have $\varphi(C_{24}) = C_{12}$. Therefore,

$$V_{12} = C_1^{-1} C_2^{-1} \Upsilon = C_1^{-1} C_2^{-1} \varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})) C_{12},$$

as stated in part (2) of Theorem 1.1. □

In [WKO4] V is computed to degree 4 using the techniques of this section. Matching this result to the Alekseev–Enriquez–Torossian formula of [AET, Theorem 4.1] is technical, and not used anywhere else in the paper, hence we defer this to Appendix A.

$f(x, y) \in \text{cyc}\langle x, y \rangle$ denote the logarithm of $\|T\|$. In turn, $c \in \text{cyc}\langle x \rangle \cong \mathbb{Q}[[x]]$, thus we write $c = c(x)$.

We claim that

$$(C_{12})^{T^{-1}} = (\exp(c(x+y)))^{T^{-1}} = T \exp(c(x+y)) T^{-1} = \exp(c(\log(e^x e^y))).$$

The last equality follows from the fact that T satisfies the (R4) equation; this is explained in detail in [WKO2], Section 4.4, in particular in and around Fig. 19.

With this reduction in place (6) reduces to

$$e^{f(x,y)} = (e^{c(\log(e^x e^y))} e^{c(x)} e^{c(y)})^2,$$

and since all exponents (wheels) commute, this is equivalent to

$$f(x, y) = 2(c(x) + c(y) + c(\log(e^x e^y))). \quad (7) \quad \text{eq:Unitari}$$

which we need to solve for $c \in \mathbb{Q}[[x]]$.

To do so, we use that $Z^{\text{tree}}(\alpha(K)) = \pi \alpha Z^u(K)$, specifically for the associator graph $K = \phi$ shown in Figure ???. Recall that $Z^u(\phi)$ is a Drinfel'd associator Φ . On the other hand, $\alpha(\phi)$ is a circuit algebra product of four vertices, therefore

$$\alpha(\Phi) = Z^{\text{tree}}(\alpha(\phi)) = T_{12,3}^{-1} T_{12}^{-1} T_{23} T_{1,23}.$$

Rearranging, we obtain

$$T_{12} T_{12,3} \alpha(\Phi) = T_{23} T_{1,23}.$$

Applying the "norm" to both sides, keeping in mind that $\|\alpha(\phi)\| = 1$ by Lemma 3.5 property (6), and using the exponent notation for the adjoint action on a norm, we get

$$\|T_{12}\| \|T_{12,3}\|^{T_{12}^{-1}} = \|T_{23}\| \|T_{1,23}\|^{T_{23}^{-1}}. \quad (8) \quad \text{eq:Unitari}$$

Recall that $\|T_{12,3}\| = \exp(f(x+y, z))$, $\|T_{23}\| = \exp(f(y, z))$, $\|T_{1,23}\| = \exp(f(x, y+z))$.

As before, we have

$$\|T_{12,3}\|^{T_{12}^{-1}} = (\exp(f(x+y, z)))^{T_{12}^{-1}} = T_{12} \exp(f(x+y, z)) T_{12}^{-1} = \exp(f(\log(e^x e^y), z)),$$

since T satisfies the (R4) equation. Similarly,

$$\|T_{1,23}\|^{T_{23}^{-1}} = (\exp(f(x, y+z)))^{T_{23}^{-1}} = \exp(f(x, \log(e^y e^z))).$$

Thus, Equation (8) reduces to

$$e^{f(x,y)} e^{f(\log(e^x e^y), z)} = e^{e^{f(y,z)}} e^{f(x, \log(e^y e^z))},$$

and since all the exponentials (wheels) commute, this is equivalent to

$$f(x, y) = f(y, z) + f(x, \log(e^y e^z)) - f(\log(e^x e^y), z) \quad (9) \quad \text{eq:Unitari2}$$

The fact that there exists a solution $c(x)$ to Equation 7 follows from this equation via a short cohomology calculation as in [AET, Proposition 27], and [AT, Appendix A, Proof of Theorem 2.8]. \square

In order to use Theorem 3.6, it is necessary to show that the value V^{tree} constructed in Section 3.1.1 satisfies the (R4) equation; unfortunately this is hard. The equation (R4) is an equality between different circuit algebra – in fact, planar algebra – compositions of crossings. Hence, the proof would be much easier if Z^u were to be a circuit algebra – or planar algebra – map. This unfortunately makes no sense, as $sKTG$ is not a circuit or planar algebra but a different, more complicated algebraic structure.

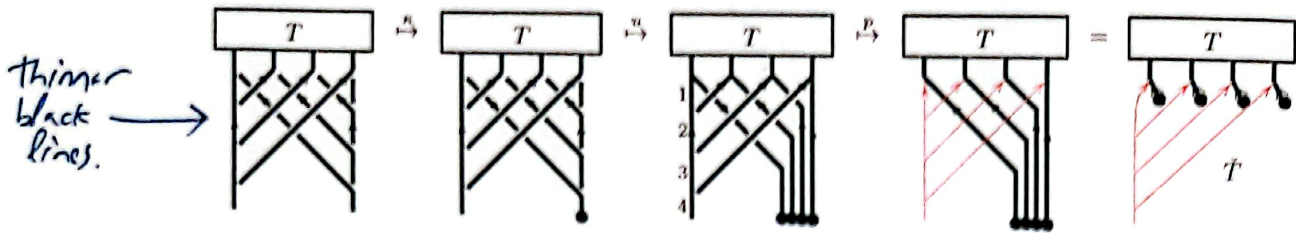


Figure 17. The cap attachment, unzips and punctures. While these operations are applied in \mathcal{A}^{sw} – there are arrows on these skeleta – for simplicity the figure only shows the effect on the skeleton.

perform $(n - 1)$ consecutive disc unzips on the capped strand, as shown in Figure 17. Then puncture the left hand tree, for example by puncturing the left vertical strands marked “1, 2, ...” in Figure 17 (these punctures also affect the connecting diagonal strands, as in Figure 5). Note that since the punctured tree had originally crossed over the capped tree, these crossings become virtual after puncturing, hence the last equality in Figure 17.

Denote the composition of the maps and operations shown in Equation (10) by ξ , that is,

$$\xi := \varphi \circ p \circ u \circ \kappa \circ \alpha \circ Z^u \circ \varphi : \mathcal{MT} \rightarrow \mathcal{A}^{sw}. \quad (11)$$

Then, $\xi(T) \in \mathcal{A}^{sw}(T)$. We first show that $\xi(T)$ is well-defined, that is, it doesn't depend on the choice of binary trees in $\varphi(T)$.

Lemma 3.7. *The choice of binary trees in the double tree construction does not affect $\xi(T)$.*

Proof. Any binary tree can be changed into any other binary tree via a sequence of “I to H” moves, as shown on the right. Hence, it is enough to analyze how an I to H move on one of the trees affects the value of $Z^u(\varphi(T))$, and prove that the difference vanishes after the capping, unzip, and puncture operations.

Suppose τ_1 and τ_2 are two binary trees which differ by a single I to H move, and let φ_{τ_1} and φ_{τ_2} denote the two resulting double-tree maps, assuming the “other side tree” is unchanged. The I to H move can be realised by inserting¹⁵ an associator, followed by unzipping the edge marked ‘1’ on the right, then the edge marked ‘2’. By the homomorphicity of Z^u , the values $Z^u(\varphi_{\tau_2}(T))$ and $Z^u(\varphi_{\tau_1}(T))$, only differ in an inserted horizontal chord associator Φ on the three strands involved, we indicate this by writing $Z^u(\varphi_{\tau_2}(T)) = Z^u(\varphi_{\tau_1}(T)) * \Phi$. If the I to H move was done on the left side tree, then all the strands involved are later punctured, killing any arrow diagram that lived on them by the TF relation. As a result, the only surviving part of Φ is its constant term, 1, and the resulting values of ξ are equal.

If the I to H move is done on the right side tree, then the all participating strands are capped and disk unzipped. If $\alpha(\Phi)$ is immediately adjacent to the caps, then it cancels by the CP relation. However, it is a priori possible that there are other arrow ending separating Φ from the caps. Note that in \mathcal{A}^u , any chord endings can be commuted from below the associator to above, using VI relations and the invariance property of chord diagrams shown in Figure 18 [BN2, Lemma 3.4]. Thus, one can assume that $\alpha(\Phi)$ is adjacent to the caps and hence cancels. This concludes the proof. \square

¹⁵See [WKO2, Section 4.6] for a detailed description of the tangle insertion operation in $sKTG$.

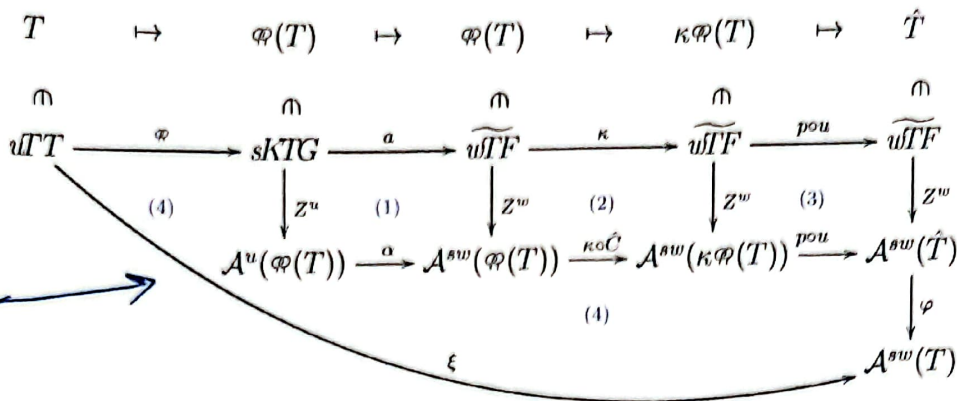


Figure 21. Comparing ξ and Z_ξ^w , assuming that Z_ξ^w exists.

fig:BigCom

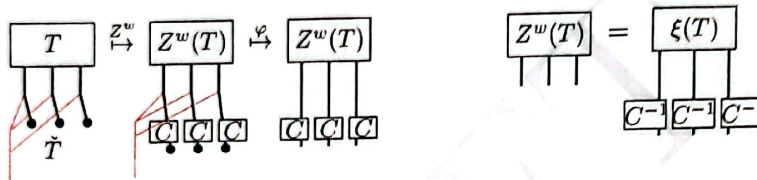
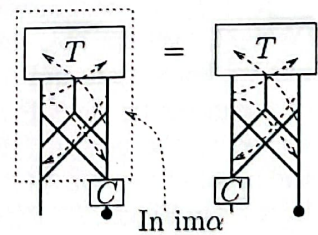


Figure 22. Computing $Z_\xi^w(\tilde{T})$ and $Z_\xi^w(T)$.

fig:ZFChec

It remains to show that the diagram in Figure 21 commutes. The square (1) is the assumed compatibility of Z^u and Z_ξ^w . In square (2), recall the map κ denotes the circuit algebra operation of attaching a cap at the bottom right end of the w -foam. The map \hat{C} denotes the circuit algebra operation which attaches a value $C = Z(\downarrow)$ at the end of the strand. Thus, the commutativity of square (2) is implied by the homomorphicity of Z_ξ^w with respect to circuit algebra composition (as a binary operation). The square (3) is commutative due to the homomorphicity of Z_ξ^w with respect to punctures and disc unzips.

The commutativity of the heptagon (4) would be true by definition, if not for the map \hat{C} (multiplication by the cap value). We show that, in fact, the value C cancels after punctures, by a property of arrow diagrams in the image of α , called *tail-invariance*, shown in Figure 18 (see [WKO2], Remark 3.14 and early in Section 3.3). In the current situation tail invariance means that the value C , which has only arrow tails, can be moved from one tangle end to the other, as shown on the right. Consequently, C cancels when the left strand is punctured. \square



orientations

Remark 3.10. In Lemma 3.9 we assume by convention that all tangle ends of T are oriented upwards (towards T). If k tangle ends are oriented down, the corresponding cap values appear with their orientations switched: $Z_\xi^w(aT) = \xi(T) \cdot (C^{-1})^{n-k} (S(C)^{-1})^k$.

Treefromxi

Corollary 3.11. If there exists a homomorphic expansion Z^w for wTF compatible with Z^u , then $\pi(V) = \pi(\xi(\wedge))$, where V is the Z^w -value of the vertex, and π is the tree projection.

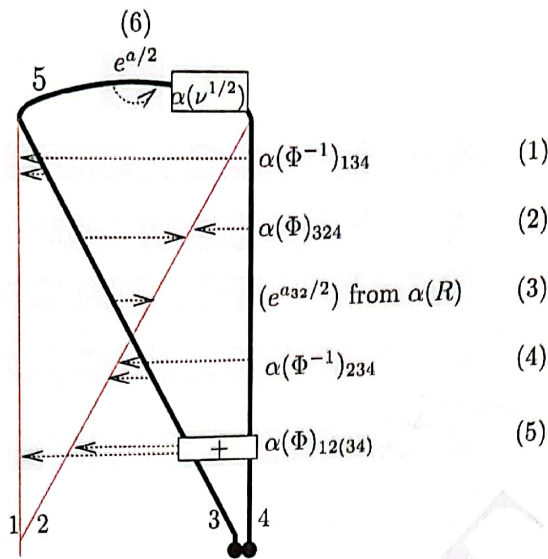


Figure 24. The value of $p_1 p_2 \alpha Z^u \Phi(\gamma)$: the numbering (1) through (6) refer to the paragraphs where each component is explained.

fig:dtstra

c_{12}, c_{23} and c_{13} . For example, $\Phi(c_{12}, c_{23}) = \Phi(c_{12}, -c_{12} - c_{23})$. For each associator we chose the presentation in which $p_1 p_2 \alpha(\Phi)$ is of the simplest form.

- (1) The top associator of Figure 23, after applying a VI relation, is written as $\Phi_{13(24)}^{-1}$ in the strand numbering of Figure 24. We write this in terms of c_{13} and $c_{1(24)} = c_{12} + c_{14}$, since after the pictures $p_1 \alpha(c_{13}) = a_{31}$ and $p_1 p_2 \alpha(c_{1(24)}) = a_{41}$, thus

$$p_1 p_2 \alpha \Phi^{-1}(c_{13}, -c_{13} - c_{1(24)}) = \Phi^{-1}(a_{31}, -a_{31} - a_{41}).$$

This is reflected in Figure 24 in drawing only the a_{31} and a_{41} arrows. In turn, the associator cancels, essentially by Fact (3) of Lemma 3.2: strand 5 acts as a concatenation of strands 3 and 4, as the tail of a_{41} can be "pulled over the top along strand 5" using the VI relations and the fact that $e^{a/2} \alpha(\nu)$ is a local arrow diagram on a single strand, hence it is central. Thus, $a_{41} = a_{31}$ and $\Phi^{-1}(a_{31}, -a_{31} - a_{41}) = \Phi^{-1}(a_{31}, -2a_{31}) = 1$ as the arguments commute. Therefore

→ *Im sceptic* $p_1 p_2 \alpha(\Phi_{13(24)}^{-1}) = 1.$

- (2) Second from top we have

$$p_1 p_2 \alpha(\Phi_{324}) = p_1 p_2 \alpha(\Phi(c_{23}, c_{24})) = \Phi(a_{32}, a_{42}) = 1,$$

by the same modified version of Fact (3) (concatenation).

- (3) For the exponential,

$$p_1 p_2 \alpha(R_{23}) = p_1 p_2 \alpha(e^{c_{23}/2}) = e^{a_{32}/2}.$$

- (4) Next,

$$p_1 p_2 \alpha(\Phi_{234}^{-1}) = p_1 p_2 \alpha(\Phi^{-1}(c_{23}, -c_{23} - c_{24})) = \Phi(a_{32}, -a_{32} - a_{42}) = 1$$

by the modified Fact (3), noting that the arrow tails also commute with the arrow tails of the exponential.

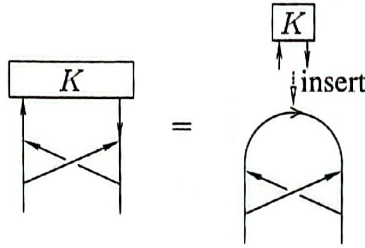


Figure 25. For $K \in sKTG$, $\varphi(K)$ is K inserted into $\varphi(\uparrow)$.

fig:Insert



(5) By Fact (2) of Lemma 3.2

$$p_1 p_2 \alpha(\Phi)_{12(34)} = 1.$$

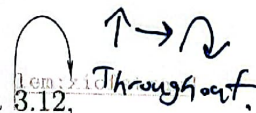
(6) Finally, move the top exponential $e^{a/2}$ to strands 3 and 2, using the VI relation at both vertices. The tail of each arrow moves freely from strand 5 to strand 3. The heads commute with $\alpha(\nu)$, they are killed on strand 4 due to the CP relation, so they slide onto strand 2 but acquire a negative sign due to opposite orientations. Hence, $(e^{a_{55}/2}) = (e^{-a_{32}/2})$, and this cancels $\alpha(R)$. In summary, $\xi(\curvearrowright) = \alpha(\nu^{1/2})$, as claimed. \square

As an aside, Lemma 3.12 enables a quick computation of the even part of $C = e^c = Z_\xi^w(\downarrow)$. Recall that c is a linear combination of wheels: $c = \sum_{n=2}^{\infty} \gamma_n w_n$. Let $c = c_0 + c_1$, where c_0 denotes the even part of c (sum of all even wheels), and c_1 denotes the odd part, that is, $c = c_0 + c_1$. Let $C_0 = e^{c_0}$, the even part of C . Corollary 3.13 shows in particular that the even part of the C is independent of the choice of Z^b (that is, the choice of Drinfel'd associator) and Z^u .

cor:CapValue

Corollary 3.13. If $C = Z_\xi^w(\downarrow)$, and C_0 is the even part of C , then $C_0 = \alpha(\nu^{1/4})$, regardless of the choice of expansion Z^u used to construct Z_ξ^w .

Proof. By Lemma 3.9 and Remark 3.10, we have $Z_\xi^w(\uparrow) = C^{-1} \xi(\uparrow) S(C^{-1})$: see the figure on the right for the orientations. Note that $S(w_{2k}) = w_{2k}$ and $S(w_{2k+1}) = -w_{2k+1}$, and hence $S(C) = e^{c_0 - c_1}$. Also, by homomorphicity, $Z_\xi^w(\uparrow) = 1$. Thus, by Lemma 3.12, $1 = e^{c_0 + c_1} \alpha(\nu^{1/2}) e^{c_0 - c_1}$, and therefore $\alpha(\nu^{1/2}) = e^{2c_0}$, which gives $C_0 = e^{c_0} = \alpha(\nu^{1/4})$. \square



Next we prove that Z_ξ^w is indeed compatible with Z^u :

compatibility

Proposition 3.14. For any $K \in sKTG$, $Z_\xi^w(a(K)) = \alpha(Z^u(K))$.

Proof. Note that $sKTG \subseteq \mathcal{u}TT$, and for $K \in sKTG$, $\varphi(K)$ can be obtained by inserting K into the top strand of $\varphi(\uparrow)$: see Figure 25. Since Z^u is compatible with insertions, $Z^u(\varphi(K))$ can be obtained by $Z^u(K)$ inserted into $Z^u(\varphi(\uparrow))$. Through the sequence of α , capping, puncturing, φ and multiplications by C^{-1} , all of $Z^u(\varphi(\uparrow))$ cancels, as in Lemma 3.12. Note that the cancellations still go through despite the fact that $\alpha(Z^u(K))$ is inserted on the top strand: this follows from the fact that $\alpha(Z^u(K))$ is in the α -image of \mathcal{A}^u , and the appropriate "commutativity" property holds in \mathcal{A}^u . Hence, $Z_\xi^w(K) = \alpha(Z^u(K))$ as required. \square

Now, we show that Z_ξ^w is a planar algebra homomorphism on $a(\mathcal{u}TT)$. This is technically challenging, but it implies the R4 equation immediately. For the proof, it will be

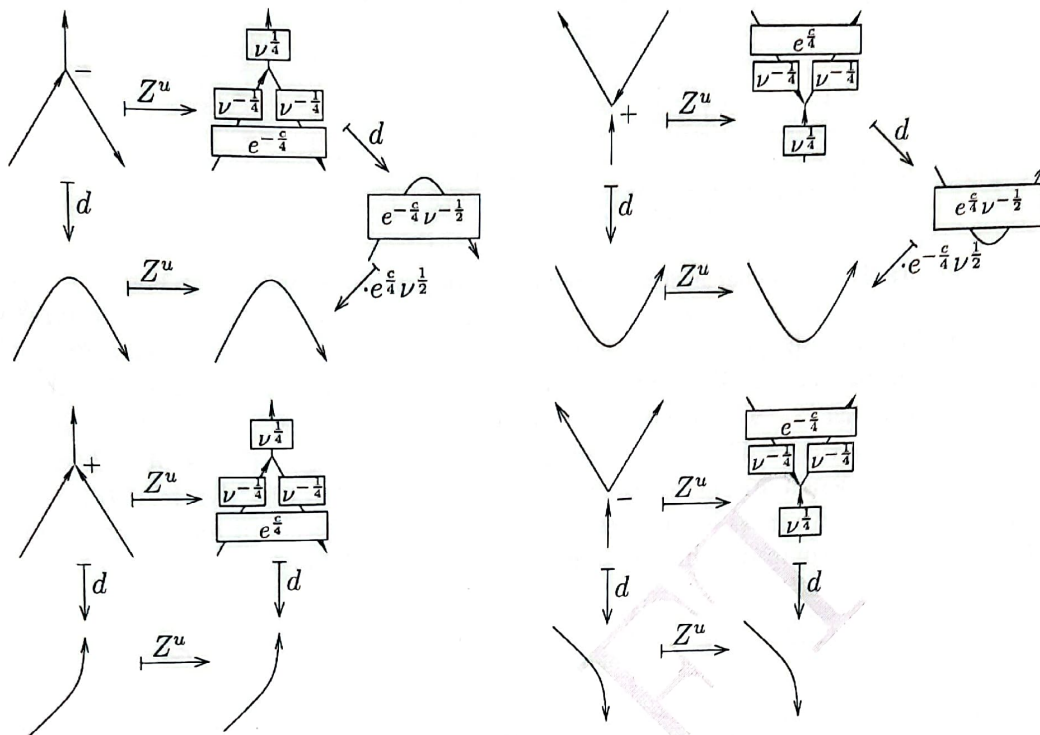


Figure 26. In [WKO2, Section 4.6.1] Z^u is constructed from an invariant Z^{old} by applying vertex normalizations, which depend on vertex signs: these are shown along the top horizontal arrow of each diagram (see also [WKO2, Figure 29]). It follows Z^u is only homomorphic up to a correction term when deleting the top edge of a positive vertex (first in the total ordering around the vertex) or the bottom edge of a negative vertex: see the top two diagrams. In other edge deletions the normalizations cancel, and hence Z^u is homomorphic with respect to these edge deletions, as for example in the bottom two diagrams.

necessary to know the behaviour of Z^u with respect to edge deletions. When an edge e of a knotted trivalent graph $K \in sKTG$ is deleted, the two vertices at each end of e cease to be vertices. The associated graded operation on chord diagrams deletes skeleton edge e , and chord diagrams with any chord endings on e are set equal to 0.

Fact 3.15. Z^u commutes with edge deletions up to a possible correction term of $e^{\pm c/4} \nu^{1/2}$ depending on the position of the edge, as in Figure 26.

Proof sketch. In [WKO2, Section 4.6.1] Z^u is constructed from an invariant Z^{old} by adding vertex normalizations, as shown in Figure 26. Note that the top two diagrams in figure differ from [WKO2, Figure 29] in a single edge orientation switch, which switches the vertex sign and accordingly the normalization¹⁸. In fact, Z^{old} commutes with edge deletions [Da, Proposition 6.7], so the edge deletion error (and hence, the correction term) for Z^u arises from the vertex normalisations implemented, as shown in Figure 26. \square

¹⁸The point of the normalization is to make Z^u commute with unzips. The reader might wonder, why normalize so that the expansion respects unzips, rather than deletions? The answer is that for finite generation of knotted trivalent graphs, unzips are crucial but deletions are not.

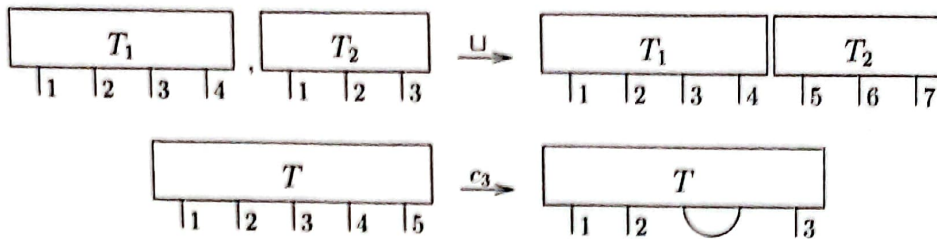


Figure 27. Basic planar algebra operations: disjoint union and contraction.

Remark 3.16. There is also an “edge delete” operation of \widetilde{wITF} : this is not required for the finite presentation of \widetilde{wITF} or \mathcal{A}^{sw} , but it is necessary for the proof of Theorem 3.17. When deleting an edge in \widetilde{wITF} – which can be either a tube or a string – the vertices at either end¹⁹ cease being vertices. The associated graded operation $d_e : \mathcal{A}^{sw} \rightarrow \mathcal{A}^{sw}$ deletes the skeleton edge e and sends any arrow diagram with arrow endings on the deleted strand to zero. The crucial fact we need is that edge delete operations for chord and arrow diagrams are compatible via the map α , which is immediate from the definitions:

$$\begin{array}{ccc}
 \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^{sw} \\
 \downarrow d_e & & \downarrow d_e \\
 \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^{sw}
 \end{array}$$

Theorem 3.17. The map Z_ξ^w is a planar algebra map on $a(\widetilde{wITF})$.

Proof. Planar algebra operations can be written as compositions of two simpler, basic operations: disjoint unions and contractions. In the disjoint union of two tangles T_1 and T_2 , the ends of $T_1 \sqcup T_2$ are ordered by declaring that the ordered ends of T_1 come first, followed by the ordered ends of T_2 . The contraction operation c_i applies to any tangle with at least $i + 1$ ends: it acts by joining the i -th and $(i + 1)$ -st ends of T and re-numbering the rest, resulting in a tangle with two less ends. Both operations are shown in Figure 27.

Thus, we only need to show that Z_ξ^w commutes with these two operations, that is, $Z_\xi^w(T_1 \sqcup T_2) = Z_\xi^w(T_1) \sqcup Z_\xi^w(T_2)$, and $Z_\xi^w(c_i(T)) = c_i(Z_\xi^w(T))$. Note that the right sides of these equalities make sense: arrow diagrams on the skeleta of $a(\widetilde{wITF})$, where Z_ξ^w takes values, also form a planar algebra, and in particular disjoint union and concatenation of arrow diagrams is well defined.

Disjoint unions. We need to compute $\xi(T_1 \sqcup T_2)$, where $T_1, T_2 \in \widetilde{wITF}$. The value $\varphi(T_1 \sqcup T_2)$ is shown in Figure 28. The binary trees in φ can be chosen arbitrarily by Lemma 3.7: Figure 28 shows the most convenient trees for this proof.

Observe that $\varphi(T_1 \sqcup T_2)$ can be obtained as an *sKITG* by inserting $\varphi(T_1)$ and $\varphi(T_2)$ into a simpler *sKITG*, denoted H , as shown in the same figure (up to orientation switches which don’t impact what follows and will be ignored). Hence, $Z^u(\varphi(T_1 \sqcup T_2))$ is given by inserting $Z^u(\varphi(T_1))$ and $Z^u(\varphi(T_2))$ into $Z^u(H)$.

One could compute $Z^u(H)$ explicitly using the same algorithm as before, but we can avoid this work, as follows. All chords in $Z^u(H)$ can be assumed to be located in the

¹⁹It is also possible to delete a capped edge.

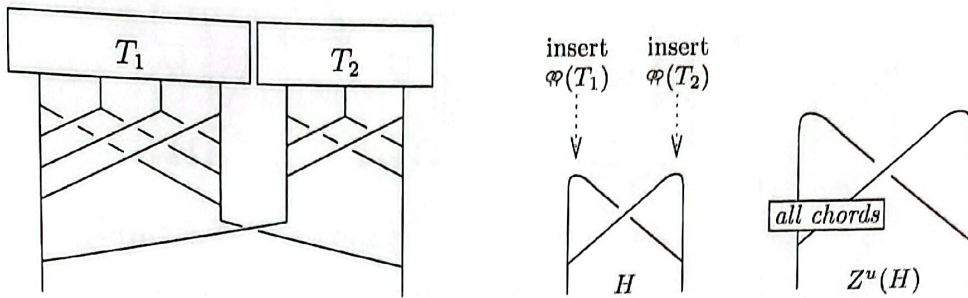


Figure 28. The double tree map applied to a disjoint union of uTT -s is the same as inserting the double tree of each individual uTT into the $sKTG$ H . In $Z^u(H)$ all chords can be pushed into the rectangle shown, using VI relations when necessary.

fig:DisjUn

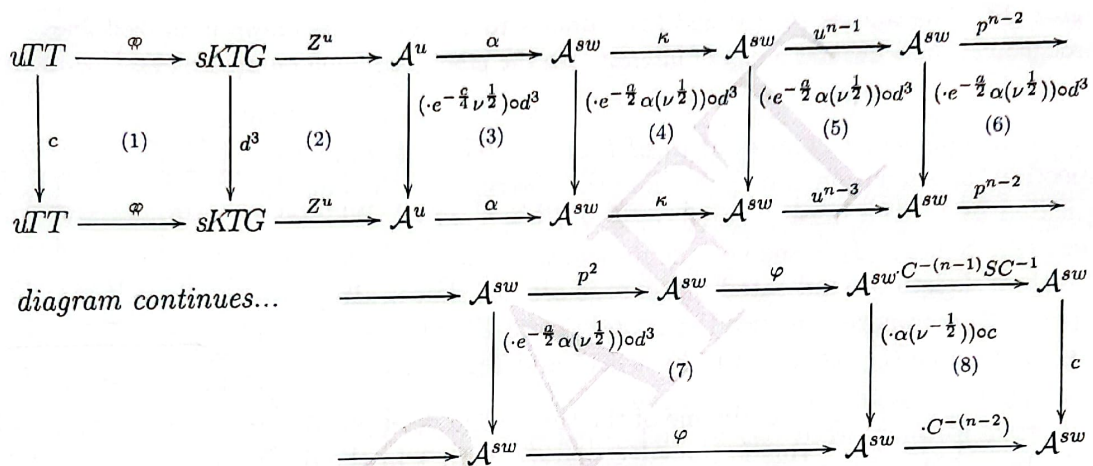


Figure 29. Summary of the proof that Z_ξ^w commutes with contractions: Z_ξ^w is the composition along the entire top and entire bottom horizontal edge of the diagram.

fig:Contra

rectangle shown in Figure 28 (using VI relations, if necessary). During the computation of ξ both supporting strands are punctured, and therefore $p^2\alpha(Z^u(H)) = 1$. This implies that $\xi(T_1 \sqcup T_2) = \xi(T_1) \sqcup \xi(T_2)$, and it follows via Lemma 3.9 that $Z_\xi^w(T_1 \sqcup T_2) = Z_\xi^w(T_1) \sqcup Z_\xi^w(T_2)$.

Contractions. Proving that Z_ξ^w commutes with contractions is more involved. By Lemma 3.8, we can assume that the ends contracted are the last (rightmost) two ends of the n ends of T . Hence we will drop the subscript from c_i and denote this operation simply by c .

We need to show that $Z_\xi^w(cT) = cZ_\xi^w(T)$, for any $T \in uTT$. Since Z_ξ^w is given by the composition of many maps, ~~so~~ this can be restated as the commutativity of the perimeter of a large diagram (shown in Figure 29), which in turn can be broken down to its smaller parts. Throughout this proof, let $T \in uTT$ denote an arbitrary trivalent tangle.

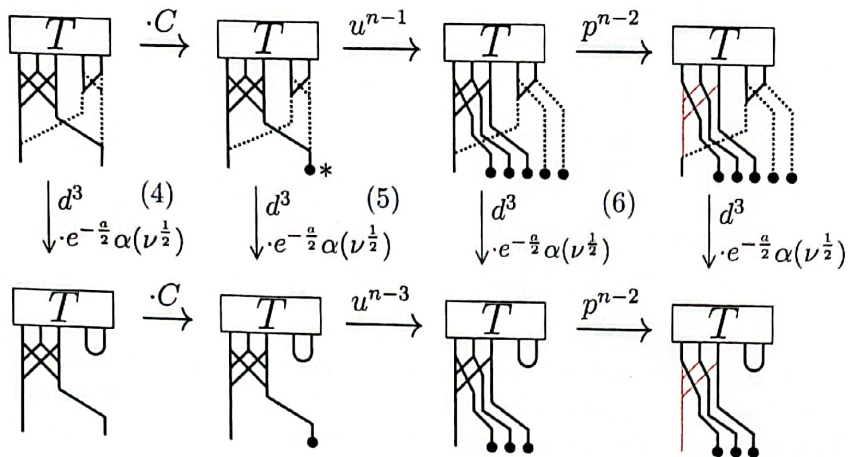
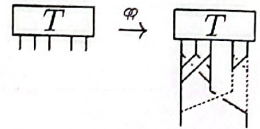


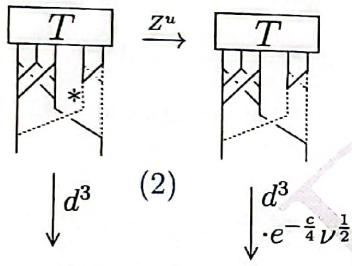
Figure 30. The squares (4) (5) and (6). Strands to be deleted are drawn in dashed lines throughout. The * denotes a cap of interest: see the proof paragraph on square (5).

fig:456

Square (1). This square plays out in \mathcal{MT} and $sKTG$, and commutes by inspection, as shown on the right. The three strands to be deleted are indicated by broken lines. Therefore, $d^3 \varphi(T) = \varphi c(T)$.



Square (2). Square (2) is shown schematically below on the left: for the Z^u -values skeleta are indicated but chords are not shown. To prove that square (2) commutes, we use the properties of Z^u with respect to deleting edges in $sKTG$, as stated in Fact 3.15 and Figure 26.



Only one of the three edge deletions requires a correction term: this is the edge marked with * in the diagram on the left. This edge ends in a $e^{-c/4} \nu^{1/2}$ inserted at the place of the vertex, where c stands for a single chord. In square (2), this correction term appears at the bottom right corner of the square, where the two ends of T are contracted (see in the diagram showing skeleta in Figure 29). In summary: $(d^3 Z^u \varphi(T)) \cdot (e^{-c/4} \nu^{1/2}) = Z^u \varphi c(T)$.

Square (3). Square (3) is essentially the commutativity of edge deletions stated in Remark 3.16, combined with applying α to the correction term. So we have:

$$(d^3 \alpha Z^u \varphi(T)) \cdot (e^{-c/2} \alpha(\nu^{1/2})) = \alpha Z^u \varphi c(T).$$

Square (4). Squares (4), (5), and (6) are shown in detail in Figure 30. Square (4) plays out in \mathcal{A}^{sw} and it is commutative as the deletions and the cap attachments (denoted by κ) affect different strands: see the diagram on the right. Therefore,

$$(d^3 \kappa \alpha Z^u \varphi(T)) \cdot (e^{-c/2} \alpha(\nu^{1/2})) = \kappa \alpha Z^u \varphi c(T).$$

Square (5). The only difference between $d^3 \circ u^{n-1}$ and $u^{n-3} \circ d^3$ is what happens to arrows on the capped skeleton edge marked by * in Figure 30. Following the diagram right and down, this edge is unzipped $n - 1$ times, then the last two of its daughter edges are deleted. On the other hand, following the diagram down and right, the same edge is unzipped $n - 3$ times. The results of these compositions are the same by definition of the unzip and delete

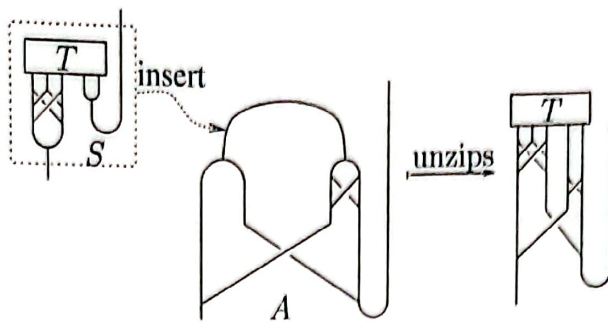


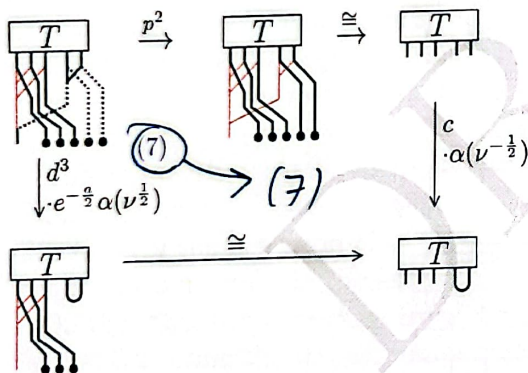
Figure 31. Computing the top left corner of Square 7, Step 1: $\varphi(T)$ can be expressed as the *sKTG* denoted S inserted into the *sKTG* denoted A , followed by unzips, as shown. Z^u respects insertions, hence computing $Z^u(A)$ determines the value of $Z^u(\varphi(T))$ outside of S .

operations. Thus, we have

$$(d^3 u^{n-1} \kappa \alpha Z^u \varphi(T)) \cdot (e^{-\frac{\alpha}{2}} \alpha(\nu^{\frac{1}{2}})) = u^{n-3} \kappa \alpha Z^u \varphi c(T).$$

Square (6). The deletions and punctures occur on different strands, as shown in Figure 30, hence these operations commute. One detail to note is that when a tube strand is deleted at a “tube-and-string” vertex, all that is left is a string (as in the case of puncturing the tube at a tube-string vertex, see Figure 5). In summary:

$$(d^3 p^{n-2} u^{n-1} \kappa \alpha Z^u \varphi(T)) \cdot (e^{-\frac{\alpha}{2}} \alpha(\nu^{\frac{1}{2}})) = p^{n-2} u^{n-3} \kappa \alpha Z^u \varphi c(T).$$



Pentagon (7). The pentagon (7) is shown on the left. This is the most delicate part of the proof. We first show that – for the specific input of $p^{n-2} u^{n-1} \kappa \alpha Z^u \varphi(T)$ – the pentagon (7) commutes up to a single possible error on the contracted (u-shaped) strand, and later prove that this error is necessarily zero.

revers
?
multiplication
←

To begin, a better understanding of the arrow diagram $p^{n-2} u^{n-1} \kappa \alpha Z^u \varphi(T)$ in the top left corner is necessary. All of the operations performed on

T , with the exception of Z^u , are “easy” in the sense that we have a complete understanding of their effect. Z^u is “hard”, but we can compute the relevant part of its value using the finite generation of *sKTG* ([WKO2, Proposition 4.13]). The computation is shown in Figures 31 and 32 and their captions.

In summary, $Z^u(\varphi(T))$ is given by inserting $Z^u(A)$ into the chord diagram D of Figure 32. Now we need to analyze what happens when one applies α , the cap attachment, unzips and punctures to this value: this is an exercise similar to what has been done for Lemma 3.12 for example. The result is shown in Figure 33, and explained below. First note that the n value in D of Figure 32 cancels after punctures by the tail-invariance property (Figure 18), as in the last paragraph of the proof of Lemma 3.9; so does the bottom Φ^{-1} in D , by Fact (3) of Lemma 3.2. These components are not shown in Figure 33.

Working downwards from the top left of the pentagon in Figure 33, the three edge deletions cancel both buckle (β^w) values. The value b value at the top of the diagram D is pulled

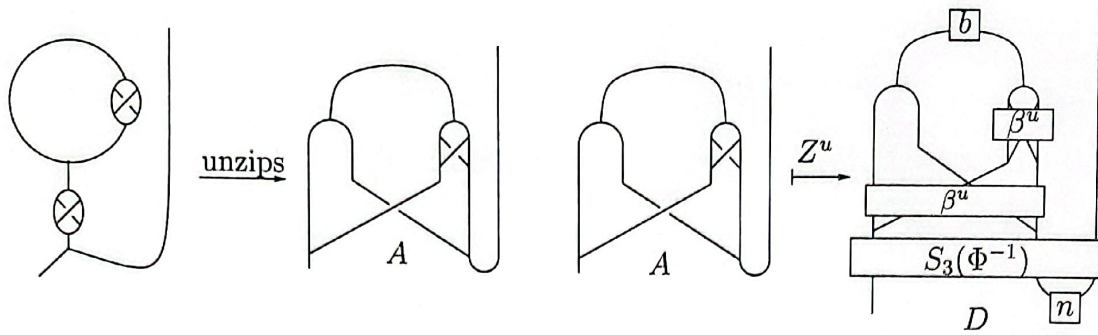


Figure 32. Computing the top left corner of Square 7, Step 2: computing $Z^u(A)$. The *sKTG* A can be obtained by inserting the *buckle sKTG* twice into a simpler *sKTG*, and unzipping, as shown on the left. The value of the buckle was computed in Figure 14. Using this value—denoted β^u —and the algorithm in [WKO2, Section 5.2], one computes $Z^u(A)$. The result is denoted D and shown on the right.

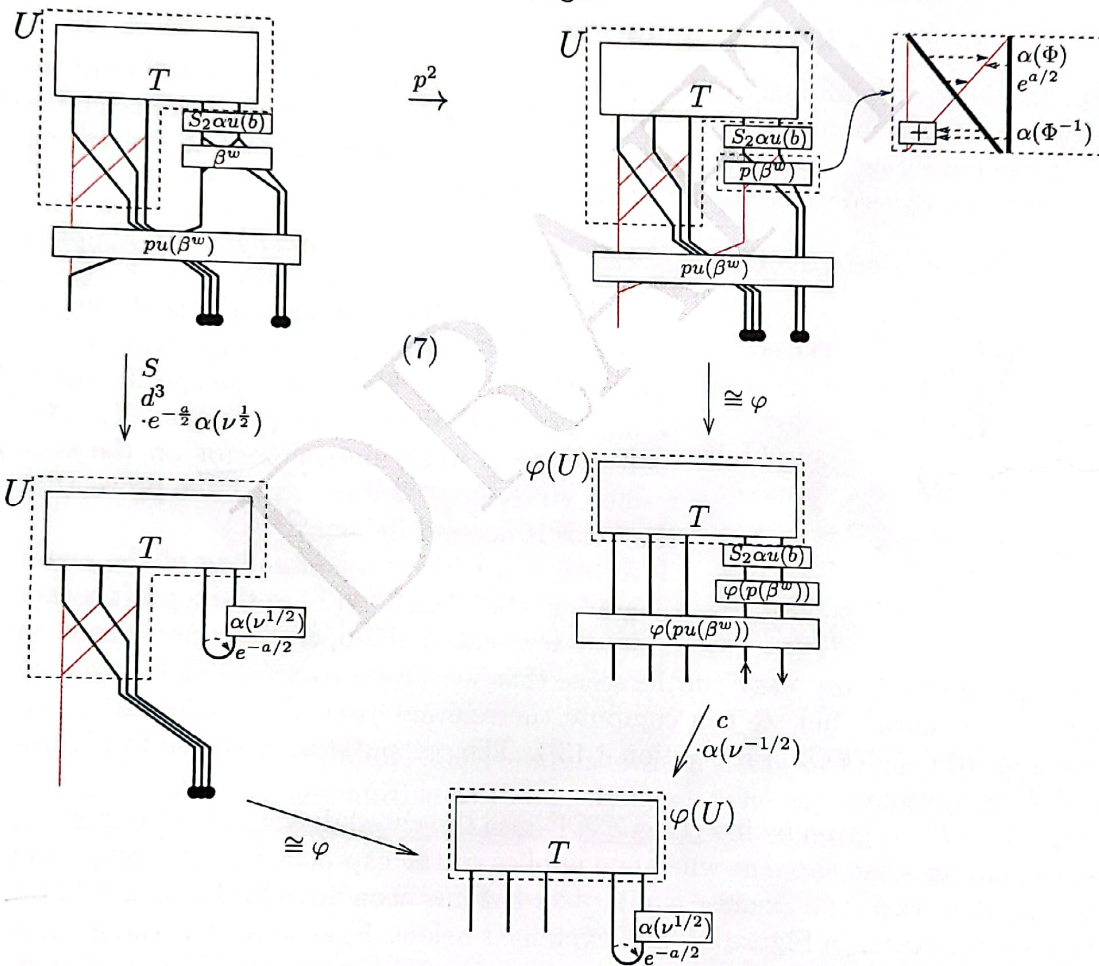


Figure 33. The more detailed picture of the Pentagon 7 .
l.c.



Figure 34. Unzip, switch orientation and connect kills arrows on the strand.

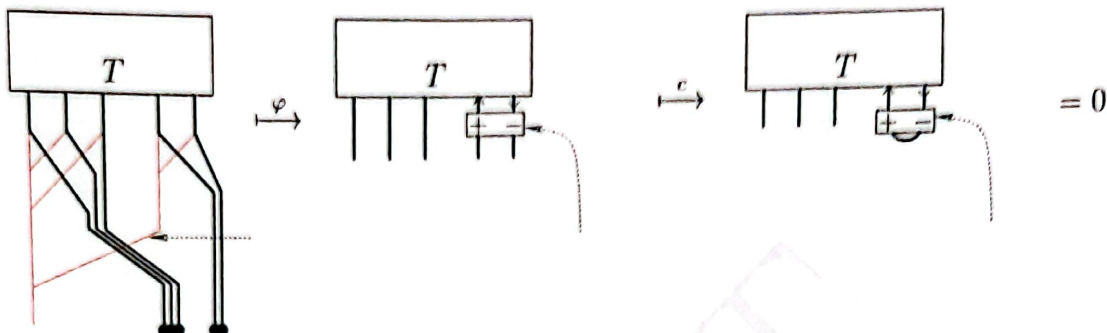


Figure 35. The cancellation of $\varphi(pu(\beta^w))$.

down across the vertex using a VI relation: this has the same effect as an unzip and an orientation switch on the second stand (as this is oriented downwards). The resulting value $S\alpha u(b)$ cancels by the following reasoning – essentially by definition of unzips and orientations switches – which is illustrated in Figure 34:

Given an arrow 'a' ending on strand 'e', unzipping e produces a sum of two arrows $a_1 + a_2$: one ending on each daughter strand. Reversing the orientation of the first daughter strand gives $-a_1 + a_2$. Contracting the two daughter strands to form a U-shape identifies a_1 and a_2 , making $a_1 - a_2$ vanish.

This is exactly what happens to the arrow diagram $S_2\alpha u(b)$ (just under the tangle T in Figure 33) after the edge deletions or contraction, hence this component cancels it in both directions of the pentagon (7).

The bottom arrow on the left is the Sorting Isomorphism φ of Lemma 2.5.

On the other hand, working from the top left corner of the pentagon (7) to the right: from Section 3.2, using the strand numbering convention of Figure 15, we have

$$p_1 p_3 \beta^w = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}).$$

This is shown in the enlarged rectangle at the top right corner of Figure 33. The same is true for $pu(\beta^w)$ at the bottom, except with more unzips.

From here, the first downward arrow applies the Sorting Isomorphism φ of Lemma 2.5, followed by the contraction and correction term along the second downward arrow. At the bottom of the diagram, $\varphi(pu(\beta^w))$ cancels altogether after contraction, in a similar fashion to Figure 34. Namely, φ and the contraction annihilates any arrow ending on the diagonal red strand, or the double capped strand on the right, as shown in Figure ?? for the diagonal red strand. This cancels each factor of $pu(\beta^w)$.

Of the top $\varphi(p(\beta^w))$ shown in the enlarged rectangle, the Φ^{-1} component cancels by the same contraction argument; only the exponential and the Φ component remains. The arrow in the exponent of $e^{a/2}$ switches sign due to the reversed orientation: this is the $e^{-a/2}$

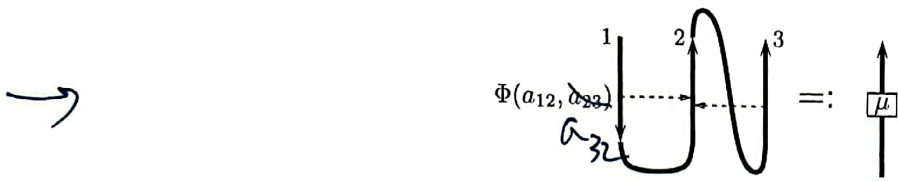
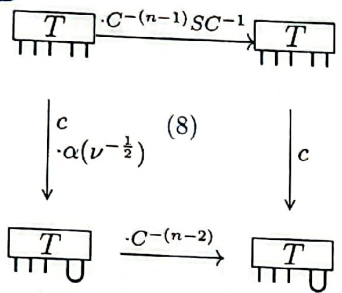


Figure 36. The Φ component of $\varphi(p(\beta^w))$ after contraction.

component at the bottom of the pentagon in Figure 33. After contraction, the Φ component gives rise to a "local" arrow diagram on a single strand shown in Figure 36 and denoted μ . In summary, we see that the pentagon (7) commutes if and only if $\mu = \alpha(\nu)$, and otherwise commutes up to a localised error on the contracted strand, of value $\alpha(\nu)^{-1}\mu$.



Square (8). Finally, for square (8) we need to check that $C^{-1}S(C^{-1}) = \alpha(\nu^{-1/2})$. Note that the orientation switch negates odd wheels and preserves even wheels, therefore in $C^{-1}S(C^{-1})$ the odd part of C^{-1} cancels, and $C^{-1}S(C^{-1}) = C_0^{-2} = \alpha(\nu^{-1/2})$ by Corollary 3.13. This verifies Square (8).

We have therefore shown that Z^w commutes with contraction up to an error $\alpha(\nu)^{-1}\mu$ on the contracted strand. It remains to show that this error is 1. This follows from the facts that $Z^w(\uparrow) = 1$, and that Z^w commutes with disjoint unions:

$$1 = c(Z^w(\uparrow\downarrow)) = \alpha(\nu)^{-1}\mu Z^w(c(\uparrow\downarrow)) = \alpha(\nu)^{-1}\mu \cdot Z^w(\curvearrowright) = \alpha(\nu)^{-1}\mu.$$

This completes the proof. □

Note that as a side result we have proven the following curious fact about associators:

Proposition 3.18. For any Φ horizontal chord associator, μ defined from Φ as in Figure 36, and ν the Kontsevich integral of the unknot, $\mu = \alpha(\nu)$. □

The value $V = Z_\xi^w(\curvearrowright)$ differs, on first glance, from the value V_B constructed in Section 3.1. In the next lemma we show that in fact $V = V_B$, this serves both as a reality check, and as a technical tool for showing - in Theorem 3.22 - that Z_ξ^w satisfies the Cap equation.

Lemma 3.19. The two vertex values from the buckle and the double tree constructions coincide: $V_\beta = V$.

Proof. Notice that $\varphi(\lambda)$ - as in Figure 37 - can be obtained from simpler *sKTGs* by inserting the buckle B^u into $\varphi(\curvearrowright)$ followed by an unzip. We computed $\xi(\curvearrowright)$ in Lemma 3.12. Since Z^u is compatible with insertions and unzips, $Z^u(\lambda)$ can be computed by inserting the buckle value β^u (computed in Section 3.2) into $Z^u(\varphi(\lambda))$.

To compute $\xi(\lambda)$ we then apply α and the cap, unzip and puncture operations. Because $\beta^w = \alpha(\beta^u)$ is local (is confined to the skeleton of the inserted B^u graph) and is in the image of α , all other arrow endongs commute with it by the head- and tail-invariance properties of A^{sw} (Figure 18). Therefore, all of the cancellations in the computation of $\xi(\curvearrowright)$ still occur, and $\xi(\lambda)$ is as shown in Figure 37. Writing this in $A^{sw}(\uparrow_2)$ we obtain $\varphi(p_1p_3(\beta^w))u(\alpha(\nu^{1/2}))$.

shrink!

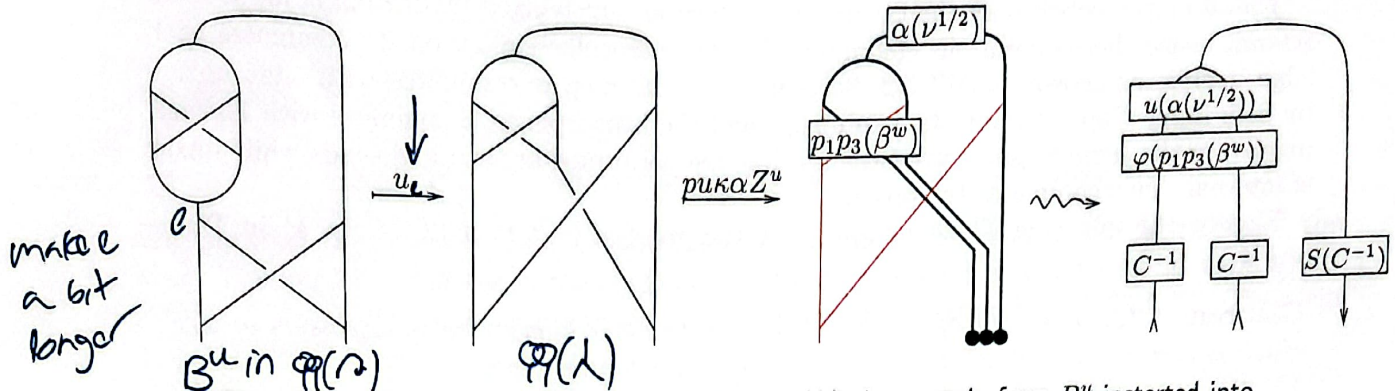


Figure 37. On the left we show how to obtain $\varphi(\lambda)$ via an unzip from B^u inserted into $\varphi(\lambda)$. From this we compute $\xi(\lambda)$, and finally V on the right.

fig:DTandB

To obtain $V = Z_\xi^w(\lambda)$, one multiplies $\xi(\lambda)$ at each end by C^{-1} or $S(C^{-1})$ depending on orientation, as shown in Figure 37 on the right. Thus,

$$V = C_1^{-1} C_2^{-1} \varphi(p_1 p_3 \beta^w) u(\alpha \nu^{1/2}) u(S(C^{-1})).$$

On the other hand, from Section 3.2, we have:

$$V^\beta = C_1^{-1} C_2^{-1} \varphi(p_1 p_3 \beta^w) u(C).$$

Thus, we need to show that:

$$C_1^{-1} C_2^{-1} \varphi(p_1 p_3 \beta^w) u(\alpha \nu^{1/2}) u(S(C^{-1})) = C_1^{-1} C_2^{-1} \varphi(p_1 p_3 \beta^w) u(C),$$

in $\mathcal{A}^{sw}(\uparrow_2)$. Multiplying with $(\varphi(p_1 p_3 \beta^w))^{-1} C_1 C_2$ on the left (bottom) and by $u(S(C))$ on the right (top), this simplifies:

$$u(\alpha \nu^{1/2}) = u(C) u(S(C)).$$

Since unzips commute with orientation switches, it is sufficient to prove that

$$CS(C) = \alpha \nu^{1/2}.$$

Recall that in $CS(C)$ all odd wheels cancel, hence $CS(C) = (C_0)^2$, where C_0 denotes the even part of C . Indeed, by Corollary 3.13, $C_0^2 = \alpha \nu^{1/2}$. \square

SameResult

Corollary 3.20. The "buckle" and "double tree" constructions lead to the same result, that is, $Z_\xi^w = Z_\beta^w$.

Proof. Since any homomorphic expansion Z^w of wTF is uniquely determined by $Z^w(\lambda)$, this is immediate from Lemma 3.19. \square

The next lemma implies that V satisfies the Unitarity (U) equation:

lem:Unzip

Lemma 3.21. The map Z_ξ^w commutes with strand unzips in wTT .

Proof. We first note that in wTT unzip is only defined for internal edges, that is, edges which end in a vertex²⁰ at both ends. By construction, Z_ξ^w is a composition of several maps. We show that edge unzips commute with every one of these, hence with Z_ξ^w .

²⁰In fact, there are further restrictions, eg the two vertices must be of opposite signs, but this is not important for the proof.

The φ map involves only the tangle ends, which are unchanged by unzipping an internal strand, hence these operations commute. The homomorphic expansion Z^u commutes with edge unzips, as shown in [WKO2, Section 4.6]. The map α commutes with edge unzips by definition. Cap attachments, cap unzips and the isomorphism φ commute with internal unzips for the same reason as φ , because they are performed at the tangle ends while unzip is internal. This completes the proof. \square

Finally, the following Theorem completes the proof of part Part (3) of the Main Theorem 1.1:

Theorem 3.22. *The map $Z_\xi^w : \widetilde{wTF} \rightarrow \mathcal{A}^w$ is the unique homomorphic expansion of \widetilde{wTF} , which is compatible with Z^u in the sense of the commutative diagram (2).*

Proof. By Proposition 3.9, Z_ξ^w is compatible with Z^u . By Part (1) (Section 3.1) and Corollary 3.20, $Z_\xi^w = Z_\beta^w$ and uniquely determined by Z^u .

To show that Z_ξ^w is a homomorphic expansion, by Fact 2.6 and Theorem 2.7, one only needs to verify that it satisfies the (R4), Unitarity (U) and Cap (C) equations of Fact 2.6. Of these, R4 follows from the fact that Z_ξ^w is a planar algebra map, Theorem 3.17. The Unitarity equation (U) is equivalent to the statement that Z_ξ^w commutes with strand unzips [WKO2, Section 4.3], and hence it is satisfied by Lemma 3.21. *why do we need to show R4?*

This leaves the Cap Equation, \square

APPENDIX A. THE ALEKSEEV-ENRIQUEZ-TOROSSIAN FORMULA

This appendix is mainly for readers familiar with the Alekseev-Enriquez-Torossian formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators [AET].

For a quick re-cap of [AET] notions, let \mathfrak{lie}_2 denote the degree completed free Lie algebra on two generators x and y . Let \mathfrak{tder}_2 denote tangential derivations of this Lie algebra, that is, derivations d with the property that $d(x) = [x, a_1]$ and $d(y) = [y, a_2]$, where $a_1, a_2 \in \mathfrak{lie}_2$. Let $\text{TAut}_2 := \exp(\mathfrak{tder}_2)$ denote the group of tangential automorphisms of \mathfrak{lie}_2 .

There is a linear "interpretation map" (not a map of Lie algebras) $\theta : \mathfrak{lie}_2^2 \rightarrow \mathfrak{tder}_2$, sending a pair (a_1, a_2) to the derivation d given by $d(x) = [x, a_1], d(y) = [y, a_2]$. The kernel of this map consists only of pairs of the form $(\alpha x, \beta y)$ for α, β constants. A one-sided inverse $\eta : \mathfrak{tder}_2 \rightarrow \mathfrak{lie}_2^2$ which sends a tangential derivation to a pair whose first component has no linear x term and second component has no y term. We denote the exponential of θ by $\Theta : \mathcal{U}(\mathfrak{lie}_2^2)_{\text{exp}} \rightarrow \text{TAut}_2$. For an element $(e^{\lambda_1}, e^{\lambda_2}) \in \mathcal{U}(\mathfrak{lie}_2^2)_{\text{exp}}$, we have $G = \Theta((e^{\lambda_1}, e^{\lambda_2})) \in \text{TAut}_2$ given by $G(x) = e^{-\lambda_1} x e^{\lambda_1}$, and $G(y) = e^{-\lambda_2} y e^{\lambda_2}$. Just as θ is not a Lie algebra map, Θ is not a group homomorphism: composition in TAut_2 is not given by piecewise multiplication of the conjugators. However, θ and Θ present a convenient way to denote tangential derivations and tangential automorphisms as pairs in \mathfrak{lie}_2^2 and $(\exp(\mathfrak{lie}_2))^2$, respectively.

Given a Drinfel'd associator Φ , [AET, Theorem 4] gives an explicit formula for a Kashiwara-Vergne solution F as follows²¹:

$$F = (\Phi^{-1}(x, -x - y), e^{-(x+y)/2} \Phi(-x - y, y) e^{y/2}). \quad (12)$$

²¹There are some notational differences between [AT] and [AET]. There are sign differences between the formula (12) and [AET] due to notational misalignment, for example our Φ is [AET]'s Φ^{-1} . Our notation is consistent with the other papers in this series and the formulas are computationally verified in [WKO4].

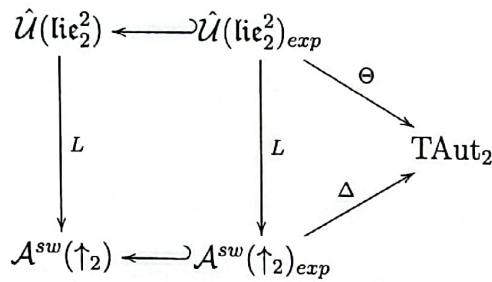


Figure 38. The connection between $\mathcal{A}^{sw}(\uparrow_2)$ and TAut_2 .

fig:ATInte

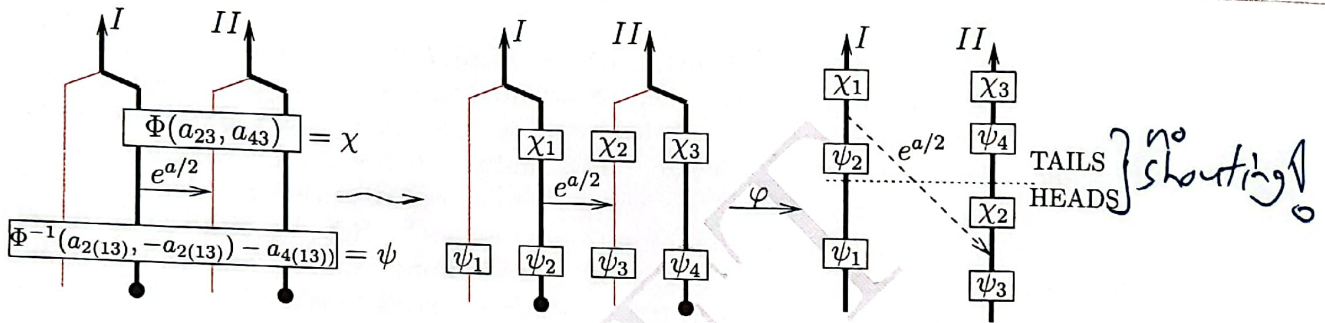


Figure 39. To compute $\varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}))$ we switch to a placement notation in which we mark on each skeleton strand the elements that have arrows ending on it. For this purpose we denote $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) =: \psi$ and $\Phi(a_{23}, a_{43}) =: \chi$.

fig:Valuev

To accomplish this, we need to compute $V^{tree} = \varphi(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}))$ more explicitly, as shown in Figure 39, and explained in the caption. The first strand of $\mathcal{A}^{sw}(\uparrow_2)$ joins strands 1 and 2 in a vertex, and the second strand of $\mathcal{A}^w(\uparrow_2)$ joins strands 3 and 4. Strands 1 and 3 are punctured and strands 2 and 4 are capped. Let us call the two strands of $\mathcal{A}^w(\uparrow_2)$ strand I and strand II to avoid confusion. Recall from the construction of φ that one first slides arrow tails from the capped strands "up" through the vertices, then slides all the heads up from the punctured strands 1 and 3. Thus one obtains an element of $\mathcal{A}^w(\uparrow_2)$ in which all arrow heads are below all tails on both strands.

Now we are ready to compute how $\pi(V) \in \mathcal{A}^w(\uparrow_2)$ acts on the generator x of \mathfrak{lie}_2 and match this to the formula (12). Recall the value of $\pi(V)$ shown in Figure 39. The generator x is represented by an arrow from the first strand to the added third strand, and the action is by conjugation, as shown in Figure 40. To compute this, one commutes the tail of x to the top of the strand across $\pi(V)$ using \overline{STU} relations, thereby $\pi(V)$ and $\pi(V)^{-1}$ cancel, and the result of the action remains. Observe that due to the TC relation, only arrows with heads on strand I act nontrivially on x , in other words only ψ_1 matters, which came from $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. The arrows a_{23} and a_{43} act trivially on x , so, more simply stated, the action on x is by $\varphi(\Phi^{-1}(a_{21}, -a_{21} - a_{41}))$. Note that $L(\Phi^{-1}(x, -x - y), 0) = \varphi(\Phi^{-1}(a_{21}, -a_{21} - a_{41}))$, so Theorem 1.1 agrees with Formula (12) in the first component.

One can proceed similarly for the second component: the action on y is by

$$\varphi(\Phi^{-1}(a_{23}, -a_{23} - a_{43})e^{a_{23}}\Phi(a_{23}, a_{43})) = L(0, \Phi^{-1}(x, -x - y)e^{x/2}\Phi(x, y)).$$

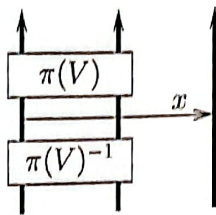


Figure 40. The action of $\pi(V)$ on the generator x of lic_2 .

fig:ActOnx

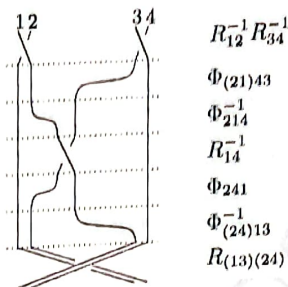


Figure 41. A different expression of β^b .

fig:BetaBraid

While this does not match the second component of Formula (12), it only differs from it by a hexagon relation. Alternatively, note that one can obtain the second component of the Formula (12) "on the nose" by starting from an equivalent (isotopic) expression²² of β^b , as shown in Figure 41. This completes the proof.

A

REFERENCES

- [AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations*, Publications Mathématiques de L'IHÉS, 112-1 (2010) 143–189, arXiv:0903.4067.
- [AKKN1] A. Alekseev, N. Kazumi, Y. Kuno, F. Naef, *The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem*, arXiv:1703.05813.
- [AKKN2] A. Alekseev, N. Kazumi, Y. Kuno, F. Naef, *Goldman-Turaev formality implies Kashiwara-Vergne*, Quantum Topology 11-4 (2020) 657–689.
- [AM] A. Alekseev and E. Meinrenken, *On the Kashiwara-Vergne conjecture*, Inventiones Mathematicae, 164 (2006) 615–634, arXiv:0506499.
- [AN] A. Alekseev, F. Naef, *Goldman-Turaev formality from the Knizhnik-Zamolodchikov connection* Comptes Rendus Mathématique 355-11 (2017) 1138–1147
- [AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Annals of Mathematics 175 (2012) 415–463, arXiv:0802.4300.
- [BN1] D. Bar-Natan, *On associators and the Grothendieck-Teichmüller Group I*, in *Selecta Mathematica*, New Series 4, 183–212, June 1996.
- [BN2] D. Bar-Natan, *Non-associative tangles*, in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.
- [BND1] D. Bar-Natan and Z. Dancso, *Homomorphic expansions for knotted trivalent graphs*, Journal of Knot Theory and its Ramifications Vol. 22, No. 1 (2013) arXiv:1103.1896

²²We thank Karene Chu for this idea.

A: A sequel paper [wko4] verifies the results in this appendix by explicit computations in low degrees.

tSolutions
dmanTuraev
:GTRreverse
orenken:KV
vNaef:GTKZ
waraVergne
-Natan:GT1
-Natan:JAT
Dancso:KTG