We need an index of notation (Dror)

Re-read the two mass-killing arguments.

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: W-FOAMS, THE KASHIWARA-VERGNE THEOREM AND DRINFEL'D ASSOCIATORS

DROR BAR-NATAN AND ZSUZSANNA DANCSO

ABSTRACT. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological solution to the Kashiwara-Vergne problem. In particular we recover via a topological argument the Alkeseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of *2-dimensional foams* and various associated features in four-dimensioanl space. We use a topological construction which we name the double tree construction to show that every *expansion* (also known as *universal finite type invariant*) of parenthesized braids extends first to an expansion of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is uniquely determined by *Drinfel'd associator* Φ, and an expansion for w-knotted objects is uniquely determined by a solution V of the Kashiwara-Vergne problem [KV], as reformulated by Alekseev and Torossian [AT]. Hence our result provides a topological framework for the result of [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof of the Kashiwara-Vergne Theorem and the Alekseev-Enriquez-Torossian formula.

CONTENTS

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~~Dror: Fix WK04 page 26, 39, 45. Dror Ek.~~

In Sections 2.1 to 2.3 of [WKO2] we discuss associated graded structures and expansions for general algebraic structures. For any algebraic structure (think braids, or tangles with tangle composition), one allows formal linear compositions of elements of the same *kind* (think, same skeleton). Associated graded structures are taken with respect to the filtration by powers of the *augmentation ideal*. For the spaces *uPaB*, *sKTG* and $\widetilde{wTF}$, the associated graded spaces $\mathcal{A}^{hor}$, $\mathcal{A}^u$ and $\mathcal{A}^{sw}$ are the spaces of "horizontal chord diagrams on parenthesized strands", "chord diagrams on trivalent skeleta", and "arrow diagrams", as described in [BN1], [WKO2, Section 4.6], and Section 2 of this paper, respectively. As a result, the associated graded structure of $\mathcal{K}$ is

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{\mathrm{cl}} \mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and a, respectively. More specifically, cl is the "closure of chord diagrams", and α is "replacing each chord with the sum of its two possible orientations", see [WKO2, Section 3.3].

An expansion [WKO2, Section 2.3] is a filtration-respecting map from an algebraic structure to its associated graded structure, whose associated graded map is the identity. In knot theory, expansions are also called universal finite type invariants. A homomorphic expansion is an expansion which behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the associated graded structure; for a detailed definition and introduction see [WKO2, Section 2.3]. Hence, a homomorphic expansion $Z : \mathcal{K} \to \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u, and Z^w for $\mathcal{K}^b :=$ *uPaB*, $\mathcal{K}^u :=$ *sKTG* and $\mathcal{K}^w := \widetilde{wTF}$, respectively, so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K}: \arrow[d, "Z"] & \mathcal{K}^b \arrow[r, "\mathrm{cl}"] \arrow[d, "Z^b"] & \mathcal{K}^u \arrow[r, "a"] \arrow[d, "Z^u"] & \mathcal{K}^w \arrow[d, "Z^w"] \\
\mathcal{A}: & \mathcal{A}^{hor} \arrow[r, "\mathrm{cl}"] & \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^w
\end{tikzcd} \tag{2}$$

eq:MainD

We recall (see [BN1]) that a homomorphic expansion Z^b for parenthesized braids is determined by a "horizontal chord associator" $\Phi = Z^b(|\!/|)$. A homomorphic expansion Z^u of *sKTG* is also determined[2] by a Drinfel'd associator (horizontal chords or not; see [WKO2, Section 4.6]), so the significance of the left commutative square is to force the associator corresponding to Z^u to be a horizontal chord associator. In turn, Z^w is determined by a solution F (a close cousin of $V = Z^w(\text{⤴}))$ to the Kashiwara-Vergne problem (see [WKO2, Section 4.4 – 4.5]). The goal of this paper is to prove the following theorem, which, via the correspondence above, implies the KV conjecture:

thm:main

Theorem 1.1. (1) *Assuming that $Z : \mathcal{K} \to \mathcal{A}$ exists, it is determined*[3] *by Z^u.*

what is the formula for C?

(2) *There is a formula for $V = Z^w(\text{⤴})$ and $C = Z^w(\downarrow)$ in terms of the Drinfel'd associator Φ associated to Z^u:*

$$V = C_1^{-1}C_2^{-1}\varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2}\Phi(a_{23}, a_{43})\right) C_{(12)}, \tag{3}$$

eqn:AET

[2]With the exception of some minor normalization, see [WKO2], Lemma 4.14 and the paragraph after.

[3]In fact, almost entirely determined by Z^b, with the exception of some minor normalization of Z^u which is not determined by an associator.

where a denotes a single arrow[4]*. This agrees*[5] *with the formula proven in* [AET].

(3) *Every Z^b extends to a Z.*

The key to the proof of the theorem is to show that the generator of $\widetilde{wTF}$ can be expressed in terms of the generator of $uPaB$ and the operations of $\mathcal{K}$. Assuming that Z exists, this yields a formula for V in terms of Φ.

1.3 A

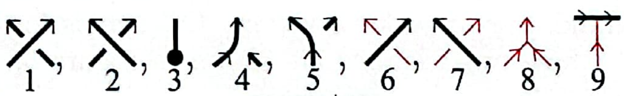
1.4 ~~1.3.~~ **Paper Structure.** In Section 2 we provide an overview of the space wTF^o of (oriented) w-foams and its extension with strings $\widetilde{wTF}$. We provide a brief review of definitions and crucial facts from [WKO2], and details of the extension. We prove that homomorphic expansions for wTF^o extend uniquely to homomorphic expansions for $\widetilde{wTF}$.

Section 3 makes up the bulk of the paper and is devoted to the proof of Theorem 1.1. In Section 3.1 we prove part (1). In Section 3.2 we deduce the formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators, proving part (2). In Section 3.3 we prove statement (3), the hardest part of the proof.

Section ?? is a short section of closing remarks, and in Appendix A we give an explicit comparison and equivalence between our formula in Part (2) and the ALekseev–Enriquez–Torossian [AET] formula.

2. The spaces $\widetilde{wTF}$ and $\mathcal{A}^{sw}$ in more detail

As mentioned in the introduction, $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4]. It can be introduced as a planar algebra or as a circuit algebra; we will do the latter as it is simpler and more concise. Circuit algebras are defined in [WKO2, Section 2.4]; in short, they are similar to planar algebras but without the planarity requirement for "connecting strands". As in [WKO2], each generator and relation of $\widetilde{wTF}$ has a local topological interpretation. Recall from [WKO2, Sections 1.2, 3.4, 4.1] that wTF^o diagrams represent certain ribbon knotted tubes with foam vertices in $\mathbb{R}^4$, and the circuit algebra wTF^o is conjecturally a Reidemeister theory for this space (i.e., there is a surjection δ from the circuit algebra wTF^o to ribbon knotted tubes with foam vertices, and δ is conjectured to be an isomorphism). The space $\widetilde{wTF}$ extends wTF^o by adding one-dimensional strands to the picture. Note that in themselves, one dimensional strands in $\mathbb{R}^4$ are never knotted, however, they can be knotted *with* the two-dimensional tubes. In figures two-dimensional tubes will be denoted by thick lines and one dimensional strings by thin red lines. With this in mind, we define $\widetilde{wTF}$ as a circuit algebra defined in terms of generators and relations, and with some extra operations beyond circuit algebra compositions. Each generator, relation and operation has a local topological interpretation which provides much of the intuition behind the proofs. However, the corresponding Reidemeister theorem is only conjectural.

$\widetilde{wTF} = \text{CA}\Big\langle$  | relations as in Section 2.2 | auxiliary operations as in Section 2.3 $\Big\rangle$

generators

[4]The notation is explained in detail in Section 3.2

[5]Although the two formulas are written in different languages, and checking that they agree takes effort. See Section 3.2 and Appendix A.

A: 1.3 computations. We note that this paper is "abstract", yet everything difficult in it occurs within graded spaces, and can be computed explicitly up to a certain degree (which depends on what exactly is computed). In a continuation paper [WKO4] some of these computations are carried out.

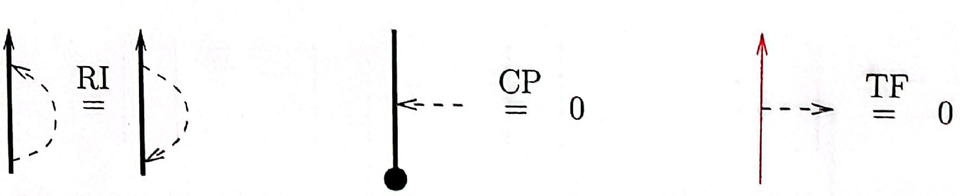


Figure 8. The RI and CP relations, and the TF relation (which is not really a relation).

over FTW

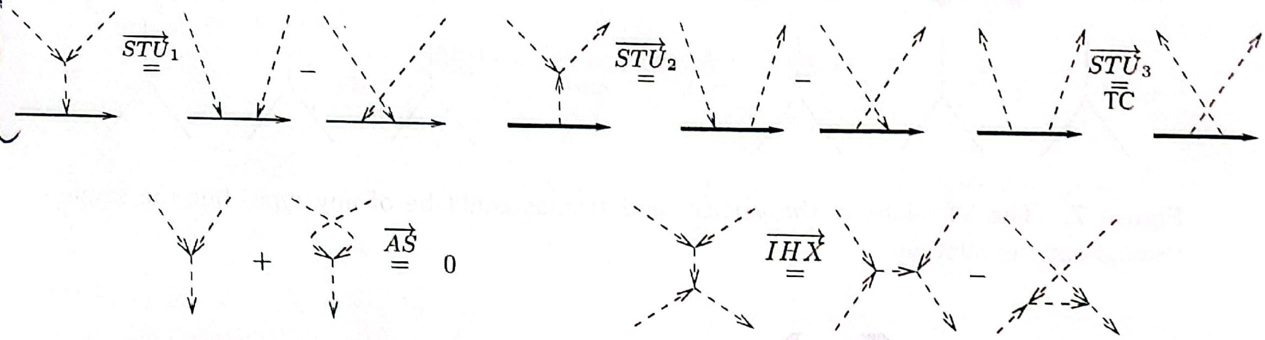
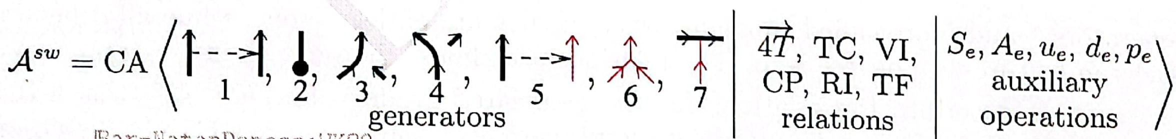


Figure 9. The $\overrightarrow{AS}$, $\overrightarrow{IHX}$ and the three $\overrightarrow{STU}$ rerations. Note that in $\overrightarrow{STU}_1$, the skeleton strand can be thin red or thick black, and that $\overrightarrow{STU}_3$ is the same as the TC relation.

for wTF^{oo}. We quickly recall these here, for details see [WKO2, Section 4.2.2]. The orientation switch S_e reverses the orientation of the skeleton strand e, and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow heads and tails on } e\}}$. The adjoint operation also reverses the skeleton strand e and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow heads on } e\}}$. Given a skeleton S with a distinguished strand e, unzip (or disc unzip, if e is capped) is an operation $u_e : \mathcal{A}^{sw}(S) \to \mathcal{A}^{sw}(u_e(S))$ which maps each arrow ending on e to a sum of two arrows, one ending on each of the two new strands which replace e. Deleting a long strand e kills all arrow diagrams with any arrow ending on e. The operation induced by puncture, denoted p_e, turns the formerly thick black e into a thin red strand, and kills any arrow diagram with any arrow tails on e.

To summarise:



As in [WKO2, Definition 3.7], we define a "w-Jacobi diagram" (or just "arrow diagram") by also allowing trivalent chord vertices, each of which is equipped with a cyclic orientation, and modulo the $\overrightarrow{STU}$ relations of Figure 9. Denote the circuit algebra of formal linear combinations of these w-Jacobi diagrams by $\mathcal{A}^{swt}$. Then, as in [WKO2, Theorem 3.8], we have the following "bracket-rise" theorem:

Theorem 2.1. *The natural inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$. Furthermore, the $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ relations of Figure 9 hold in $\mathcal{A}^{swt}$.*

The proof is identical to the proof of [WKO2, Theorem 3.8]. In light of this isomorphism, we will drop the extra "t" from the notation and use $\mathcal{A}^{sw}$ to denote either of these spaces. As in [WKO2], the primitive elements of $\mathcal{A}^{sw}$ are connected diagrams, denoted $\mathcal{P}^{sw}$, and

In the other direction, consider an arrow diagram on the capped/stringed vertex. One may assume that there are only arrow tails on the capped strand under the vertex: any arrow head may be commuted using $\overrightarrow{STU}$ relations towards the cap, where it is killed by the CP relation[6]. On the thin red strand there are only arrow heads. To construct φ, first "push" the arrow tails (denoted "t") from the capped strand up across the vertex using the VI relation. Since tails vanish on the thin red strand, they simply slide past the vertex. Once the capped side is cleared, continue by sliding the arrow heads "h" up from the thin red string to the strand above the vertex. Now the cap relation kills any arrow heads on the capped strand, so once again they simply slide past the vertex. The result placed on a single thick black strand is shown in Figure 11.

It is clear that ψ is well-defined, we leave it to the reader to check that so is φ as a short exercise. Given that both maps are well-defined, it is clear that they are inverses of each other. $\square$

Observe that in the image of φ, all arrow tails are above arrow heads along the strand. Arrow diagrams of this form appear in the context of "over-then-under" tangles, which have applications in several contexts, including virtual braid classification [BDV].

2.5. The homomorphic expansion. As discussed in [WKO2, Section 2.3], an expansion for $\widetilde{wTF}$ is a map $Z^w : \widetilde{wTF} \to \mathcal{A}^{sw}$ with the property that the associated graded map $\operatorname{gr} Z^w : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ is the identity map $\mathrm{id}_{\mathcal{A}^{sw}}$. A homomorphic expansion is an expansion which also intertwines each operation of $\widetilde{wTF}$ with its arrow diagrammatic counterpart. In [WKO2, Theorems 4.9 and 4.11] we proved that the existence of solutions for the Kashiwara–Vergne equations implies that there exsists a homomorphic expansion for $\widetilde{wTF}^o$. In fact that homomorphic expansions[7] for $\widetilde{wTF}^o$ are in one-to-one correspondence with solutions to the Kahiwara-Vergne problem.

The point of this paper is to provide a topological construction for such a homomorphic expansion (and hence for a solution of the Kashiwara–Vergne conjecture), and this is easier to do for the slightly more general space $\widetilde{wTF}$.

Let $\mathcal{A}^{osw} \subseteq \mathcal{A}^{sw}$ denote arrow diagrams on $\widetilde{wTF}^o$ skeleta, the associated graded space of $\widetilde{wTF}^o$. One of the key results of [WKO2, Section 4.3] is the characterisation of homomorphic expansions of $\widetilde{wTF}^o$. For any (group-like) homomorphic expansion $Z^{ow} : \widetilde{wTF}^o \to \mathcal{A}^{osw}$, the value $Z^{ow}(\overcrossing)$ is uniquely determined and equals $R = e^{a_{12}}$, where a_{12} denotes a single arrow from the over strand 1 to the under strand 2.

To state the full characterisation, we use co-simplicial notation in subscripts. For example, for $R = e^{a_{12}} =\in \mathcal{A}^{sw}(\uparrow_2)$, $R_{13} = e^{a_{13}}$ and $R_{23} = e^{a_{23}}$ in $\mathcal{A}^{sw}(\uparrow_3)$ are the diagrams where R is placed on strands 1 and 3, and 2 and 3, respectively. $R_{(12)3} \in \mathcal{A}^{sw}(\uparrow_3)$ is obtained by doubling the first strand of R and placing it on strands 1 and 2, and placing the second strand of R on strand 3, that is, $R_{(12)3} = e^{a_{13}+a_{23}}$. Similarly for $V \in \mathcal{A}(\uparrow_2)$, $V_{12} \in \mathcal{A}(\uparrow_3)$ denotes V placed on the first two starnds, et cetera.

Fact 2.5. *A filtered, group-like map $Z^{ow} : \widetilde{wTF}^o \to \mathcal{A}^{osw}$ is a homomorphic expansion if and only if the Z^{ow}-values $V = Z^{ow}(\curlywedge)$ and $C = Z^{ow}(\downarrow)$ satisfy the following equations:*

(1) R4 Equation:

$$V_{12}R_{(12)3} = R_{23}R_{13}V_{12} \quad \text{in } \mathcal{A}^{sw}(\uparrow_3). \tag{R4}$$

[6]This argument also appears in [WKO2], for example as the basic idea for the proof of Fact 2.2.

[7]Subject to the minor technical condition that the value of the vertex doesn't contain isolated arrows.

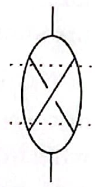
$A_1A_2(p_1(b)) = p_1(b)$. Each arrow has one head, so $A_1A_2(p_1(t)) = -p_1(t)$. Hence, using commutativity, $p_1V \cdot A_1A_2(p_1V) = e^{2p_1(b)} = 1$, which implies that $p_1(b) = 0$. As for $p_1(t)$, one can show that there are no arrows pointing from the right to the left strand by a direct computation in degree 1. □

3. Proof of Theorem 1.1

3.1. Proof of Part (1). We prove Part 1 in two steps: first verifying the easier "tree level" case, which nonetheless contains the main idea, then in general.

3.1.1. *Tree level proof of Part (1).* Let $\mathcal{A}^{tree}$ denote the quotient of $\mathcal{A}^{sw}$ by all wheels, and let $\pi : \mathcal{A}^{sw} \to \mathcal{A}^{tree}$ denote the quotient map (cf [WKO2, Section 3.2]). Part (1) of the main theorem is the same as stating that Z^w is determined by Z^u. Z^w, in turn is determined by the values V and C of the positive vertex and the cap [WKO2, Sections 4.3 and 4.5], so one only needs to show that V and C are determined by Z^u. Proving this "on the tree level" means showing only that $\pi(V)$ and $\pi(C)$ are determined by Z^u. In particular, observe that since C is a linear combination of products of wheels (Fact 2.2), we have $\pi(C) = 1$, so we only need to show that $\pi(V)$ is determined by Z^u.

Let B^u denote the "buckle" *sKTG*, as shown on the right (ignore the dotted lines for now). All edges are oriented up, and by the drawing conventions of [WKO2, Section 4.6] all the vertices in the bottom half of the picture are negative and all the ones in the top half are positive. Let $B^w = a(B^u) \in \widetilde{wTF}$, and $\beta^u := Z^u(B^u)$.* Note that β^u can be thought of as a chord diagram on four strands: use VI relations to move all chord endings to the "middle" of the skeleton, between the dotted lines on the picture. Hence, we write $\beta^u \in \mathcal{A}^u(\uparrow_4)$. Let $\beta^w = \alpha(\beta^u)$, and note that by the compatibility of Z^u and Z^w we have $\beta^w = Z^w(B^w)$. We will perform a series of operations on B^w and $\pi(\beta^w)$ to recover $\pi(V)$ from it.

$B^u =$ 

First, connect (a circuit algebra operation in $\widetilde{wTF}$) a positive vertex to the bottom of B^w, as shown in Figure 12. Then unzip the edge marked by u, and puncture the edges marked e and e'. Then attach a cap (once again a circuit algebra operation) to the thick black end at the bottom. Finally, unzip the capped strand.

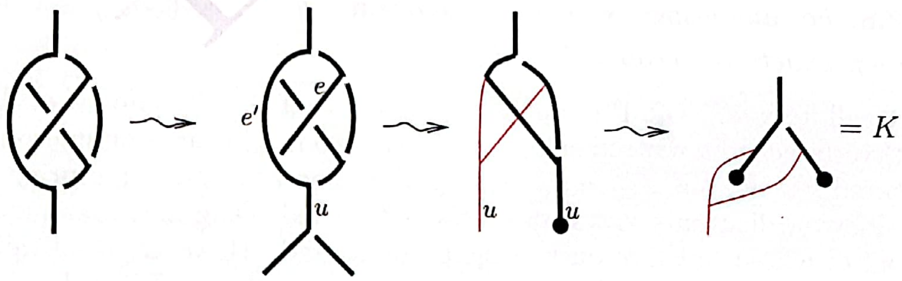


Figure 12. From the "buckle" β^w to the (modified) vertex.

Call the resulting w-foam K, as shown at the right in Figure 12. What is $Z^w(K)$? Due to the homomorphicity of Z, it is obtained from β^w by performing the same series of operations in the associated graded: a circuit algebra composition with V, unzip, punctures, circuit algebra composition with C, and disc unzip. Notice that the left strand of that attached

* β^u is computed explicitly to degree 4 in [WKO4]

vertex got punctured, and hence by Lemma 2.8 the attached value V cancels.[9] $Z^w(K)$ still depends on the value C. At the tree level, since $\pi(C)=1$, $\pi(Z^w(K))$ can be computed from β^w by performing punctures and unzips. Since $\beta^w=\alpha(\beta^u)$, this means that $\pi(Z^w(K))$ is determined by Z^u.

On the other hand, note that the space of chord diagrams on the skeleton of K is the space $\mathcal{A}(\uparrow_2)$ by Lemma 2.4 and VI. Note also that K is a circuit algebra combination of a vertex, two left-punctured right-capped vertices and an all-red-strings vertex, and the Z^w-values of the latter three are trivial. So $\pi(Z^w(K))=\pi(V)\in\mathcal{A}^{tree}(\uparrow_2)$. Hence, $\pi(V)$ is determined by Z^u as needed. $\square$

3.1.2. *Complete proof of Part (1).* In the previous subsection we showed that Z^u determines $\pi(V)\in\mathcal{A}^{tree}(\uparrow_2)$. The proof of Part (1) is completed by the following Lemma:

Lemma 3.1. *For any homomorphic expansion Z^w of wTF* s.t. *$V=Z^w(\curlyvee)$ and $C=Z^w(\downarrow)$*, Then *$\pi(V)\in\mathcal{A}^{tree}(\uparrow_2)$ determines both V and C uniquely.*

Proof. We use a perturbative argument. By contradiction, assume this is not the case, in particular, first assume that there exist $V\neq V'$, both of which are vertex values of Z^u-compatible homomorphic expansions, such that $\pi(V)=\pi(V')$. Let v denote the lowest degree term of $V-V'$. Note that v is primitive and $v\in\ker\pi$, so v is a homogeneous linear combination of wheels. By the Unitarity Equation of Fact 2.5, we have $A_1A_2(v)=-v$. Recall that A_i reverses the direction of the strand i and multiplies each arrow diagram by (-1) to the number of heads on that strand. Since v has only tails, $A_1A_2(v)=v$, so $v=-v$, so $v=0$, a contradiction. Therefore, $\pi(V)$ determines V uniquely.

Now we show that V determines C uniquely. Assume there are different values C and C' in $\mathcal{A}^{sw}(\downarrow)$ so that (V,C) and (V,C') are both vertex-cap value pairs of Z^u-compatible homomorphic expansions. Let c denote the lowest degree term of $C-C'$, then c is a scalar multiple of a single wheel. The Cap Equation of Fact 2.5 implies $c_{(12)}=c_1+c_2$ in $\mathcal{A}^{sw}(\downarrow_2)$.

There is a well-defined linear map $\omega:\mathcal{A}^{sw}(\downarrow_2)\to\mathbb{Q}[x,y]$ sending an arrow diagram – which has arrow tails only on each strand – to "x to the power of the number of tails on strand 1, times y to the power of the number of tails on strand 2". Assume $c=\alpha w_r$, where w_r denotes the r-wheel, and $\alpha\in\mathbb{Q}$. Then $0=\omega(c_{(12)}-c_1-c_2)=\alpha((x+y)^r-x^r-y^r)$, so either $r=1$ or $\alpha=0$. But $w_1=0$ in $\mathcal{A}^{sw}$ by the RI relation, hence $\alpha=0$ and thus $c=0$, a contradiction. $\square$

3.2. **Proof of Part (2).** In this section we compute V, the value of the vertex, from Φ, the Drinfel'd associator determining Z^b, using the construction of Part (1). In Appendix A we also show that this result translates to the [AET] formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators.

In the computation of V from Φ, as well as later in the paper, we use two facts about Drinfel's associators. We summarise these in the following Lemma:

Lemma 3.2. *Let $\Phi=\Phi(c_{12},c_{23})\in\mathcal{A}^u(\uparrow_3)$ be a Drinfel'd associator, where c_{ij} denotes a chord between strands i and j. Then, the following facts hold for $\alpha(\Phi)\in\mathcal{A}^w(\uparrow_3)$:*

(1) *$p_ip_j\alpha(\Phi)=1$, whenever $i,j\in\{1,2,3\}$, $i\neq j$, and p_i denotes puncture of the i-th strand.*

[9] Any short arrows would also cancel when the right strand is capped.

C_{23} is a chord?!
use m_{23}?

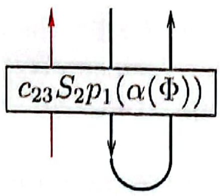


Figure 13. A concatenated associator.

(2) $c_{23}S_2p_1(\alpha(\Phi)) = 1$, *where S_2 stands for orientation switch of strand 2, and c_{23} is* ~~*concatenation*~~ multiplication (concatenation) *of strands 2 and 3 (a circuit algebra operation), as shown in Figure 13.*

Proof. To show Property (1), recall that is if $\Phi(x, y)$ is a Drinfel'd associator, ~~then~~, ~~by the anti-symmetry property of associators, the image of Φ in the quotient where $xy = yx$ is 1. In particular,~~ $\Phi(0, y) = \Phi(x, 0) = 0$.
Similarly ~~Therefore~~ $p_1p_2\alpha(\Phi) = 1$, because $p_1p_2\alpha(c_{12}) = p_1p_2(a_{12}+a_{21}) = 0$, ~~therefore, $p_1p_2\alpha(\Phi) = 1$~~. The same reasoning shows that $p_2p_3\alpha(\Phi) = 1$.

Finally, $p_1p_3\alpha(\Phi(c_{12}, c_{23})) = \Phi(a_{21}, a_{23})$, and since $[a_{21}, a_{23}] = 0$ by the TC relation, $\Phi(a_{21}, a_{23}) = 1$.

For Property 2, note that $p_1(\alpha(\Phi)) = \Phi(a_{21}, -a_{21} - a_{31})$. Thus, strands 2 and 3 ~~support~~ carry only tails, and these commute by the TC relation, and $S_2p_1(\alpha(\Phi)) = \Phi(-a_{21}, a_{21} - a_{31})$. Furthermore, tails on strand 3 can be pulled to strand 2 through the concatenation, which identifies a_{21} with a_{31}. Therefore, $\Phi(-a_{21}, a_{21} - a_{31}) = \Phi(-a_{21}, 0) = 1$. □

To compute V and prove Part (2) of Theorem 1.1, consider once again the w-tangled foam K on the right of Figure 12.

On one hand, $Z^w(K)$ can be computed directly from the generators: $Z^w(K) = C_1C_2V_{12} \in \mathcal{A}^{sw}(\uparrow_2)$, since the values of the left-punctured vertices are trivial. Hence, if we know $Z^w(K)$, we know V.

On the other hand, we can compute $Z^w(K)$, using the compatibility with Z^u, as follows. Note that B^u is the closure – in the sense of (1) – of the parenthesised braid B^b shown in Figure 14, $B^w = a(B^u)$. Using the notation $\beta^u = Z^u(B^u)$, and $\beta^w = Z^w(B^w)$, and by the compatibility of Z^w with Z^u, we have

$$\beta^w = Z^w(B^w) = \alpha(Z^u(B^u)) = \alpha(\beta^u).$$

How does $Z^w(K)$ differ from β^w? To obtain K, a vertex and a cap were attached to B^w, two strands were punctured and the cap unzipped, as in Figure 12. The Z^w-value of the added vertex cancels when its left strand is punctured, however, the value of the cap remains and is unzipped. Thus, in loose notation, $Z^w(K) = u(C) \cdot p^2(\beta^w)$, where p^2 denotes the two punctures – we will compute this value explicitly in terms of associators shortly.

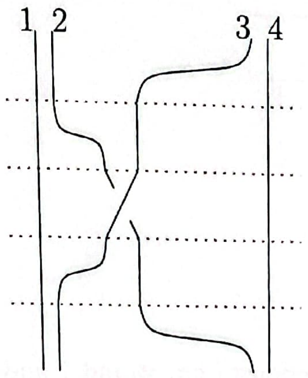
To equate the two approaches, we need to express $u(C) \cdot p^2(\beta^w)$ as an element of $\mathcal{A}^{sw}(\uparrow_2)$, by applying the *sorting isomorphism* φ of Lemma 2.4. By doing so, we obtain

$$C_1C_2V_{12} = \varphi(u(C)p^2(\beta^w)). \tag{4}$$

Through a careful analysis of the right hand side, this will imply formula (7) stated in Theorem 1.1. In summary, we want to compute

$$\Upsilon := \varphi(u(C)p^2(\beta^w)).$$

Center

$B^b =$ 

$$\overset{Z^b}{\longmapsto} \quad \begin{array}{ll} \Phi_{(12)34} & \overset{p^2\alpha}{\longmapsto} \Phi(a_{23}, a_{43}) \\ \Phi^{-1}_{123} & \longmapsto 1 \\ R_{32} & \longmapsto e^{a_{23}/2} \\ \Phi_{132} & \longmapsto 1 \\ \Phi^{-1}_{(13)24} & \longmapsto \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \end{array}$$

Figure 14. Computing β^b. Strands are numbered at the top and multiplication is read from bottom to top; the rightmost column lists the images of the factors under $p_1p_3\alpha$.

fig:Buck

To achieve this, we use that $\beta^w = \alpha(\beta^u)$, and compute β^u in terms of the Drinfel'd associator Φ associated to Z^u. By the compatibility of Z^u and Z^b, it is enough to compute $\beta^b := Z^b(B^b)$. The result can be read from the picture in Figure 14:

$$\beta^b = \Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34}.$$

Recall that the cosimplicial notation used in the subscripts show which strands the diagrams are placed on, for example, $\Phi^{-1}_{(13)24} = \Phi^{-1}(c_{12} + c_{32}, c_{24})$. Also recall that $R = e^{c/2}$, so $R_{32} = e^{c_{23}/2}$.

As β^u is the tree closure of β^b, it is given by the same formula interpreted as an element of $\mathcal{A}^u(\uparrow_4)$. One then applies α to obtain $\beta^w = \alpha(\beta^u)$. After the vertex and cap attachment, of Figure 12, strands 1 and 3 are punctured and strands 2 and 4 are capped, and in this strand numbering, $u(C) = C_{24}$ Therefore, we have

$$\Upsilon = \varphi\left(C_{24} \cdot p_1p_3\alpha(\Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34})\right).$$

Next, we analyse how the puctures and α act on factors of β^b. First observe that $p_3\alpha(R_{32}) = e^{a_{23}/2}$, where a_{ij} is a single arrow pointing from strand i to strand j.

Observe that $p_1p_3\alpha(\Phi^{-1}_{123}) = p_1p_3\alpha(\Phi^{-1}_{123}) = 1$ by Fact (1) of Lemma 3.2.

Since strands 1 and 3 are both punctured, no arrows can be supported between these two strands, hence $p_1p_3\alpha(\Phi_{(12)34}) = \Phi(a_{23}, a_{43})$.

This can go in Lemma 3.2

A basic property of associators is that $\Phi(x, y) = \Phi(x, -x - z)$ whenever $(x + y + z)$ is central. Using this property we deduce $\Phi^{-1}_{(13)24} = \Phi^{-1}(c_{(13)2}, c_{24}) = \Phi^{-1}(c_{(13)2}, -c_{(13)2} - c_{(13)4})$, so $p_1p_3\alpha\Phi^{-1}_{(13)24} = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. To summarise,

$$\Upsilon = \varphi\left(C_{24} \cdot \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})\right).$$

Note that the expression $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})$ has only arrow tails on strands 2 and 4, and therefore commutes with C_{24} by the TC relation. Hence, by the definition of φ,

$$\begin{aligned} \Upsilon &= \varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}) \cdot C_{24}\right) \\ &= \varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})\right) \cdot \varphi(C_{24}). \end{aligned}$$

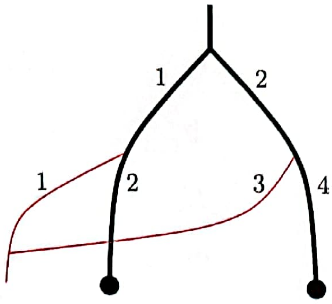


Figure 15. Strand numbering convention for K and V: arrow endings from strand 1 and 2 of K are "pushed" to strand 1 of V when applying φ, and arrow endings from strands 3 and 4 are pushed to strand 2.

Furthermore, by the strand numbering convention shown in Figure 15, we have $\varphi(C_{24}) = C_{12}$.
→ Terefore

$$V_{12} = C_1^{-1}C_2^{-1}\Upsilon = C_1^{-1}C_2^{-1}\varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})\right) C_{12},$$

A as stated in part (2) of Theorem 1.1. □

Matching this result to the Alekseev–Enriquez–Torossian formula of [AET, Theorem 4.] is technical, and not used anywhere else in the paper, hence we defer this to Appendix A.

3.3. Proof of part (3): the double tree construction.

Given a homomorphic expansion Z^u of *sKTG*, in Section 3.1.2 we showed that if there is to exist a homomorphic expansion Z^w of $\widetilde{wTF}$ compatible with Z^u, then $V = Z^w(\curlywedge)$ and $C = Z^w(\downarrow)$, and hence Z^w itself, are uniquely determined by Z^u. From here on we denote these unique values – which arose from the "buckle" construction – by V_β and C_β, and the candidate homomorphic expansion determined by them Z^w_β.

It remains to prove that Z^w_β is indeed a homomorphic expansion of $\widetilde{wTF}$: in other words, we need to show that V_β and C_β satisfy the equations (R4), (U), and (C) of Fact 2.5. Unfortunately, doing this directly seems difficult.

Note that (R4), which is in some sense the "main equation", is an equality between different circuit algebra – in fact, planar algebra – compositions of crossings. Hence, the proof would be much easier if Z^u were to be a circuit algebra – or planar algebra – map. This unfortunately makes no sense, as *sKTG* is not a circuit or planar algebra but a different, more complicated algebraic structure. The reader might ask, why work with a space as inconvenient as *sKTG* instead of, say, a planar algebra of trivalent tangles? The answer is that the existence of a homomorphic expansion is a highly non-trivial property, and in particular ordinary trivalent tangles do not have one. Even without trivalent vertices, ordinary tangles, or u-tangles, do not have have a homomorphic expansion as a planar algebra[10]. *Parenthesized tangles* (a.k.a. *q-tangles*) [LM, BN2] do have homomorphic expansions, yet in fact these are almost equivalent to *sKTG* [T, BND1, Da].

[10] We only mention that the planar algebra of u-tangles does not have a homomorphic expansion Z^t so as to explain why we are not using one. This said, the non-existance of Z^t is easy to prove: by an explicit calculation in degree 2 one shows that there is no linear combination of chord diagrams that can serve as $Z^t(\times)$, which satisfies the R3 relation.

A in [WKO4], V is computed to degree 4 using the techniques of this section.

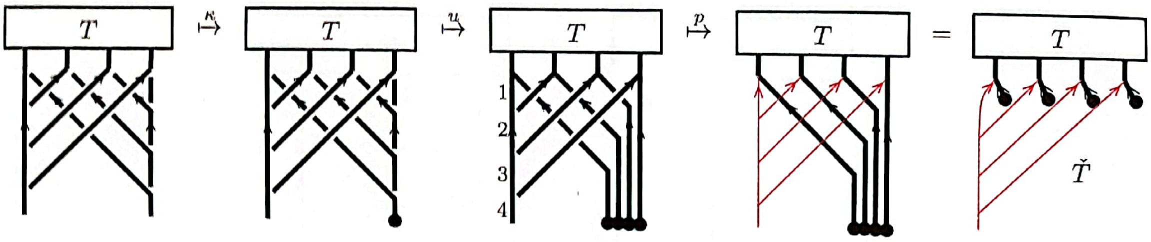


Figure 17. The cap attachment, unzips and punctures. While these operations are applied in $\mathcal{A}^{sw}$ – there are arrows on these skeleta – for simplicity the figure only shows the effect on the skeleton. fig:pCu

Here T stands for an arbitrary tangle in $u\!T\!T$. The double tree map sends T into $sKTG$, and by applying Z^u one obtains a value in $\mathcal{A}^u$, namely a chord diagram on the skeleton of $\varphi(T)$. We denote the space of chord diagrams on this skeleton by $\mathcal{A}^u(\varphi(T))$. Now α maps this to arrow diagrams on the skeleton of $\varphi(T)$, that is, to $\mathcal{A}^{sw}(\varphi(T))$. In order to revert the skeleton back to that of T, we apply some operations in $\mathcal{A}^{sw}$: a cap attachment κ, unzips and punctures (as shown in Figure 17 and explained below), resulting in a slightly modified version of the desired skeleton, denoted $\check{T}$. Finally, we use that $\mathcal{A}^{sw}(\check{T}) \cong \mathcal{A}^{sw}(T)$ via the sorting isomorphism φ of Lemma 2.4, and hence we obtain a value in $\mathcal{A}^{sw}(T)$, as needed, which, we will later see, is *almost* $Z^w(a(T))$. (Although the punctured strands connect in a single binary tree, VI relations can be used as part of the sorting isomorphism.)

The cap attachment, unzip and puncture operations are done in the order shown in Figure 17. First attach a cap – a capped strand with no arrows on it – to the end of the right vertical strand in $\alpha(\varphi(T))$: this is a circuit algebra operation in $\mathcal{A}^{sw}$. If T has n ends, perform $(n-1)$ consecutive disc unzips on the capped strand, as shown in Figure 17. Then puncture the left hand tree, for example by puncturing the left vertical strands marked "1, 2,..." in Figure 17 (these punctures also affect the connecting diagonal strands, as in Figure 5). Note that since the punctured tree had originally crossed over the capped tree, these crossings become virtual after puncturing, hence the last equality in Figure 17.

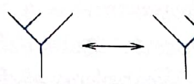
Denote the composition of the maps and operations shown in Equation (5) by ξ, that is,

$$\xi := \varphi \circ p \circ u \circ \kappa \circ \alpha \circ Z^u \circ \varphi. : uTT \longrightarrow A^{sw} \qquad (6)$$ eq:xidef

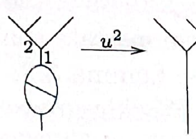
Then, $\xi(T) \in \mathcal{A}^{sw}(T)$. We first show that $\xi(T)$ is well-defined, that is, it doesn't depend on the choice of binary trees in $\varphi(T)$.

eeChange **Lemma 3.3.** *The choice of binary trees in the double tree construction does not affect $\xi(T)$.*

Proof. Any binary tree can be changed into any other binary tree via a sequence of "I to H" moves, as shown on the right. Hence, it is enough to analyze how an I to H move on one of the trees affects the value of $Z^u(\varphi(T))$, and prove that the difference vanishes after the caping, unzip, and puncture operations.



Suppose τ_1 and τ_2 are two binary trees which differ by a single I to H move, and let φ_{τ_1} and φ_{τ_2} denote the two resulting double-tree maps, assuming the "other side tree" is unchanged. The I to H move can be realised by inserting[11] an associator, followed by unzipping the edge marked '1' on



[11]See [WKO2, Section 4.6] for a detailed description of the tangle insertion operation in $sKTG$.

$$
\begin{tikzcd}
T \arrow[r, mapsto] & \wp(T) \arrow[r, mapsto] & \wp(T) \arrow[r, mapsto] & \kappa\wp(T) \arrow[r, mapsto] & \hat{T} \\
\Cap & \Cap & \Cap & \Cap & \Cap \\
u\mathcal{T}T \arrow[r, "\wp"] \arrow[dr, phantom, "(4)"] \arrow[ddrrrr, bend right, "\xi"'] & sKTG \arrow[r, "a"] \arrow[d, "Z^u"] \arrow[dr, phantom, "(1)"] & \widetilde{w\mathcal{T}F} \arrow[r, "\kappa"] \arrow[d, "Z^w"] \arrow[dr, phantom, "(2)"] & \widetilde{w\mathcal{T}F} \arrow[r, "pou"] \arrow[d, "Z^w"] \arrow[dr, phantom, "(3)"] & \widetilde{w\mathcal{T}F} \arrow[d, "Z^w"] \\
 & \mathcal{A}^u(\wp(T)) \arrow[r, "\alpha"] & \mathcal{A}^{sw}(\wp(T)) \arrow[r, "\kappa\circ\hat{C}"] & \mathcal{A}^{sw}(\kappa\wp(T)) \arrow[r, "pou"] \arrow[d, phantom, "(4)"] & \mathcal{A}^{sw}(\hat{T}) \arrow[d, "\varphi"] \\
 & & & & \mathcal{A}^{sw}(T)
\end{tikzcd}
$$

improved (Dror)

Figure 21. Comparing ξ and Z^w, assuming that Z^w exists. fig:BigC

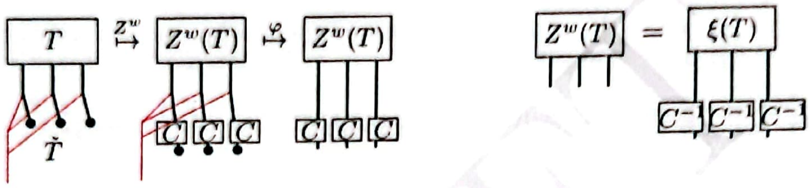


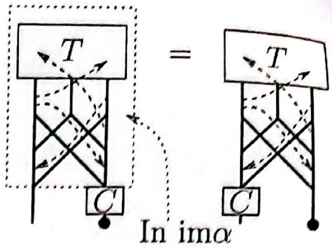
Figure 22. Computing $Z^w(\check{T})$ and $Z^w(T)$. fig:ZTCh

Proof. Assume there exists a homomorphic expansion Z^w compatible with Z^u. We use, as in Figure 21, the homomorphicity of Z^w and its compatibility with Z^u to show that $\xi(T) = Z^w(\check{T})$, where $\check{T}$ is as in Equation 5 and shown in Figure 22 on the left.

If the diagram in Figure 21 commutes, then for any $T \in u\mathcal{T}T$ and any Z^u-compatible Z^w, we have $\varphi(Z^w(\check{T})) = \xi(T)$. Since Z^w is a circuit algebra homomorphism, $Z^w(\check{T})$ can be obtained from $Z^w(T)$ by attaching the Z^w-value of a left-punctured right-capped vertex at each tangle end, as illustrated in Figure 22. By Lemma 2.8 we have $Z^w\Big($$\Big) = 1$, so the only additions are C values at each capped end, as shown in Figure 22. This can then be interpreted as a value in $\mathcal{A}^{sw}(T)$ via the isomorphism φ of Lemma 2.4. This implies the statement of the Lemma.

It remains to show that the diagram in Figure 21 commutes. The square (1) is the assumed the compatibility of Z^u and Z^w. In square (2), recall the map κ denotes the circuit algebra operation of attaching a cap at the bottom right end of the w-foam. The map $\hat{C}$ denotes the circuit algebra operation which attaches a value $C = Z(\downarrow)$ at the end of the strand. Thus, the commutativity of square (2) is implied by the homomorphicity of Z^w with respect to circuit algebra composition (as a binary operation). The square (3) is commutative due to the homomorphicity of Z^w with respect to punctures and disc unzips.

The commutativity of the heptagon (4) would be true by definition, if not for the map $\hat{C}$ (multiplication by the cap value). We show that, in fact, the value C cancels after punctures, by a property of arrow diagrams in the image of α, called *tail-invariance*, shown in Figure 18 (see [WKO2], Remark 3.14 and early in Section 3.3). In the current situation tail invariance means that the value C, which has only arrow tails, can be moved from one tangle end to the other, as shown on the right. Consequently, C cancels when the left strand is punctured.



□

Remark 3.6. In Lemma 3.5 we assume by convention that all tangle ends of T are oriented upwards (towards T). If k tangle ends are oriented down, the corresponding cap values appear with their orientations switched: $Z^w(aT) = \xi(T)\cdot(C^{-1})^{n-k}(S(C)^{-1})^k$.

Corollary 3.7. *If there exists a homomorphic expansion Z^w for $\widetilde{w\mathit{TF}}$ compatible with Z^u, then $\pi(V) = \pi(\xi(\text{⅄}))$, where V is the Z^w-value of the vertex, and π is the tree projection. This uniquely determines Z^w.*

Proof. The first statement is an immediate consequence of Lemma 3.5. The second was shown in Section 3.1.2. □

I don't understand how Z^w_ξ is defined. In particular, what's $Z^w_\xi(\downarrow)$?

Thus, the map ξ uniquely determines Z^w, assuming that Z^w exists: from here on we denote this candidate homomorphic expansion by Z^w_ξ. We have shown how to explicitly compute $\pi(V)$ from Z^b through ξ. What remains to be proven is that:

(1) Z^w_ξ is compatible with Z^u: see Proposition 3.10 below.
(2) The restriction of Z^w_ξ to $a(u\mathit{T})$ is a planar algebra map (see Theorem 3.13 below), and thus Z^w_ξ satisfies the (R4) equation.
(3) Z^w_ξ satisfies the (U) and (C) equations, hence it is a homomorphic expansion of $\widetilde{w\mathit{TF}}$ compatible with Z^u: see Theorem 3.18 below.
(4) The expansion Z^w_ξ is the same as Z^w_β obtained from the "buckle" construction of Section 3.1.1: $V = Z^w_\xi(\text{⅄}) = V_\beta$ and $C = Z^w_\xi(\downarrow) = C_\beta$. This is done in Lemma 3.15 below.
(5) Thus, there exists a unique homomorphic expansion $Z^w = Z^w_\xi = Z^w_\beta$, compatible with Z^u, see Proposition 3.18 below.

While the construction of Z^w_ξ enables us to prove the (R4) and (U) equations, Z^w_β is computationally simpler, and thus we use step (4) and C_β to prove that the (C) equation is satisfied.

3.3.2. *Z^w_ξ is a homomorphic expansion.* The goal for this subsection is to complete the sketch above. We begin by showing that Z^w_ξ is compatible with Z^u. This requires a technical lemma, in which we compute the ξ-value of a vertical strand:

Lemma 3.8. *For a single un-knotted strand, $\xi(\frown) = \alpha(\nu^{1/2})$, where $\nu \in \mathcal{A}^u(\frown)$ denotes the Kontsevich integral of the un-knot*[13].

Proof. We apply φ to $\frown$, as shown in Figure 23, and comupte $Z^u(\varphi(\frown))$ using the finite generation property of *sKTG* and the homomorphicity of Z^u. In [WKO2, Section 5.2] we

[13]The value of ν was conjectured in [BGRT] and proven in [BLT]. Note that ν involves wheels only.

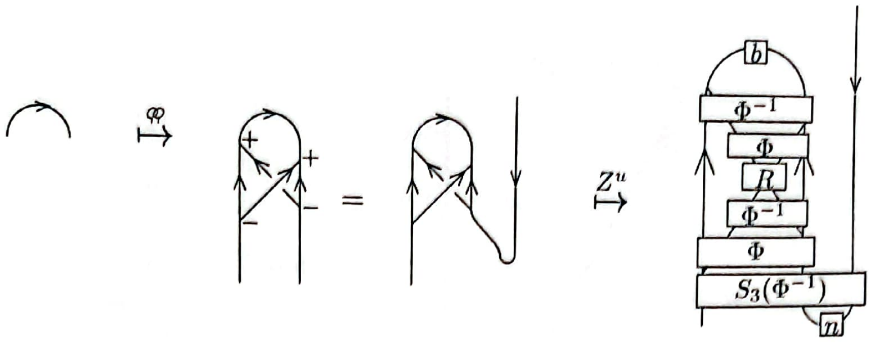


Figure 23. The double tree map composed with Z^u, applied to a single strand. To compute $Z^u(\varphi(\frown))$, we write $\varphi(\frown)$ as a composition of generators; this requires first expressing it as a bottom-top tangle. See [WKO2, Proposition 4.13] for details.

gave an algorithm for writing any *sKTG* as an *sKTG*-composition of generators (the primary operation in *sKTG* is *tangle insertion*, see [WKO2, Figure 22]). Feeding $\varphi(\frown)$ into this algorithm, one needs to "curve up" one strand as in Figure 23, in this case the strand on the right (the choice of strand doesn't affect the outcome).

The chord diagram $Z^u(\varphi(\frown))$ is shown in Figure 23, expressed in terms of the generators of *sKTG* described in [WKO2, Proposition 4.13]: the value Φ of the *associator graph*, which is a Drinfel'd associator; the value $R = e^{c/2}$ of the *twist graph*, where c is a single chord; and the values n and b of the *noose* and *balloon* graphs, respectively.

In $\xi(\frown)$, Z^u is followed by α, a cap attachment, unzips and punctures. As explained in [WKO2, Section 4.6], there is possibly a one-parameter freedom in the values of n and b, but we know that and $\alpha(b) = e^{a/2}\alpha(\nu)^{1/2}$, and $\alpha(n) = e^{-a/2}\alpha(\nu)^{1/2}$. The exponential part of n cancels by the CP relation once the cap is attached. The wheels part $\alpha(\nu)^{1/2}$ can be moved to the bottom left end by the tail invariance property (as in the last paragraph of the proof of Lemma 3.5), where it cancels after punctures. For the associator $S_3(\Phi^{-1})$ at the bottom in Figure 23, there is an orientation switch S_3 applied as the third strand is oriented downwards. In fact, $S_3(\Phi^{-1})$ cancels by Fact (2) of Lemma 3.2.

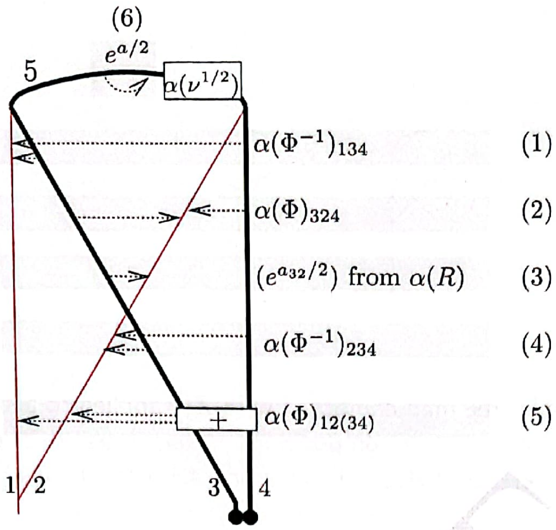
Taking these cancellations into account, the value $pu\kappa\alpha Z^u\varphi(\frown) \in \mathcal{A}^{sw}$ is shown in Figure 24 and explained below. Recall that α maps a chord to the sum of its two possible orientations. However, when one supporting strand is punctured, only one of these orientations survive. Hence, for example, $p_2(\alpha(R_{23})) = (e^{a_{32}/2})$. Figure 24 shows a schematic picture of $pu\kappa\alpha Z^u\varphi(\frown)$ with exponentials and associators indicated by single arrows. Recall that $\Phi \in \mathcal{A}^{hor}(\uparrow_3)$ can be written as a power series in any two of the three generators of $\mathcal{A}^{hor}(\uparrow_3)$: c_{12}, c_{23} and c_{13}. For example, $\Phi(c_{12}, c_{23}) = \Phi(c_{12}, -c_{12} - c_{23})$. For each associator we chose the presentation in which $pu\kappa\alpha(\Phi)$ is of the simplest form.

(1) The top associator of Figure 23, after applying a VI relation, is written as $\Phi^{-1}_{13(24)}$ in the strand numbering of Figure 24. We write this in terms of c_{13} and $c_{1(24)} = c_{12}+c_{14}$, since after the puctures $p_1\alpha(c_{13}) = a_{31}$ and $p_1p_2\alpha(c_{1(24)}) = a_{41}$, thus

$$p_1p_2\alpha\Phi^{-1}(c_{13}, -c_{13} - c_{1(24)}) = \Phi^{-1}(a_{31}, -a_{31} - a_{41}).$$

This is reflected in Figure 24 in drawing only the a_{31} and a_{41} arrows. In turn, the associator cancels, essentially by Fact (2) of Lemma 3.2: strand 5 acts as a

I Think it's for a different reason.



Yay!!

Figure 24. The value of $pu\kappa\alpha Z^u\wp(\frown)$: the numbering (1) through (6) refer to the paragraphs where each component is explained. fig:dtst

concatenation of strands 3 and 4, as the tail of a_{41} can be "pulled over the top along strand 5" using the VI relations and the fact that $e^{a/2}\alpha(\nu)$ is a local arrow diagram on a single strand, hence it is central. Thus, $a_{41} = a_{31}$, and $\Phi^{-1}(a_{31}, -a_{31} - a_{41}) = \Phi^{-1}(a_{31}, -2a_{31}) = 1$ as the arguments commute. Therefore ?

I'm sceptic
$$p_1p_2\alpha(\Phi^{-1}_{13(24)}) = 1.$$

(2) Second from top we have
$$p_1p_2\alpha(\Phi_{324}) = p_1p_2\alpha(\Phi(c_{23}, c_{24})) = \Phi(a_{32}, a_{42}) = 1,$$
by the same modified version of ~~Fact (2)~~ Lemma 3.2(2) lem:Concat (concatenation). ✓

(3) For the exponential,
$$p_1p_2\alpha(R_{23}) = p_1p_2\alpha(e^{c_{23}/2}) = e^{a_{32}/2}.$$

(4) Next,
$$p_1p_2\alpha(\Phi^{-1}_{234}) = p_1p_2\alpha(\Phi^{-1}(c_{23}, -c_{23} - c_{24})) = \Phi(a_{32}, -a_{32} - a_{42}) = 1$$
by the modified ~~Fact 2~~ Lemma 3.2(2) lem:Concat, noting that the arrow tails also commute with the arrow tails of the exponential.

(5) By ~~Fact (1)~~ Lemma 3.2(1) lem:TwoPunctures
$$p_1p_2\alpha(\Phi)_{12(34)} = 1.$$

(6) Finally, move the top exponential $e^{a/2}$ to strands 3 and 2, using the VI relation at both vertices. The tail of each arrow moves freely from strand 5 to strand 3. The heads commute with $\alpha(\nu)$, they are killed on strand 4 due to the CP relation, so they slide onto strand 2 but acquire a negative sign due to opposite orientations. Hence, $(e^{a_{55}/2}) = (e^{-a_{32}/2})$, and this cancels $\alpha(R)$. In summary, $\xi(\frown) = \alpha(\nu^{1/2})$, as claimed. □

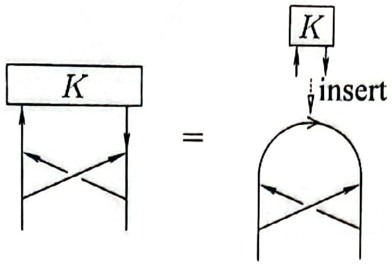


Figure 25. For $K \in sKTG$, $\varphi(K)$ is K inserted into $\varphi(\uparrow)$.

fig:Inse

As an aside, Lemma 3.8 enables a quick computation of the even part of $C = e^c = Z^w_\xi(\downarrow)$. Recall that c is a linear combination of wheels: $c = \sum_{n=2}^{\infty} \gamma_n w_n$. Let $c = c_0 + c_1$, where c_0 denotes the even part of c (sum of all even wheels), and c_1 denotes the odd part, ~~that is, $c = c_0 + c_1$~~. Let $C_0 = e^{c_0}$, the even part of C. Corollary 3.9 shows in particular that the even part of the C is independent of the choice of Z^b (that is, ~~the~~ below choice of Drinfel'd associator) and Z^u.

CapValue **Corollary 3.9.** *If $C = Z^w_\xi(\downarrow)$, and C_0 is the even part of C, then $C_0 = \alpha(\nu^{1/4})$, regardless of the choice of expansion Z^u used to construct Z^w_ξ.*

Does this belong with the construction of Z^w_ξ?

I'm confused between C & C^{-1}.

Proof. By Lemma 3.5 and Remark 3.6, we have $Z^w_\xi(\times) = C^{-1}\xi(\times)S(C^{-1})$: see the figure on the right for the orientations. Note that $S(w_{2k}) = w_{2k}$ and $S(w_{2k+1}) = -w_{2k+1}$, and hence $S(C) = e^{c_0 - c_1}$. Also, by homomorphicity, $Z^w_\xi(\uparrow) = 1$. Thus, by Lemma 3.8, $1 = e^{c_0+c_1}\alpha(\nu^{1/2})e^{c_0-c_1}$, and therefore $\alpha(\nu^{1/2}) = e^{2c_0}$, which gives $C_0 = e^{c_0} = \alpha(\nu^{1/4})$. □

Next we prove that Z^w_ξ is indeed compatible with Z^u:

tibility **Proposition 3.10.** *For any $K \in sKTG$, $Z^w_\xi(a(K)) = \alpha(Z^u(K))$.*

Proof. Note that $sKTG \subseteq uTT$, and for $K \in sKTG$, $\varphi(K)$ can be obtained by inserting K into the top strand of $\varphi(\uparrow)$: see Figure 25. Since Z^u is compatible with insertions, $Z^u(\varphi(K))$ can be obtained by $Z^u(K)$ inserted into $Z^u(\varphi(\uparrow))$. Through the sequence of α, capping, puncturing, φ and multiplications by C^{-1}, all of $Z^u(\varphi(\uparrow))$ cancels, as in Lemma 3.8. Note that the cancellations still go through despite the fact that $\alpha(Z^u(K))$ is inserted on the top strand: this follows from the fact that $\alpha(Z^u(K))$ is in the α-image of $\mathcal{A}^u$, and the appropriate "commutativity" property holds in $\mathcal{A}^u$. Hence, $Z^w_\xi(K) = Z^w_\xi(K)$ as required. □

Throughout

Now, we show that Z^w_ξ it is a planar algebra homomorphism on $a(uTT)$. This is technically challenging, but it implies the R4 equation immediately. For the proof, it will be necessary to know the behaviour of Z^u with respect to edge deletions. When an edge e of a knotted trivalent graph $K \in sKTG$ is deleted, the two vertices at each end of e cease to be vertices. The associated graded operation on chord diagrams deletes skeleton edge e, and chord diagrams with any chord endings on e are set equal to 0.

ZuDelete **Fact 3.11.** *Z^u commutes with edge deletions up to a possible correction term of $e^{\pm c/4}\nu^{1/2}$ depending on the position of the edge, as in Figure 26.*

Proof sketch. In [WKO2, Section 4.6.1] Z^u is constructed from an invariant Z^{old} by adding vertex normalizations, as shown in Figure 26. Note that the top two diagrams in ~~figure~~ the

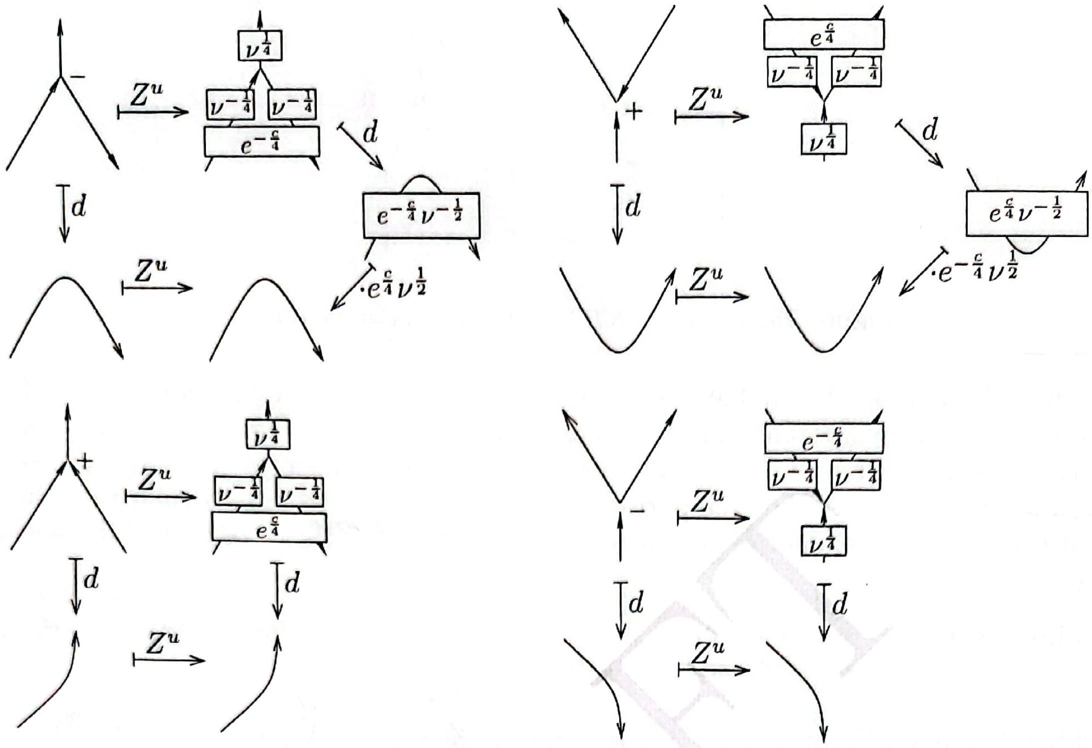


Figure 26. In [WKO2, Section 4.6.1] Z^u is constructed from an invariant Z^{old} by applying vertex normalizations, which depend on vertex signs: these are shown along the top horizontal arrow of each diagram (see also [WKO2, Figure 29]). It follows Z^u is only homomorphic up to a correction term when deleting the top edge of a positive vertex (first in the total ordering around the vertex) or the bottom edge of a negative vertex: see the top two diagrams. In other edge deletions the normalizations cancel, and hence Z^u is homomorphic with respect to these edge deletions, as for example in the bottom two diagrams.

differ from [WKO2, Figure 29] in a single edge orientation switch, which switches the vertex sign and accordingly the normalization[14]. In fact, Z^{old} commutes with edge deletions [Da, Proposition 6.7], so the edge deletion error (and hence, the correction term) for Z^u arises from the vertex normalisations implemented, as shown in Figure 26. □

Remark 3.12. There is also an "edge delete" operation of $\widetilde{w\mathcal{T}F}$: this is not required for the finite presentation of $\widetilde{w\mathcal{T}F}$ or $\mathcal{A}^{sw}$, but it is necessary for the proof of Theorem 3.13 below. When deleting an edge in $\widetilde{w\mathcal{T}F}$ – which can be either a tube or a string – the vertices at either end[15] cease being vertices. The associated graded operation $d_e : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ deletes the skeleton edge e and sends any arrow diagram with arrow endings on the deleted strand to zero. The crucial fact we need is that edge delete operations for chord and arrow diagrams

[14]The point of the normalization is to make Z^u commute with unzips. The reader might wonder, why normalize so that the expansion respects unzips, rather than deletions? The answer is that for finite generation of knotted trivalent graphs, unzips are crucial but deletions are not.

[15]It is also possible to delete a capped edge.

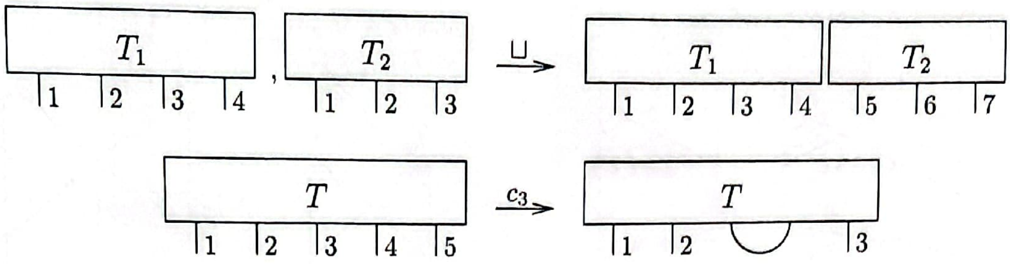


Figure 27. Basic planar algebra operations: disjoint union and contraction.

fig:Atom

are compatible via the map α, which is immediate from the definitions:

$$\begin{tikzcd}
\mathcal{A}^u \arrow[r, "\alpha"] \arrow[d, "d_e"'] & \mathcal{A}^{sw} \arrow[d, "d_e"] \\
\mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^{sw}
\end{tikzcd}$$

hm:PAMap **Theorem 3.13.** *The restriction of Z^w_ξ to $a(\mathit{uT\!T})$ is a planar algebra map.*

as in Fig 27

Proof. Planar algebra operations can be written as compositions of two simpler, basic operations: disjoint unions and contractions. In the disjoint union of two tangles T_1 and T_2, the ends of $T_1 \sqcup T_2$ are ordered by declaring that the ordered ends of T_1 come first, followed by the ordered ends of T_2. The contraction operation c_i applies to any tangle with at least $i+1$ ends: it acts by joining the i-th and $(i+1)$-st ends of T and re-numbering the rest, resulting in a tangle with two less ends. ~~Both operations are shown in Figure 27.~~

Thus, we only need to show that Z^w_ξ commutes with these two operations, that is, $Z^w_\xi(T_1 \sqcup T_2) = Z^w_\xi(T_1) \sqcup Z^w_\xi(T_2)$, and $Z^w_\xi(c_i(T)) = c_i(Z^w_\xi(T))$. Note that the right sides of these equalities make sense: arrow diagrams on the skeleta of $a(\mathit{uT\!T})$, where Z^w_ξ takes values, also form a planar algebra, and in particular disjoint union and concatenation of arrow diagrams is well defined.

Disjoint unions. We need to compute $\xi(T_1 \sqcup T_2)$, where $T_1, T_2 \in \mathit{uT\!T}$. The value $\wp(T_1 \sqcup T_2)$ is shown in Figure 28. The binary trees in $\wp$ can be chosen arbitrarily by Lemma 3.3: Figure 28 shows the most convenient trees for this proof.

Observe that $\wp(T_1 \sqcup T_2)$ can be obtained as an *sKTG* by inserting $\wp(T_1)$ and $\wp(T_2)$ into a simpler *sKTG*, denoted H, as shown in the same figure (up to orientation switches which don't impact what follows and will be ignored). Hence, $Z^u(\wp(T_1 \sqcup T_2))$ is given by inserting $Z^u(\wp(T_1))$ and $Z^u(\wp(T_2))$ into $Z^u(H)$.

One could compute $Z^u(H)$ explicitly using the same algorithm as before, but we can avoid this work, as follows. All chords in $Z^u(H)$ can be assumed to be located in the rectangle shown in Figure 28 (using VI relations, if necessary). During the computation of ξ both supporting strands are punctured, and therefore $p^2\alpha(Z^u(H)) = 1$. This implies that $\xi(T_1 \sqcup T_2) = \xi(T_1) \sqcup \xi(T_2)$, and it follows via Lemma 3.5 that $Z^w_\xi(T_1 \sqcup T_2) = Z^w_\xi(T_1) \sqcup Z^w_\xi(T_2)$.

Contractions. Proving that Z^w_ξ commutes with contractions is more involved. By Lemma 3.4, we can assume that the ends contracted are the last (rightmost) two ends of the n ends of T. Hence we will drop the subscript from c_i and denote this operation simply by c.

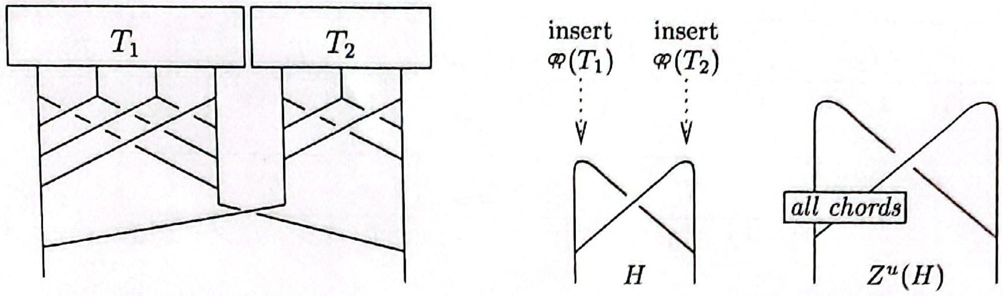


Figure 28. The double tree map applied to a disjoint union of $u\mathcal{T}T$-s is the same as inserting the double tree of each individual $u\mathcal{T}T$ into the $sKTG$ H. In $Z^u(H)$ all chords can be pushed into the rectangle shown, using VI relations when necessary.

$$\begin{tikzcd}
u\mathcal{T}T \arrow[r, "\varphi"] \arrow[d, "c"] \arrow[dr, phantom, "(1)"] & sKTG \arrow[r, "Z^u"] \arrow[d, "d^3"] \arrow[dr, phantom, "(2)"] & \mathcal{A}^u \arrow[r, "\alpha"] \arrow[d, "(\cdot e^{-\frac{c}{4}\nu^{\frac{1}{2}}})\circ d^3"] \arrow[dr, phantom, "(3)"] & \mathcal{A}^{sw} \arrow[r, "\kappa"] \arrow[d, "(\cdot e^{-\frac{a}{2}\alpha(\nu^{\frac{1}{2}})})\circ d^3"] \arrow[dr, phantom, "(4)"] & \mathcal{A}^{sw} \arrow[r, "u^{n-1}"] \arrow[d, "(\cdot e^{-\frac{a}{2}\alpha(\nu^{\frac{1}{2}})})\circ d^3"] \arrow[dr, phantom, "(5)"] & \mathcal{A}^{sw} \arrow[r, "p^{n-2}"] \arrow[d, "(\cdot e^{-\frac{a}{2}\alpha(\nu^{\frac{1}{2}})})\circ d^3"] \arrow[dr, phantom, "(6)"] & {} \\
u\mathcal{T}T \arrow[r, "\varphi"] & sKTG \arrow[r, "Z^u"] & \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^{sw} \arrow[r, "\kappa"] & \mathcal{A}^{sw} \arrow[r, "u^{n-3}"] & \mathcal{A}^{sw} \arrow[r, "p^{n-2}"] & {}
\end{tikzcd}$$

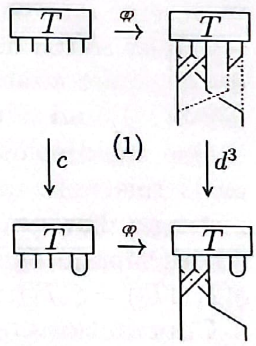
diagram continues...

$$\begin{tikzcd}
{} \arrow[r] & \mathcal{A}^{sw} \arrow[r, "p^2"] \arrow[d, "(\cdot e^{-\frac{a}{2}\alpha(\nu^{\frac{1}{2}})})\circ d^3"] \arrow[drr, phantom, "(7)"] & \mathcal{A}^{sw} \arrow[r, "\varphi"] & \mathcal{A}^{sw} \arrow[r, "C^{-(n-1)}SC^{-1}"] \arrow[d, "(\cdot\alpha(\nu^{-\frac{1}{2}}))\circ c"] \arrow[dr, phantom, "(8)"] & \mathcal{A}^{sw} \arrow[d, "c"] \\
{} \arrow[r] & \mathcal{A}^{sw} \arrow[rr, "\varphi"] & & \mathcal{A}^{sw} \arrow[r, "\cdot C^{-(n-2)}"] & \mathcal{A}^{sw}
\end{tikzcd}$$

Figure 29. Summary of the proof that Z^w_ξ commutes with contractions: Z^w_ξ is the composition along the entire top and entire bottom horizontal edge of the diagram.

We need to show that $Z^w_\xi(cT) = cZ^w_\xi(T)$, for any $T \in u\mathcal{T}T$. Since Z^w_ξ is given by the composition of many maps, ~~so~~ this can be restated as the commutativity of the perimeter of a large diagram (shown in Figure 29), which in turn can be broken down to its smaller parts. Throughout this proof, let $T \in u\mathcal{T}T$ denote an arbitrary trivalent tangle.

Square (1). This square plays out in $u\mathcal{T}T$ and $sKTG$, and commutes by inspection, as shown on the right. The three strands to be deleted are indicated by broken lines. Therefore, $d^3\varphi(T) = \varphi c(T)$.



Square (2). Square (2) is shown schematically below on the left: for the Z^u-values skeleta are indicated but chords are not shown. To prove that square (2) commutes, we use the properties of Z^u with respect to deleting edges in $sKTG$, as stated in Fact 3.11 and Figure 26.

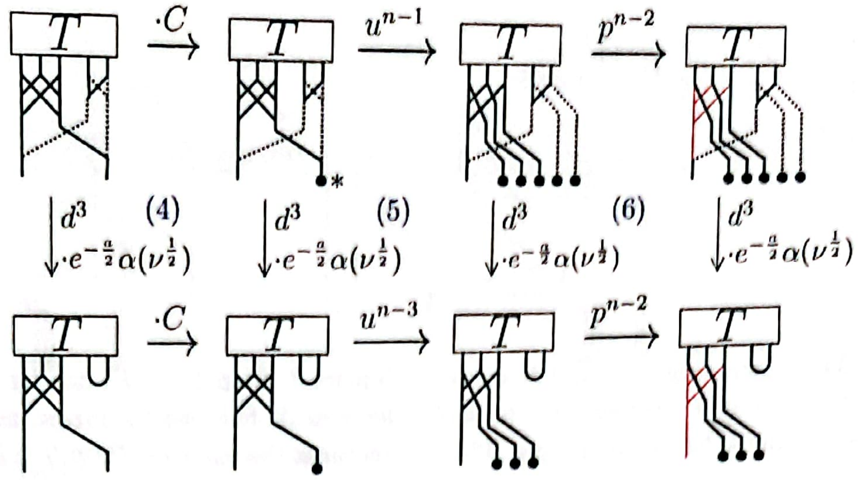
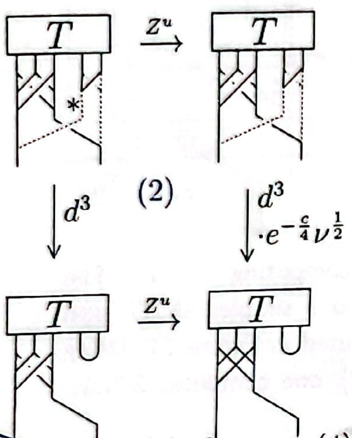


Figure 30. The squares (4) (5) and (6). Strands to be deleted are drawn in dashed lines throughout. The $*$ denotes a cap of interest: see the proof paragraph on square (5). fig:456



Only one of the three edge deletions requires a correction term: this is the edge marked with $*$ in the diagram on the left. This edge ends in a $e^{-c/4}\nu^{1/2}$ inserted at the place of the vertex, where c stands for a single chord. In square (2), this correction term appears at the bottom right corner of the square, where the two ends of T are contracted (see in the diagram showing skeleta in Figure 29). In summary: $(d^3 Z^u \varphi(T)) \cdot (e^{-\frac{c}{4}}\nu^{\frac{1}{2}}) = Z^u \varphi c(T)$.

Square (3). Square (3) is essentially the commutativity of edge deletions stated in Remark 3.12, combined with applying α to the correction term. So we have: $(d^3 \alpha Z^u \varphi(T)) \cdot (e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}})) = \alpha Z^u \varphi c(T)$.

space

Square (4). Squares (4), (5), and (6) are shown in detail in Figure 30. Square (4) plays out in A^{sw} and it is commutative as the deletions and the cap attachments (denoted by κ) affect different strands: ~~see the diagram on the right~~. Therefore,

$$(d^3 \kappa \alpha Z^u \varphi(T)) \cdot (e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}})) = \kappa \alpha Z^u \varphi c(T).$$

Square (5). The only difference between $d^3 \circ u^{n-1}$ and $u^{n-3} \circ d^3$ is what happens to arrows on the caped skeleton edge marked by $*$ in Figure 30. Following the diagram right and down, this edge is unzipped $n-1$ times, then the last two of its daughter edges are deleted. On the other hand, following the diagram down and right, the same edge is unzipped $n-3$ times. The results of these compositions are the same by definition of the unzip and delete operations. Thus, we have

$$(d^3 u^{n-1} \kappa \alpha Z^u \varphi(T)) \cdot (e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}})) = u^{n-3} \kappa \alpha Z^u \varphi c(T).$$

*Square (6).*The deletions and punctures occur on different strands, as shown in Figure 30, hence these operations commute commute. One detail to note is that when a tube strand is deleted at a "tube-and-string" vertex, all that is left is a string (as in the case of puncturing the tube at a tube-string vertex, see Figure 5). In summary:

$$(d^3 p^{n-2} u^{n-1} \kappa \alpha Z^u \varphi(T)) \cdot (e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}})) = p^{n-2} u^{n-3} \kappa \alpha Z^u \varphi c(T).$$

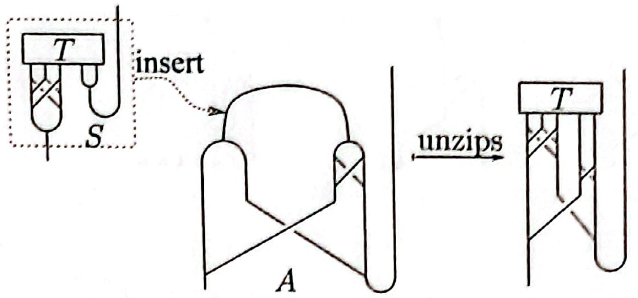


Figure 31. Computing the top left corner of Square 7, Step 1: $\varphi(T)$ can be expressed as the *sKTG* denoted S inserted into the *sKTG* denoted A, followed by unzips, as shown. Z^u respects insertions, hence computing $Z^u(A)$ determines the value of $Z^u(\varphi(T))$ outside of S.

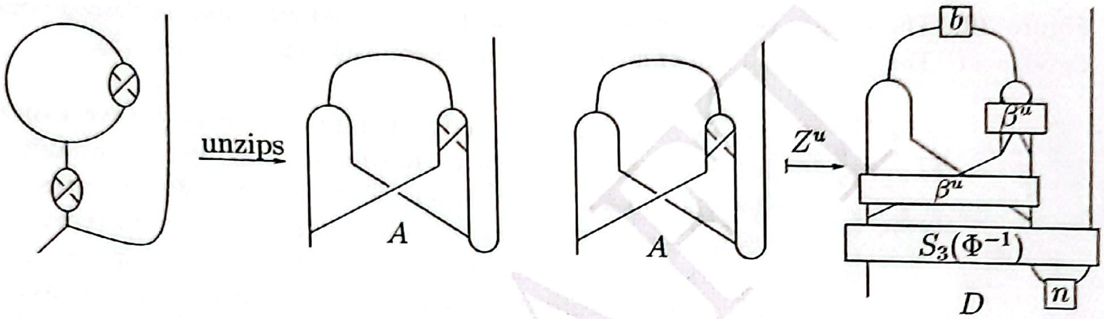
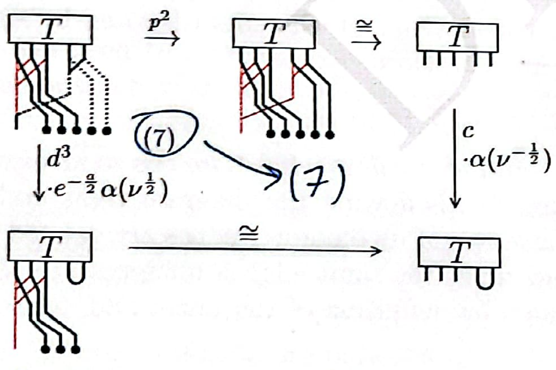


Figure 32. Computing the top left corner of Square 7, Step 2: computing $Z^u(A)$. The *sKTG* A can be obtained by inserting the *buckle sKTG* twice into a simpler *sKTG*, and unzipping, as shown on the left. The value of the buckle was computed in Figure 14. Using this value—denoted β^u—and the algorithm in [WKO2, Section 5.2], one computes $Z^u(A)$. The result is denoted D and shown on the right.



Pentagon (7). The pentagon (7) is shown on the left. This is the most delicate part of the proof. We first show that – for the specific input of $p^{n-2}u^{n-1}\kappa\alpha Z^u\varphi(T)$ – the pentagon (7) commutes up to a single possible multiplicative error on the contracted (u-shaped) strand, and later prove that this error ~~is~~ is necessarily ~~zero~~. vanish read ?

To begin, a better understanding of the arrow diagram $p^{n-2}u^{n-1}\kappa\alpha Z^u\varphi(T)$ in the top left corner is necessary. All of the operations performed on T, with the exception of Z^u, are "easy" in the sense that we have a complete understanding of their effect. Z^u is "hard", but we can compute the relevant part of its value using the finite generation of *sKTG* ([WKO2, Proposition 4.13]). The computation is shown in Figures 31 and 32 and their captions.

In summary, $Z^u(\varphi(T))$ is given by inserting $Z^u(A)$ into the chord diagram D of Figure 32. Now we need to analyze what happens when one applies α, the cap attachment, unzips and punctures to this value: this is an exercise similar to what has been done for Lemma 3.8 ~~for~~

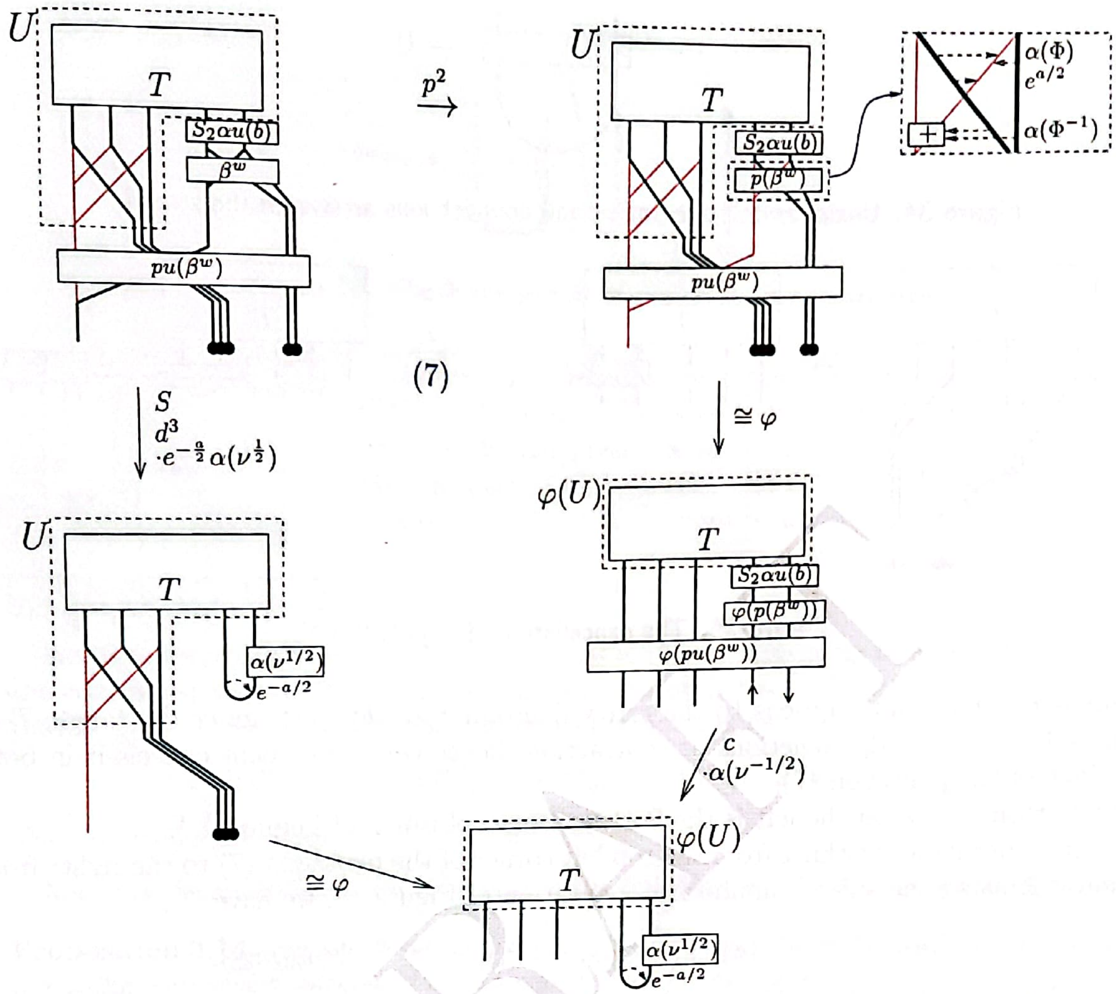


Figure 33. The more detailed picture of the Pentagon 7. fig:Squa

~~example~~. The result is shown in Figure 33, and explained below. First note that the n value in D of Figure 32 cancels after punctures by the tail-invariance property (Figure 18), as in the last paragraph of the proof of Lemma 3.5; so does the bottom Φ^{-1} in D , by Fact (2) of Lemma 3.2. These components are not shown in Figure 33.

Working downwards from the top left of the pentagon in Figure 33, the three edge deletions cancel both buckle (β^w) values. The value b value at the top of the diagram D is pulled down across the vertex using a VI relation: this has the same effect as an unzip and an orientation switch on the second stand (as this is oriented downwards). The resulting value $S\alpha u(b)$ cancels by the following reasoning – essentially by definition of unzips and orientations switches – which is illustrated in Figure 34:

Given an arrow 'a' ending on strand 'e', unzipping e produces a sum of two arrows a_1+a_2: one ending on each daughter strand. Reversing the orientation of the first daughter strand gives $-a_1+a_2$. Contracting the two daughter strands to form a U-shape identifies a_1 and a_2, making a_1-a_2 vanish.

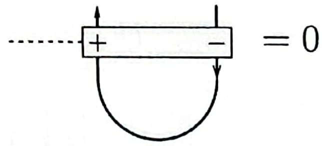


Figure 34. Unzip, switch orientation and connect kills arrows on the strand.

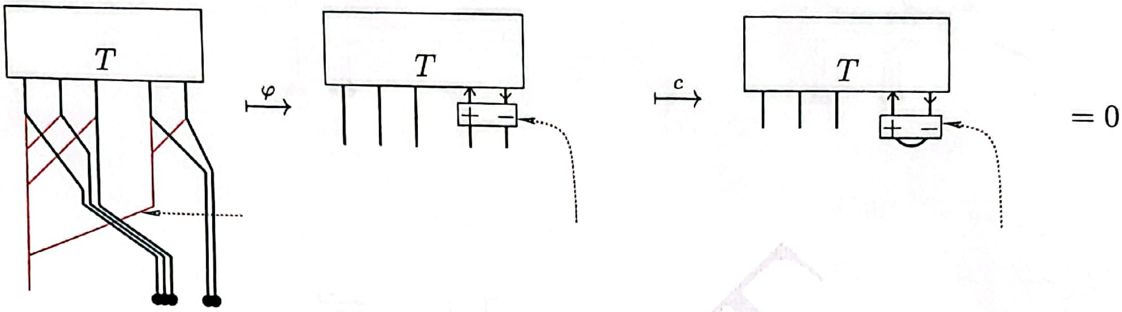


Figure 35. The cancellation of $\varphi(pu(\beta^w))$.

This is exactly what happens to the arrow diagram $S_2\alpha u(b)$ (just under the tangle T in Figure 33) after the edge deletions or contraction, hence this component cancels it in both directions of the pentagon (7).

The bottom arrow on the left is the Sorting Isomorphism φ of Lemma 2.4.

On the other hand, working from the top left corner of the pentagon (7) to the right: from Section 3.2, using the strand numbering convention of Figure 15, we have

$$p_1p_3\beta^w = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}).$$

This is shown in the enlarged rectangle at the top right corner of Figure 33. The same is true for $pu(\beta^w)$ at the bottom, except with more unzips.

From here, the first downward arrow applies the Sorting Isomorphism φ of Lemma 2.4, followed by the contraction and correction term along the second downward arrow. At the bottom of the diagram, $\varphi(pu(\beta^w))$ cancels altogether after contraction, in a similar fashion to Figure 34. Namely, φ and the contraction annihilates any arrow ending on the diagonal red strand, or the double capped strand on the right, as shown in Figure ?? for the diagonal red strand. This cancels each factor of $pu(\beta^w)$.

Of the top $\varphi(p(\beta^w))$ shown in the enlarged rectangle, the Φ^{-1} component cancels by the same contraction argument; only the exponential and the Φ component remains. The arrow in the exponent of $e^{a/2}$ switches sign due to the reversed orientation: this is the $e^{-a/2}$ component at the bottom of the pentagon in Figure 33. After contraction, the Φ component gives rise to a "local" arrow diagram on a single strand shown in Figure 36 and denoted μ.

In summary, we see that the pentagon (7) commutes if and only if $\mu = \alpha(\nu)$, and otherwise commutes up to a localised error on the contracted strand, of value $\alpha(\nu)^{-1}\mu$.

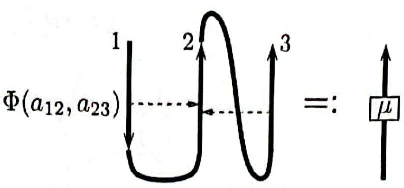
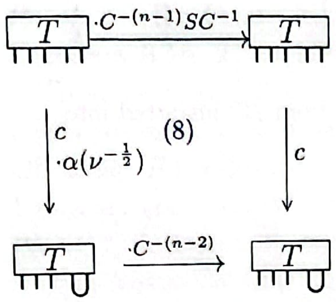


Figure 36. The Φ component of $\varphi(p(\beta^w))$ after contraction.



Square (8). Finally, for square (8) we need to check that $C^{-1}S(C^{-1}) = \alpha(\nu^{-1/2})$. Note that the orientation switch negates odd wheels and preserves even wheels, therefore in $C^{-1}S(C^{-1})$ the odd part of C^{-1} cancels, and $C^{-1}S(C^{-1}) = C_0^{-2} = \alpha(\nu^{-1/2})$ by Corollary 3.9. This verifies Square (8).

We have therefore shown that Z^w commutes with contraction up to an error $\alpha(\nu)^{-1}\mu$ on the contracted strand. It remains to show that this error is 1. This follows from the facts that $Z^w(\uparrow) = 1$, and that Z^w commutes with disjoint unions:

$$1 = c(Z^w(\uparrow\downarrow)) = \alpha(\nu)^{-1}\mu Z^w(c(\uparrow\downarrow)) = \alpha(\nu)^{-1}\mu \cdot Z^w(\frown) = \alpha(\nu)^{-1}\mu.$$

This completes the proof. $\square$

Note that as a side result we have proven the following curious fact about associators:

Proposition 3.14. *For any Φ horizontal chord associator, μ defined from Φ as in Figure 36, and ν the Kontsevich integral of the unknot, $\mu = \alpha(\nu)$.* in $\mathcal{A}^w$.. $\square$

The value $V = Z^w_\xi(\curlywedge)$ differs, on first glance, from the value V_β constructed in Section 3.1. Subscript or superscript? In the next lemma we show that in fact $V = V_\beta$: this serves both as a reality check, and as a technical tool for showing – in Theorem 3.18 – that Z^w_ξ satisfies the Cap equation.

Lemma 3.15. *The two vertex values from the buckle and the double tree constructions coincide: $V_\beta = V$.*

Proof. Notice that $\varphi(\curlywedge)$ – as in Figure 37 – can be obtained from simpler *sKTGs* by inserting the buckle B^u into $\varphi(\frown)$ followed by an unzip. We computed $\xi(\frown)$ in Lemma 3.8. Since Z^u is compatible with insertions and unzips, $Z^u(\curlywedge)$ can be computed by inserting the buckle value β^u (computed in Section 3.2) into $Z^u(\varphi(\curlywedge))$.

To compute $\xi(\curlywedge)$ we then apply α and the cap, unzip and puncture operations. Because $\beta^w = \alpha(\beta^u)$ is local (is confined to the skeleton of the inserted B^u graph) and is in the image of α, all other arrow endongs commute with it by the head- and tail-invariance properties of $\mathcal{A}^{sw}$ (Figure 18). Therefore, all of the cancellations in the computation of $\xi(\frown)$ still occur, and $\xi(\curlywedge)$ is as shown in Figure 37. Writing this in $\mathcal{A}^{sw}(\uparrow_2)$ we obtain $\varphi(p_1p_3(\beta^w))u(\alpha(\nu^{1/2}))$.

To obtain $V = Z^w_\xi(\curlywedge)$, one multiplies $\xi(\curlywedge)$ at each end by C^{-1} or $S(C^{-1})$ depending on orientation, as shown in Figure 37 on the right. Thus,

$$V = C_1^{-1}C_2^{-1}\varphi(p_1p_3\beta^w)u(\alpha\nu^{1/2})u(S(C^{-1}).$$

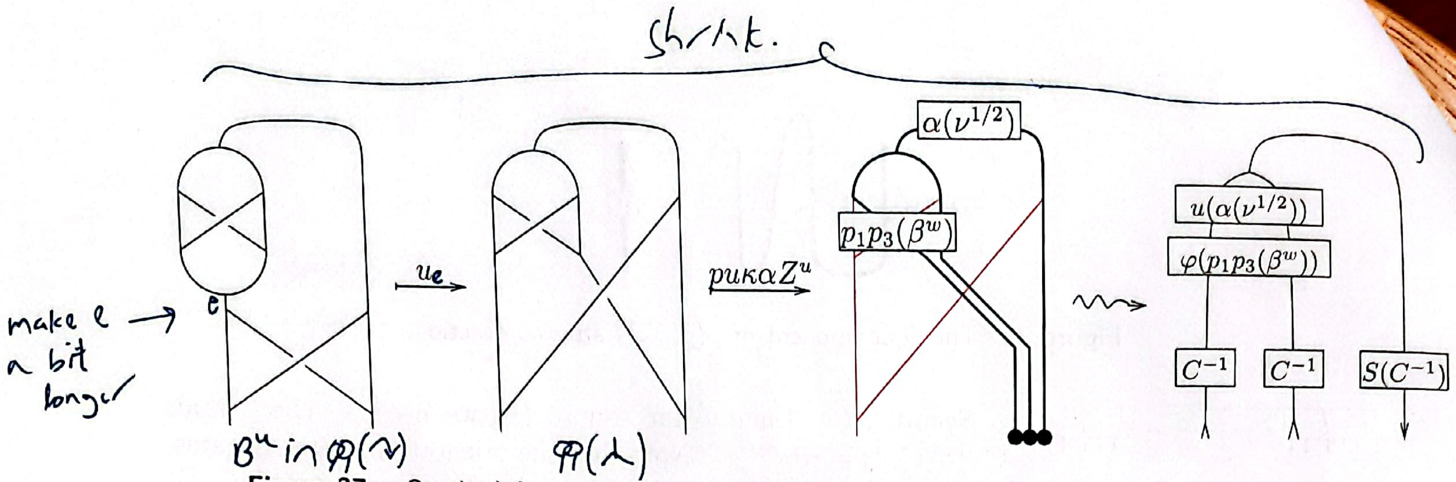


Figure 37. On the left we show how to obtain φ(⅄) via an unzip from B^u insterted into φ(⌒). From this we compute ξ(⅄), and finally V on the right. fig:DTandB

On the other hand, from Section 3.2, we have:

$$V^\beta = C_1^{-1}C_2^{-1}\varphi(p_1p_3\beta^w)u(C).$$

Thus, we need to show that:

$$C_1^{-1}C_2^{-1}\varphi(p_1p_3\beta^w)u(\alpha\nu^{1/2})u(S(C^{-1}) = C_1^{-1}C_2^{-1}\varphi(p_1p_3\beta^w)u(C),$$

in $\mathcal{A}^{sw}(\uparrow_2)$. Multiplying with $(\varphi(p_1p_3\beta^w)))^{-1}\,C_1C_2$ on the left (bottom) and by $u(S(C))$ on the right (top), this simplifies:

$$u(\alpha\nu^{1/2}) = u(C)u(S(C)).$$

Since unzips commute with orientation switches, it is sufficient to prove that

$$CS(C) = \alpha(\nu^{1/2}).$$

Recall that in $CS(C)$ all odd wheels cancel, hence $CS(C) = (C_0)^2$, where C_0 denotes the even part of C. Indeed, by Corollary 3.9, $C_0^2 = \alpha(\nu^{1/2})$. □

SameResult **Corollary 3.16.** *The "buckle" amd "double tree" constructions lead to the same result, that is,* $Z^w_\xi = Z^w_\beta$.

Proof. Since any homomorphic expansion Z^w of $\mathit{w\!T\!F}$ is uniquely determined by $Z^w(\curvearrowright)$, this is immediate from Lemma 3.15. □

The next lemma implies that V satisfies the Unitarity (U) equation:

lem:Unzip **Lemma 3.17.** *The map* Z^w_ξ *commutes with strand unzips in* $\mathit{w\!T\!T}$.

Proof. We first note that in $\mathit{w\!T\!T}$ unzip is only defined for internal edges, that is, edges which end in a vertex[16] at both ends. By construction, Z^w_ξ is a composition of several maps. We show that edge unzips commute with every one of these, hence with Z^w_ξ.

The φ map involves only the tangle ends, which are unchanged by unzipping an internal strand, hence these operations commute. The homomorphic expansion Z^u commutes with edge unzips, as shown in [WKO2, Section 4.6]. The map α commutes with edge unzips by definition. Cap attachments, cap unzips and the isomorphism φ commute with internal

[16] In fact, there are further restrictions, eg the two vertices must be of opposite signs, but this is not important for the proof.

unzips for the same reason as φ, because they are performed at the tangle ends while unzip is internal. This completes the proof. $\square$

Finally, the following Theorem completes the proof of part Part (3) of the Main Theorem 1.1:

Theorem 3.18. *The map $Z^w_\xi : \widetilde{w\mathcal{T}F} \to \mathcal{A}^w$ is the unique homomorphic expansion of $\widetilde{w\mathcal{T}F}$, which is compatible with Z^u in the sense of the commutative diagram (2).*

Proof. By Proposition 3.5, Z^w_ξ is compatible with Z^u. By Part (1) (Section 3.1) and Corollary 3.16, $Z^w_\xi = Z^w_\beta$ and uniquely determined by Z^u.

To show that Z^w_ξ is a homomorphic expansion, by Fact 2.5 and Theorem 2.6, one only needs to verify that it satisfies the (R4), Unitarity (U) and Cap (C) equations of Fact 2.5. Of these, $R4$ follows from the fact that Z^w_ξ is a planar algebra map, Theorem 3.13. The Unitarity equation (U) is equivalent to the statement that Z^w_ξ commutes with strand unzips [WKO2, Section 4.3], and hence it is satisfied by Lemma 3.17.

why do we need to show R4?

This leaves the Cap Equation, which we verify directly. By Lemma 3.15, $V = V_\beta$, therefore it is sufficient to work with V_β, which has a simpler expression computed in Section 3.1.1. Substituting this into the Cap equation (C) reads:

$$u(C)(u(C))^{-1}(\varphi(p_1p_3\beta^w))^{-1}C_1C_2 = C_1C_2,$$

in $\mathcal{A}^{sw}(\downarrow_2)$ We cancel $u(C)(u(C))^{-1}$ on the left, and multiply on the right by $C_1^{-1}C_2^{-1}$ (as $\mathcal{A}(\downarrow_2)$ is a right $\mathcal{A}(\uparrow_2)$-module by stacking). Then we only need to show that $(\varphi(p_1p_3\beta^w))^{-1} = 1$ in $\mathcal{A}(\uparrow_2)$. To show this, multiply on the right by $\varphi(p_1p_3\beta^w)$, hence it's enough to see that $1 = \varphi(p_1p_3\beta^w)$. This, in turn, is clear by the CP relation since all heads are below all tails in the image of φ. $\square$

Appendix A. The Alekseev–Enriquez–Torossian formula

This appendix is mainly for readers familiar with the Alekseev–Enriquez–Torossian formula for Kashiwara–Vergne solutions in terms of Drinfel'd associators [AET].

For a quick re-cap of [AET] notions, let $\mathfrak{lie}_2$ denote the degree completed free Lie algebra on two generators x and y. Let $\mathfrak{tder}_2$ denote *tangential derivations* of this Lie algebra, that is, derivations d with the property that $d(x) = [x, a_1]$ and $d(y) = [y, a_2]$, where $a_1, a_2 \in \mathfrak{lie}_2$. Let $\mathrm{TAut}_2 := \exp(\mathfrak{tder}_2)$ denote the group of *tangential automorphisms* of $\mathfrak{lie}_2$.

There is a linear "interpretation map" (not a map of Lie algebras) $\theta : \mathfrak{lie}_2^2 \to \mathfrak{tder}_2$, sending a pair (a_1, a_2) to the derivation d given by $d(x) = [x, a_1], d(y) = [y, a_2]$. The kernel of this map consists only of pairs of the form $(\alpha x, \beta y)$ for α, β constants. A one-sided inverse $\eta : \mathfrak{tder}_2 \to \mathfrak{lie}_2^2$ which sends a tangential derivation to a pair whose first component has no linear x term and second component has no y term. We denote the exponential of θ by $\Theta : \mathcal{U}(\mathfrak{lie}_2^2)_{exp} \to \mathrm{TAut}_2$. For an element $(e^{\lambda_1}, e^{\lambda_2}) \in \mathcal{U}(\mathfrak{lie}_2^2)_{exp}$, we have $G = \Theta((e^{\lambda_1}, e^{\lambda_2})) \in \mathrm{TAut}_2$ given by $G(x) = e^{-\lambda_1}xe^{\lambda_1}$, and $G(y) = e^{-\lambda_2}ye^{\lambda_2}$. Just as θ is not a Lie algebra map, Θ is not a group homomorphism: composition in TAut_2 is not given by piecewise multiplication of the conjugators. However, θ and Θ present a convenient way to denote tangential derivations and tangential automorphisms as pairs in $\mathfrak{lie}_2^2$ and $(\exp(\mathfrak{lie}_2))^2$, respectively.

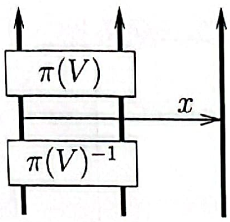


Figure 40. The action of $\pi(V)$ on the generator x of $\mathfrak{lie}_2$.

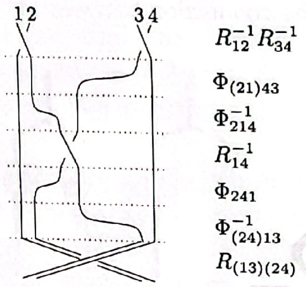


Figure 41. A different expression of β^b.

from $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. The arrows a_{23} and a_{43} act trivially on x, so, more simply stated, the action on x is by $\varphi(\Phi^{-1}(a_{21}, -a_{21} - a_{41}))$. Note that $L(\Phi^{-1}(x, -x-y), 0) = \varphi(\Phi^{-1}(a_{21}, -a_{21} - a_{41}))$, so Theorem 1.1 agrees with Formula (7) in the first component.

One can proceed similarly for the second component: the action on y is by

$$\varphi(\Phi^{-1}(a_{23}, -a_{23} - a_{43})e^{a_{23}}\Phi(a_{23}, a_{43})) = L(0, \Phi^{-1}(x, -x-y)e^{x/2}\Phi(x,y)).$$

While this does not match the second component of Formula (7), it only differs from it by a hexagon relation. Alternatively, note that one can obtain the second component of the Formula (7) "on the nose" by starting from an equivalent (isotopic) expression[18] of β^b, as shown in Figure 41. This completes the proof.

A

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[18]We thank Karene Chu for this idea.

A: ~~A~~ sequel paper [WKO4] verifies the results of this appendix by explicit computations in low degrees.