FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: THE DOUBLE TREE CONSTRUCTION

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Abstract. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological proof of the Kashiwara-Vergne problem, more specifically of the Alkeseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of "2-dimensional foams" and various associated features in four-dimensioanl space. We use a "double tree construction" to show that every "expansion" (also called "universal finite type invariant" or "UFTI") of parenthesized braids extends first to an expansion/UFTI of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion/UFTI of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is the same as a "Drinfel'd associator" Φ, and an expansion for the aforementioned w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem [KV] as reformulated by Alekseev and Torossian [AT]. Hence our result amounts to a topological framework for the result of [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof that the said formula works — namely that the equations satisfied by V follow from the equations satisfied by Φ.

Contents

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1. Introduction

This is the third in a sequence of papers studying finite type invariants of w-knotted objects, and contains the strongest result: a topological proof for the Kashiwara-Vergne conjecture (more precisely, the [AET] formula for solutions of the Kashiwara-Vergne equations in terms of Drinfel'd associators. Readers familiar with finite type invariants in general should have no trouble reading [WKO2] and this paper without having read [WKO1], however the setup and main results of [WKO2] are used heavily in this paper.

1.1. **Topology.** We begin by describing a chain of maps from "parenthesized braids" to "(signed) knotted trivalent graphs" to "w-tangled foams":

$$\mathcal{K} := \{uPB \xrightarrow{\mathrm{cl}} sKTG \xrightarrow{a} \widetilde{wTF}\}.$$

Let us first briefly elaborate on each of these spaces and maps.

Parenthesized braids are braids whose ends are ordered along two lines, the "bottom" and the "top", along with parenthetizations of the endpoints on the bottom and on the top. Two examples are shown in Figure 1. Parentehesized braids form a category whose objects are parenthetizations, morphisms are the parenthesized braids themselves, and composition is given by stacking. In addition to stacking, there are several operations defined on parenthesized braids: strand addition, removal and doubling. A detailed introduction to parenthesized braids is in [BN1].

Trivalent graphs are oriented graphs with three edges meeting at each vertex and whose vertices are equipped with a cyclic orientation of the edges. A knotted trivalent graph (KTG) is a framed embedding of a trivalent graph into $\mathbb{R}^3$. KTGs are studied from a finite type invariant point of view in [BND1]. In this paper we use a version of KTGs that was introduced and studied in [WKO2, Section 4.6], namely trivalent 1- and 2-tangles with some extra combinatorial information: signs assigned to the trivalent vertices. We call this space *sKTG* (for signed KTGs), as in [WKO2]. An example is shown on the right. The space *sKTG* is also equipped with several operations: tangle insertion, sticking a 1-tangle onto an edge of another tangle, disjoint union of 1-tangles, edge unzip, and edge orientation switch.

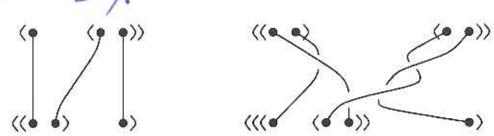
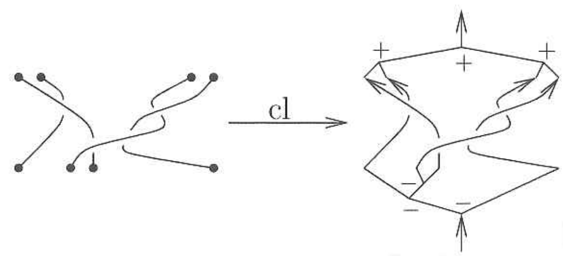


Figure 1. Two examples of parenthesized braids. Note that by convention the parenthetization can be read from the distance scales between the endpoints of the braid, and so we are going to omit the parentheses in the future.

The space $\widetilde{wTF}$ is a minor extension of wTF studied in [WKO2, Section 4.1 – 4.5], and will be introduced in detail in Section 2. It can be described as a planar algebra generated by certain features (various kinds of crossings and vertices, as well as "caps") modulo certain relations ("Reidemeister moves") and equipped with a number of auxiliary operations beyond planar algebra composition. This Reidemeister theory conjecturally represents knotted tubes in $\mathbb{R}^4$ with singular "foam vertices", caps, and attached one-dimensional strings.

The map $\text{cl} : uPB \to sKTG$ is the "closure map". Given a parenthesized braid, close up its top and bottom each by a tree according to the parentetization; this produces a *sKTG* with the convention that all strands are oriented upwards, bottom vertices are negative, and top vertices are positive. An example is shown below.



The map $a : sKTG \to \widetilde{wTF}$ arises combinatorially from the fact that all *sKTG* diagrams can be interpreted as elements of $\widetilde{wTF}$, and all $sKTG$ Reidemeister moves remain true in $\widetilde{wTF}$. Topologically, it is an extended version of Satoh's tubing map, described in Remark 3.1.1 of [WKO2].

1.2. **Algebra.** The chain of maps $\mathcal{K}$ is an example of a general "algebraic structure", as defined in [WKO2, Section 2.1]. An algebraic structure consists of a collection of objects belonging to a number of "spaces" or "different kinds", and operations that may be unary, binary, multinary or zeronary, between these spaces. In this case there are many spaces (or kinds of objects): for example, parenthesized braids with specified bottom and top parenthetizations form one space, so do knottings of a given trivalent graph (skeleton). There is also a large collection of operations, consisting of all the internal operations of *uPB*, *sKTG* and $\widetilde{wTF}$, as well as the maps a and cl.

In Sections 2.2 and 2.3 of [WKO2]we discuss projectivizations and expansions for general algebraic structures. A projectivization is the associated graded space taken with respect to the filtration by powers of the "augmentation ideal". For the spaces *uPB*, *sKTG* and $\widetilde{wTF}$, the projectivizations $\mathcal{A}^{hor}$, $\mathcal{A}^u$ and $\mathcal{A}^{sw}$ are the spaces of "horizontal chord diagrams on parenthesized strands", "chord diagrams on trivalent skeleta", and "arrow diagrams", as described in [BN1], [WKO2, Section 4.6], and Section 2 of this paper, respectively. As a result, the projectivization of $\mathcal{K}$ is the structure

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{\text{cl}} \mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and a, respectively. More specifically, cl is the "closure of chord diagrams", and α is "sending each chord to the sum of its two possible oriantations".

An expansion is a filtration-respecting map from an algebraic structure (where linear combinations of objects of the same kind are allowed) to its projectivization, satisfying a certain non-degeneracy property. Expansions are also called universal finite type invariants in knot theory. A homomorphic expansion also behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the projectivization. Hence, a homomorphic expansion $Z : \mathcal{K} \to \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u, and Z^w for *uPB*, *sKTG* and $\widetilde{wTF}$, respectively, so that the following diagram commutes:

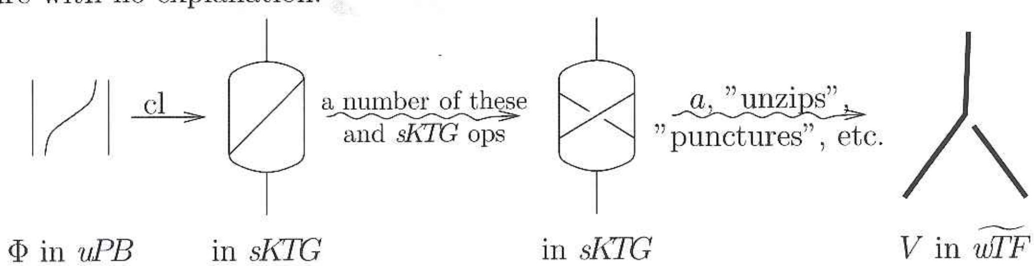
$$\begin{array}{ccccc} uPB & \xrightarrow{\text{cl}} & sKTG & \xrightarrow{a} & \widetilde{wTF} \\ \downarrow{\scriptstyle Z^b} & & \downarrow{\scriptstyle Z^u} & & \downarrow{\scriptstyle Z^w} \\ \mathcal{A}^{hor} & \xrightarrow{\text{cl}} & \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^w \end{array} \tag{1}$$

We recall (see [BN1]) that a homomorphic expansion Z^b for parenthesized braids is determined by a "horizontal chord associator" $\Phi = Z^b(|\!\diagup\!|)$. A homomorphic expansion Z^u of *sKTG* is also (almost) determined by a Drinfel'd associator (horizontal chords or not; see [WKO2, Section 4.6]), so the significance of left commutative square is to force the associator corresponding to Z^u to be a horizontal chord associator. In turn, Z^w is determined by a solution $V = Z^w(\curvearrowright)$ to the Kashiwara-Vergne problem (see [WKO2, Section 4.4 – 4.5]), and the goal of this paper is to prove the following theorem:

MORE: Clarity and advertisment: state the main theorem from WKO2 here.

Theorem 1.1. (1) *Assuming that $Z : \mathcal{K} \to \mathcal{A}$ exists, Z is determined by Z^u. Most importantly, there is a formula for V in terms of Φ, assuming that V exists.*
(2) *Said formula is the formula proven in [AET].*
(3) *Every Z^b extends to a Z.*

The key to the proof of the theorem is to show that the generator $\curvearrowright$ of $\widetilde{wTF}$ can be expressed in terms of the generator $|\!\diagup\!|$ of *uPB* and the operations of $\mathcal{K}$. Assuming that Z exists, this yields a formula for V in terms of Φ. The expression of $\curvearrowright$ in terms of $|\!\diagup\!|$ uses a "Double Tree Construction", which will be discussed in Section ??. For now, let us display a picture with no explanation:



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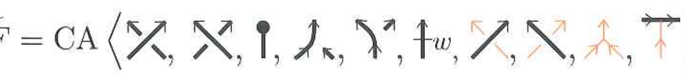
2. The spaces $\widetilde{wTF}$ and $\mathcal{A}^{sw}$ in more detail

As we mentioned in the introduction, $\widetilde{wTF}$ is a minor extension of the space *wTF* studied in [WKO2, Section 4.5]. It can be introduced as a planar algebra or as a circuit algebra, we will do the latter as it is simpler and more concise. Circuit algebras are defined in [WKO2, Section 2.4]; in short, they are similar to planar algebras but without the planarity requirement for "connecting strands". As in [WKO2], each generator and relation of $\widetilde{wTF}$ has a local topological interpretation. Recall [WKO2, Sections 1.2, 3.4, 4.1, 4.5] that *wTF* diagrams represent certain ribbon knotted tubes with foam vertices in $\mathbb{R}^4$, and the circuit algebra wTF is conjecturally a Reidemeister theory for this space (i.e., there is a surjection δ from the circuit algebra wTF to ribbon knotted tubes with foam vertices, and δ is conjectured to be an isomorphism). The space $\widetilde{wTF}$ extends wTF by adding one-dimensional strands to the picture. Note that one dimensional strands cannot be knotted in $\mathbb{R}^4$, however, they can be knotted *with* the two-dimensional tubes. In figures two-dimensional tubes will be denoted by thick lines and one dimensional strings by thin red lines. With this in mind, we define $\widetilde{wTF}$ as a circuit algebra defined in terms of generators and relations, and with some extra operations beyond circuit algebra composition. Each generator, relation and operation has a local topological interpretation which provides much of the intuition behind the proofs. However, the corresponding Reidemeister theorem is only conjectural.

$\widetilde{wTF} = \mathrm{CA}\Big\langle$  $\Big|$ relations as in Section 2.2 $\Big|$ operations as in Section 2.3 $\Big\rangle$.

2.1. **The generators of $\widetilde{wTF}$.** We begin by discussing the local topological meaning of each generator shown above.

The first six generators are as described in [WKO2, Sections 4.1.1 and 4.5], we only briefly recall their descriptions here. Knotted (more precisely, braided) tubes in $\mathbb{R}^4$ can equivalently be thought of as movies of flying rings in $\mathbb{R}^3$. The two crossings stand for movies where two rings trade places by the ring of the under strand flying through the ring of the over strand. The dotted end represents a tube "capped off" by a disk. The fourth and fifth generators stand for singular "foam vertices", and will be referred to as the positive and negative vertex, respectively. The positive vertex represents the movie shown on the left: the right ring approaches the left ring from below, flies inside it and merges with it. The negative vertex represents a ring splitting and the inner ring flying out below and to the right. To be completely precise, $\widetilde{wTF}$ as a circuit algebra has more vertex generators than shown above: the vertices appear with all possible orientations of the strands. However, all other versions can be obtained from the ones shown above using "orientation switch" operations (to be discussed in Section 2.3). Finally, the sixth generator represents the *wen*: a Klein bottle cut open, or equivalently, the movie in which a ring flips over. In band notation the wen is shown as a half twist.

The red (thin) strands denote one dimensional strings in $\mathbb{R}^4$, or "flying points in $\mathbb{R}^3$". The crossings between the two types of strands (sixth and seventh generators) denote "points flying through rings". They are both shown on the right in band notation (see [WKO2, Section 3.4] for an explanation of band notation). For example, the bottom left picture means "the point on the right approaches the ring on the left from below, flies through the ring and out to the left above it." This explains why there are no generators with a thick strand crossing under a thin red strand: a ring cannot fly through a point.

Next is a trivalent vertex of 1-dimensional strings in $\mathbb{R}^4$. Once again, this generator should be shown in all possible strand orientation combinations. Finally, tha last generator is a "mixed vertex", in other words a one-dimesional string attached to the wall of a 2-dimensional tube.

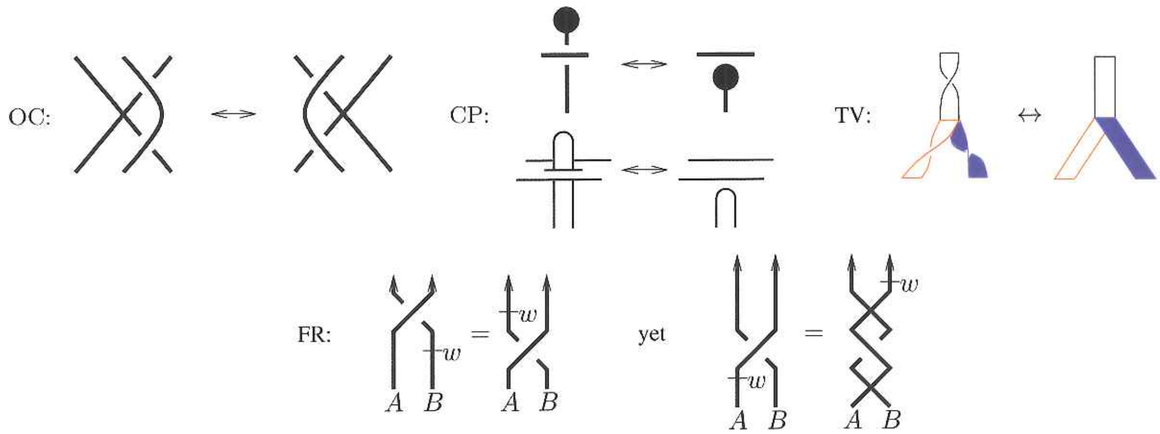
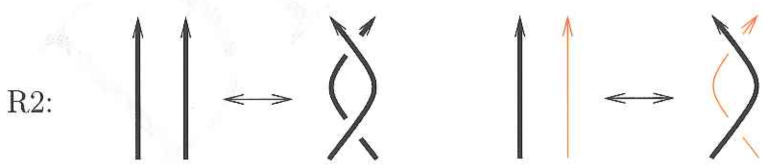


Figure 2. Some $\widetilde{wTF}$ relations.

2.2. **The relations.** As a list, the relations for $\widetilde{wTF}$ look the same as the relations for wTF [WKO2, Section 4.5]: $\{R1^s$, R2, R3, R4, OC, CP, FR, W^2, CW, TW$\}$. Recall that $R1^s$ is the weak (framed) version of the Reidemeister 1 move; R2 and R3 are the usual Reidemeister moves; R4 allowes moving a strand over or under a vertex. W^2 states that two consecutive wens can be cancelled, and similarly CW lets a wen on a capped strand disappear (since the cap can slide through the wen). OC stands for *Overcorssings Commute*, CP for *Cap Pullout*, FR for *Flip Relations*, and TW for *Twisted Vertex*; these four relations are shown in Figure 2, for a detailed explanation see [WKO2].

However, all relations should be interpreted in all possible combinations of strand types (tube or string), for example the lower strand of the R2 relation can either be thick black or thin red, as shown below. Similarly, any of the lower strands of the R3, R4, OC and FR relations may be thin red.



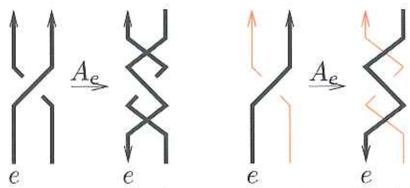
As in wTF, the relations all have local topological meaning and conjecturally $\widetilde{wTF}$ is a Reidemeister theory for ribbon knotted tubes in $\mathbb{R}^4$ with caps, singular foam vertices and attached strings. For example, Reidemeister 2 with a thin red bottom strand is imposed because a point flying in through a ring and then immediately flying back out is isotopic to not having any interaction between the point and ring at all.

2.3. **The operations.** Like wTF, $\widetilde{wTF}$ is equipped with a set of auxiliary operations in addition to the circuit algebra structure.

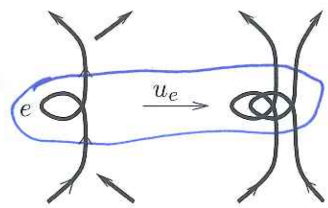
The first of these is orientation reversal. For the thin (red) strands, this simply means reversing the direction of the strand. For the thick strands (tubes), orientation switch comes in two versions. Recall from [WKO2, Section 3.4] that in the topological interpretation of wTF, each tube is oriented as a 2-dimensional surface, and also has a distinguished "core": a line along the tube which is oriented as a 1-dimensional manifold and determines the "direction" or "1-dimensional orientation" of the tube. Both of these are determined by the direction of the strand in the circuit algebra, via Satoh's tubing map.

Topologically, the operation "orientation switch", denoted S_e for a given strand e, acts by reversing both the (1-dimensional) direction and the (2-dimensional) orientation of the tube e. Diagrammatically, this corresponds to simply reversing the direction of the corresponding strand e.

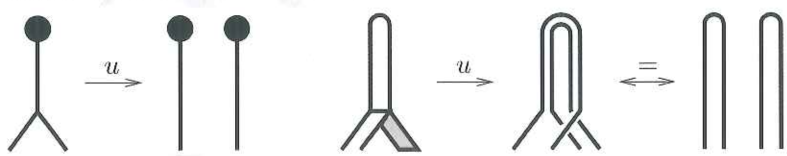
The "adjoint" operation, denoted A_e, on the other hand only reverses the (1-dimensional) direction of the tube e, not the orientation as a surface. Diagrammatically, this manifests itself as reversing the strand direction and adding two virtual crossings on either side of each crossing where e crosses *over* another strand, as shown on the right (note that the strand below e may be thick or thin). Note that virtual crossings don't appear when e crosses under another strand. For more details on orientations and orientation switches, see [WKO2, Sections 3.4 and 4.1.3].



The unzip operation u_e doubles the strand e using the blackboard framing, and then attaches the ends of the doubled strand to the connecting ones, as shown on the right. We restrict unzip to strands whose two ending vertices are of different signs. (For the definition of crossing and vertex signs, see [WKO2, Sections 3.4 and 4.1].) Topologically, the blackboard framing of the diagram induces a framing of the corresponding tube in $\mathbb{R}^4$ via Satoh's tubing map, and unzip is the act of "pushing the tube off of itself slightly in the framing direction". Note that unzips preserve the ribbon property. To unzip a strand which carries a wen, first slide the wen off the strand using the TV relation, then unzip.



A related operation, *disk unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagram world, in the direction of the blackboard framing), as before. An example in the line and band notations is shown below.



Forthermore, one can unzip thin red strands that either end in completely thin red vertices on both sides, or in a thin red vertex at one end and at "infinity" (the tangle boundary) at the other end.

So far all the operations we have introduced had already existed in *wTF*. There is also a new operation is called "puncture", denoted p_e, which diagrammatically simply turns the thick black strand e into a thin red one. The corresponding topological picture is "puncturing a tube", i.e., removing a small disk from it and retracting the rest to its core. Any crossings where e passes under another strand are not affected, while crossings in which e is the over strand turn into virtual crossings.

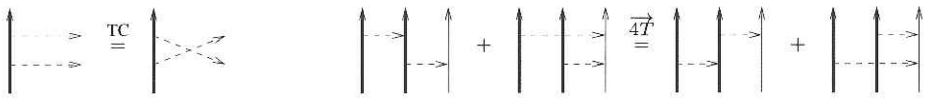
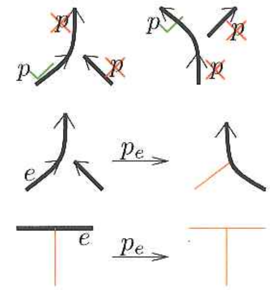
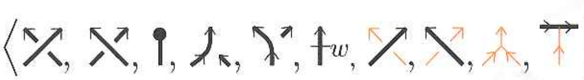


Figure 3. The TC and $\overrightarrow{4T}$ relations

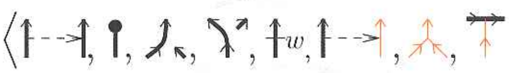


For simplicity, we place a restriction on which strands can be punctured, namely at each (fully thick black) vertex puctures are only allowed for one of the three meeting strands, as shown in the top row of the figure on the right. More general puctures could be allowed in a theory with more than one kind of "string to tube" vertex. The bottom row of the same figure shows what happens when puncturing one of the thick strands of a mixed vertex. Topologically, this is because the mixed vertex represents a string attached to a tube, so when puncturing e, the entire tube retracts to its core. Finally, a capped tube disappears when punctured, and a wen disappears when the strand which carries it is punctured.

In summary,

$\widetilde{wTF} = \mathrm{CA}\Big\langle$  $\Big|$ R1^s, R2, R3, R4, OC, CP, FR, W^2, CW, TV $\Big|$ $S_e, A_e, u_e, d_e, p_e \Big\rangle$

2.4. **The projectivization $\mathcal{A}^{sw}$.** As in [WKO2], the space $\widetilde{wTF}$ is filtered by powers of the augmentation ideal and its associated graded space or projectivization, denoted $\mathcal{A}^{sw}$, is a "space of arrow diagrams on foam skeletons with strings". As a circuit algebra, $\mathcal{A}^{sw}$ is presented as follows:

$\widetilde{wTF} = \mathrm{CA}\Big\langle$  $\Big|$ relations as below $\Big|$ operations as below $\Big\rangle$.

The first and sixth generators are called single arrows and they are of degree one, while all others are "skeleton features" of degree zero. The relations are almost the same as those for the projectivization of wTF: the sksleton relations W^2, TV and CW; $\overrightarrow{4T}$ (the 4-Term relation), TC (Tails Commute), RI (Rotation Invariance), CP (the arrow Cap Pullout), VI (Vertex Invariance), FR (the arrow Flip Relation); and the additional new relation TF (Tails Forbidden on strings). The TC and $\overrightarrow{4T}$ relations are shown in Figure 3, note that the 3rd strand in each term of the $\overrightarrow{4T}$ relation is ambiguous: it can be either thick black or thin red, the relation applies in either case. VI is shown in Figure 4: here the $\pm$ signs depend on the strand orientations. Note that the type of the vertex and the types of each strand (thick black or thin red) are left ambiguous: the VI relation applies in all cases. Figure 5 shows the other three relations: RI, CP and TF. Note that technically TF is not even a relation: there were no generators with an arrow tail on a thin red strand, so saying that such an element vanishes is superfluous. However, without TF the VI relation would have to be stated for all the sub-cases of 0, 1 or 3 thin red strands, so we prefer this cleaner way of listing the relations even if it is a slight abuse of notation.

Each opration on $\widetilde{wTF}$ induces a corresponding operation on $\mathcal{A}^{sw}$. Orientation switch, adjoint, unzip, cap unzip, and long strand deletion act exactly the same way as they do for wTF^o, we quickly recall these here, for details see [WKO2, Section 4.2.2]. The orientation switch S_e reverses the orientation of the skeleton strand e, and multiplies the arrow diagram

Figure 4. The VI relation: the vertices and strands could be of any type, but the same throughout the relation.

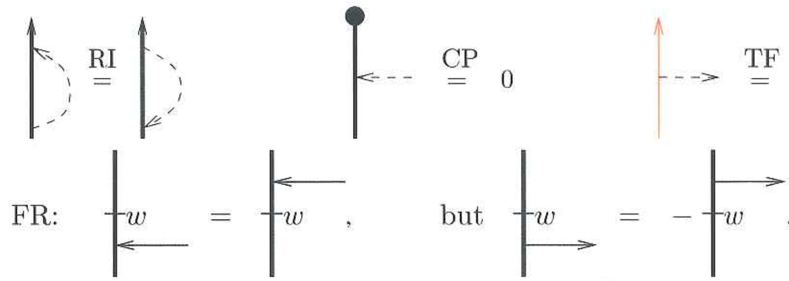
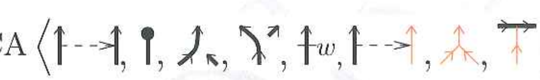


Figure 5. The RI, CP and FR relations, and the TF relation (which is not really a relation).

by $(-1)^{\#\{\text{arrow endings on } e\}}$. The adjoint operation also reverses the skeleton strand e and multiplies the arrow diagram by $(-1)^{\#\{\text{arrow heads on } e\}}$. Given a skeleton S with a distinguished strand e, unzip (or disc unzip, if e is capped) is an operation $u_e : \mathcal{A}^{sw}(S) \to \mathcal{A}^{sw}(u_e(S))$ which maps each arrow ending on e to a sum of two arrows, one ending on each of the two new strands which replace e. Deleting a long strand e kills all arrow diagrams with any arrow ending on e. The operation induced by puncture, denoted p_e, turns the formerly thick black e into a thin red strand, and kills any arrow diagram with any arrow tails on e.

To summarise:

$\widetilde{wTF} = \text{CA}\Big\langle$  $\Big|\ \begin{matrix} W^2, \text{TV}, \text{CW}, \overrightarrow{4T}, \text{TC} \\ \text{VI}, \text{CP}, \text{RI}, \text{FR}, \text{TF} \end{matrix}\ \Big|\ \begin{matrix} S_e, A_e, u_e, \\ d_e, p_e \end{matrix} \Big\rangle$

As in [WKO2, Definition 3.7], we define a "w-Jacobi diagram" (or just "arrow diagram") to be similar to by also allowing trivalent chord vertices, each of which is equipped with a cyclic orientation. Denote the circuit algebra of formal linear combinations of these w-Jacobi diagrams by, $\mathcal{A}^{swt}$. Then, as in [WKO2, Theorem 3.8], we have the following bracket-rise theorem:

Theorem 2.1. *The obvious inclusion of diagrams induces a circuit algebra isomorphism $\mathcal{A}^{sw} \cong \mathcal{A}^{swt}$. Furthermore, the $\overrightarrow{AS}$, $\overrightarrow{IHX}$ and $\overrightarrow{STU}$ relations (see Figure 6) hold in $\mathcal{A}^{swt}$.*

The proof is identical to the proof of [WKO2, Theorem 3.8]. In light of this isomorphism, we will drop the extra "t" from the notation and use $\mathcal{A}^{sw}$ to denote either of these spaces. As in [WKO2], the primitive elements of $\mathcal{A}^{sw}$ are connected diagrams, denoted $\mathcal{P}^{sw}$, and $\mathcal{P}^{sw} = \{\text{trees}\} \oplus \{\text{wheels}\}$ as a vector space. An example of trees and wheels is shown in Figure 7; for details see [WKO2, Section 3.1]. Note that the RI relation can now be rephrased (via $\overrightarrow{STU}_2$) as "the weel with a single spoke, or one-wheel, vanishes".

We recall the following two facts [WKO2, Lemmas 4.6 and 4.7]:

Fact 2.2. *$\mathcal{A}^{sw}(\uparrow)$, the part of $\mathcal{A}^{sw}$ with skeleton a single capped strand, is isomorphic as a vector space to the completed polynomial algebra freely generated by wheels w_k with $k \geq 2$.*

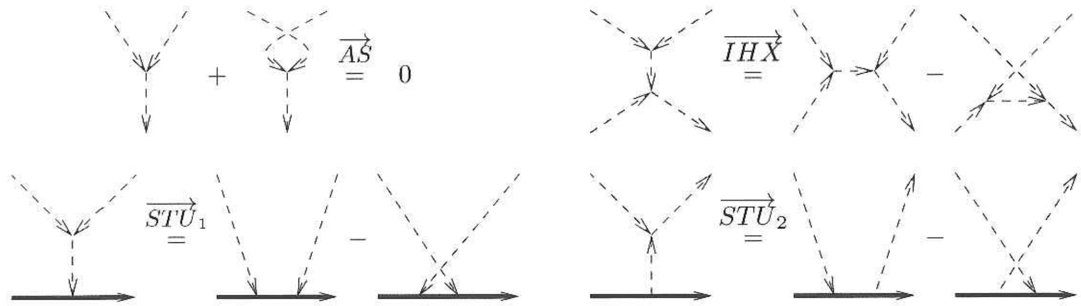


Figure 6. The $\overrightarrow{AS}$, $\overrightarrow{IHX}$ and the two $\overrightarrow{STU}$ rerations. fig:ASI

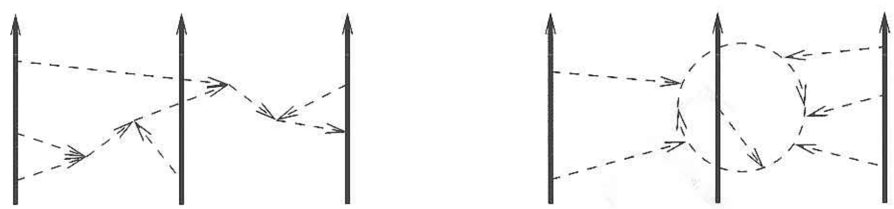


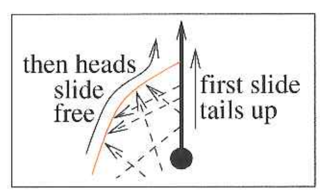
Figure 7. An example of a "tree", left, and a "wheel", right. fig:Tree

pStrands **Fact 2.3.** *$\mathcal{A}^{sw}(\curlywedge) \cong \mathcal{A}^{sw}(\uparrow_2)$, where $\mathcal{A}^{sw}(\curlywedge)$ stands for the space of arrow diagrams whose skeleton is a single vertex (the picture shows a positive vertex but the statement is true for all kinds of vertices with thick black strands), and $\mathcal{A}^{sw}(\uparrow_2)$ is the space of arrow diagramd on two (thick black) strands.*

In addition, we will make use of the following

apString **Lemma 2.4.** *$\mathcal{A}^{sw}\left(\uparrow\!\!\bullet \cdots \uparrow\!\!\bullet\right) \cong \mathcal{A}^{sw}(\uparrow_n)$, and consequently $\mathcal{A}^{sw}\left(\curlywedge\right) \cong \mathcal{A}^{sw}(\uparrow_2)$. The left hand side of the first isomorphism is the space of arrow diagrams on n strands whose bottom is capped and a thin red string is attached above the cap. Similarly, the left hand side of the second isomorphism involves a vertex whose two incoming strands are capped with a string attached.*

Proof. To prove the first isomorphism, we construct inverse maps between the two spaces. There is an obvious map $\mathcal{A}^{sw}(\uparrow_n) \to \mathcal{A}^{sw}\left(\uparrow\!\!\bullet \cdots \uparrow\!\!\bullet\right)$, namely, given an arrow diagram on two strands, place it on the capped/stringed strands above the attached string.



In the other direction, given an arrow diagram on capped/stringed strands, first note that in the same vein as Fact 2.2 fact:CapIsWheels we can assume that there are only arrow tails on the capped strand under the attached string. Also, arrow tails die on the thin red strand, hence it has only arrow heads. To produce an arrow diagram on regular strands, first push the arrow tails from the capped strand up and over the string/tube vertex. This is done using the VI relation, but since tails die on the thin red strand they slide past the vertex without a cost. Once the capped side is cleared, continue by pushing the arrow tails up from the thin red string to the strand above the vertex. Note that now the cap kills any arrow heads and hence they also slide past the vertex at no cost. Once the capped and string strands are cleared of any arrow endings, the result can be regarded as an arrow diagram on n regular strands.

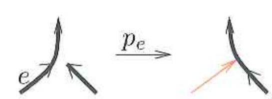
It is obvious that the two maps are inverses of each other, so we have proven the first isomorphism. For the second, do the same and combine with Fact 2.3. □

2.5. **The homomorphic expansion.** As discussed in [WKO2, Section 2.3], an expansion is a map $Z^w : \widetilde{wTF} \to \mathcal{A}^{sw}$ with the property that the associated graded map $\operatorname{proj} Z^w : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ is the identity map $\mathrm{id}_{\mathcal{A}^{sw}}$. A homomorphic expansion is an expansion which also intertwines each operation of $\widetilde{wTF}$ with its arrow diagrammatic counterpart, meaning that the appropriate squares commute. In [WKO2, Theorems 4.9 and 4.11] we proved that there exsists a homomorphic expansion for wTF. In fact that homomorphic expansions for wTF (with the minor techical condition that the value of the vertex doesn't contain isolated arrows) are in one-to-one correspondence with solutions to the Kahiwara-Vergne problem with even Duflo function. We now claim that finding a homomorphic expansion for wTF is equivalent to finding one for $\widetilde{wTF}$.

Theorem 2.5. *Homomorphic expansions for wTF are in one-to-one correspondence with homomorphic expansions for $\widetilde{wTF}$.*

Proof. For the duration of this proof we introduce the following notation: $\mathcal{A}^{sw}$ will denote the associated graded space of wTF and $Z^w : wTF \to \mathcal{A}^{ws}$ a homomorphic expansion. The associated graded space of $\widetilde{wTF}$ will be denoted by $\widetilde{\mathcal{A}}^{sw}$ and the corresponding homomorphic expansion by $\widetilde{Z}^w : \widetilde{wTF} \to \widetilde{\mathcal{A}}^{sw}$.

Note that there are obvious inclusions $wTF \to \widetilde{wTF}$ and $\mathcal{A}^{sw} \to \widetilde{\mathcal{A}}^{sw}$, namely the maps which send each generator of wTF or $\mathcal{A}^w$ to the same generator of the corresponding extended space. These maps are injective as none of the relations of $\widetilde{wTF}$ or $\widetilde{\mathcal{A}}^{sw}$ change the skeletons of the participating diagrams.



First assume that Z^w is a homomorphic expansion for wTF, we want to show that this extends uniquely to a homomorphic extension of $\widetilde{wTF}$. Z^w assigns values in $\mathcal{A}^{sw}$ to all the generators of wTF, namely the crossings, vertices and cap; hence we only need to define the value of the extension $\widetilde{Z}^w$ on the remaining generators of $\widetilde{wTF}$: those which involve thin red strands. Note that if such a *homomorphic* extension exists, then it is unique: each generator of $\widetilde{wTF}$ which involves thin red strand can be obtained via puncture operations from generators which do not, see the example in the figure on the right. Since we know the $\widetilde{Z}^w$-value of all the generators which do not involve thin red strands and $\widetilde{Z}^w$ should be homomorphic with respect to the puncture operation, this determines the value of $\widetilde{Z}^w$ on all generators.

It is straightforward to check that these values satisfy all equations arising from the relations of $\widetilde{wTF}$ and $\widetilde{Z}^w$ is indeed an expansion. Furthermore, $\widetilde{Z}^w$ is homomorphic by design, using the homomorphicity of Z^w.

Conversely, given a homomorphic expansion $\widetilde{Z}^w : \widetilde{wTF} \to \widetilde{\mathcal{A}}^{sw}$, then the corresponding homomorphic expansion Z^w for wTF is the restriction (i.e., the composition of the inclusion of wTF into $\widetilde{wTF}$ with $\widetilde{Z}^w$). □

In light of the result above and because this paper focuses on the space $\widetilde{wTF}$, we are going to drop the tildes from $\widetilde{\mathcal{A}}^{sw}$ and $\widetilde{Z}^w$. For the rest of this paper $\mathcal{A}^{sw}$ denotes the projectivization (associated graded space) of $\widetilde{wTF}$ and Z^w a homomorphic expansion $\widetilde{wTF} \to \mathcal{A}^{sw}$.

In the proof above we argued that the values of the punctured vertices can be computed from the values of the vertices which already exist in wTF. In fact, more is true: the value of the completely thin red vertex is obviously trivial (thin red strands can only support arrow heads, not tails, hence no arrow diagram can live on a completely thin red skeleton). Moreover, the value of the left punctured vertex turns out to be trivial as well. This fact will be useful later, so we prove it here.

Lemma 2.6. *For any homomorphic expansion Z^w, the Z^w-value of a left punctured vertex is trivial*[1].

Proof. Recall from [WKO2, Proof of Theorem 4.9] that the Z^w-value of the (positive, not punctured) vertex V can be written as $V = e^b e^t$, where b consists of wheels only and t (denoted uD in [WKO2]) consists of trees. Puncturing the left strand of V kills all arrow diagrams with tails on the left. This is almost everything: diagrams that survive are only wheels and short arrows supported entirely on the right strand, and arrows pointing from the right to the left. (We use that if all the tails of a tree are supported on the same strand then the tree is a single arrow, otherwise it is zero by the anti-symmetry of trivalent arrow vertices.) Note that all of the surviving arrow diagrams commute with each other.

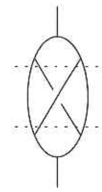
Let us then denote the value of the punctured vertex by $p_1V = e^{p_1(b)}e^{p_1(t)}$. Recall that V must satisfy "unitarity" [WKO2, Equation (12)], namely that $V \cdot A_1A_2(V) = 1$, where A_i denotes the adjoint operation performed on strand i. Hence we have $p_1V \cdot A_1A_2(p_1V) = 1$. Since wheels have only tails, $A_1A_2(p_1(b)) = p_1(b)$. Each arrow has one head, so $A_1A_2(p_1(t)) = -p_1(t)$. Hence, using commutativity, $p_1V \cdot A_1A_2(p_1V) = e^{2p_1(b)} = 1$, which implies that $p_1(b) = 0$. As for $p_1(t)$, showing that there are no arrows pointing from the right to the left strand is a direct computation in degree 1.

In [WKO2, Proof of Theorem 4.9] we showed that short arrows supported on one strand of V don't affect whether Z^w is a homomorphic expansion. In other words, if Z^w is a homomorphic expansion and a is a linear combination of such short arrows, then replacing V by e^aV gives rise to another homomorphic expansion. Hence we can assume without loss of generality that there are no short arrows, so $p_1V = 1$. □

3. Proof of Theorem 1.1

3.1. **Tree level proof of part (1).** Let $\mathcal{A}^{trees}$ denote the quotient of $\mathcal{A}^{sw}$ by all wheels, and let $\pi : \mathcal{A}^{sw} \to \mathcal{A}^{trees}$ denote the quotient map (cf [WKO2, Section 3.2]). Part (1) of the main theorem states that if a "global" homomorphic expansion Z exists, then it is determined by Z^u, in other words, Z^w is determined by Z^u. Z^w on the other hand is given by the values V and C of the positive vertex and the cap [WKO2, Sections 4.3 and 4.5], so one only needs to show that V and C are determined by Z^u. Proving this "on the tree level" means showing only that $\pi(V)$ and $\pi(C)$ are determined by Z^u. We prove this partial result first since this contains the main idea without any clouding details. In particular, observe that since C consists entirely of wheels (Fact 2.2) we have $\pi(C) = 0$, so we only need to study $\pi(V)$.

[1] Save for possibly short arrows supported on the right strand, which can be ignored without loss of generality, as explained in the last paragraph of the proof.



Let B^u denote the "buckle" *sKTG*, as shown on the right (ignore the dotted lines). All edges are oriented up, and by the drawing conventions of [WKO2, Section 4.6] all the vertices in the bottom half of the picture are negative and all the ones in the top half are positive. Let $B^w = a(B^u)$ be the corresponding element of $\widetilde{wTF}$. Let $\beta^u := Z^u(B^u)$, and note that $\beta^u \in \mathcal{A}^u(\uparrow_4)$ is a chord diagram on four strands, as VI relations can be used to push all chord endings to the "middle" of the skeleton (between the dotted lines on the picture). Let $\beta^w = \alpha(\beta^u)$, and note that by the homomorphicity of Z we also have $\beta^w = Z^w(B^w)$. We are going to perform a series of operations on B^w and $\pi(\beta^w)$ which will allow us to recover $\pi(V)$ from it.

First, 'multiply' B^w by a positive vertex at the bottom, as shown in Figure 8. This is a circuit algebra operation, and correspondingly $\pi(\beta^w)$ is circuit algebra multiplied by $\pi(V)$, where V is the value of the vertex. Then unzip the edge marked by u, and puncture the edges marked p and pp (in that order). Then attach a cap to the thick black end at the bottom. This is a circuit algebra operation; keep in mind that image of the value of the cap is trivial in $\mathcal{A}^{trees}$. Finally, perform a disc unzip on the cap, followed by unzipping the red strand marked by u.

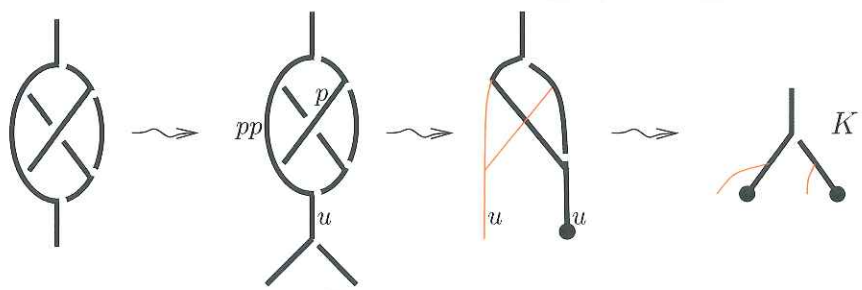


Figure 8. From the "buckle" to the vertex.

Let us call the resulting w-tangled foam K, as in Figure 8. What is the $Z^w(K)$? Due to the homomorphicity of Z, it is obtained from β^w by performing the series of operations described above. Unfortunately, one of these operations was multiplication by $\pi(V)$, which is what we are trying to compute. However, notice that the left strand of the added vertex later got punctured, and hence by Lemma 2.6 we only multiplied by 1. (We also multiplied by caps, but as discussed earlier this doesn't matter for now as $\pi(C) = 1$. Hence, $Z^w(K)$ can be cmputed from β^w by performing punctures and unzips, and since $\beta^w = \alpha(\beta^u)$, this means that $Z^w(K)$ is determined by Z^u.

On the other hand, note that the space of chord diagrams on the skeleton of K is the space $\mathcal{A}(\uparrow_2)$ by Lemma 2.4. Also note that the Z^w-value of K in $\mathcal{A}^{trees}(\uparrow_2)$ is exactly $\pi(V)$. Hence, $\pi(V)$ is determined by Z^u as needed, and hence Z^u determines Z^w on the tree level. In Part (2) we'll write a formula for B^u in terms of associators, which in turn produces a formula for V. □

3.2. **Complete proof of part (1).** The main idea of the complete proof is the same as in the tree level proof: we compute Z^w from β^u. This amounts to computing the Z^w-value of the foam K of Figure 8 two ways: from $\beta^w = \alpha(\beta^u)$ and from V and C. Then we use the resulting equation to solve for V and C. However, now with one equation and two unknowns, we need some additional information, and this is provided by the Cap Equation [WKO2,

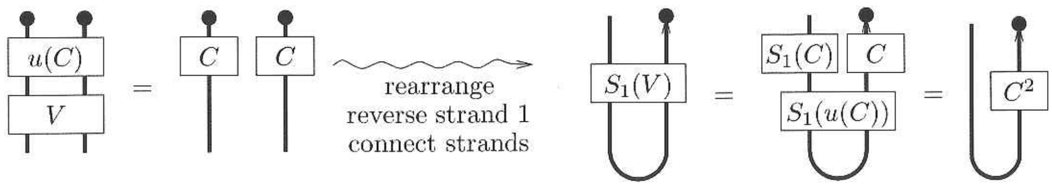


Figure 9. The Cap Equation and a deduction from it, see explanation below.

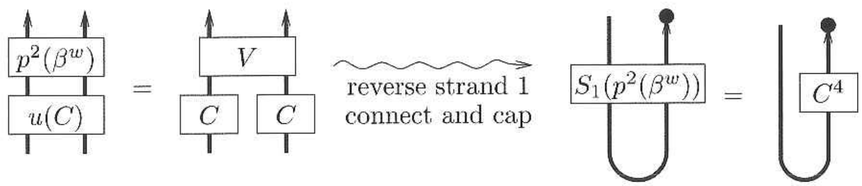


Figure 10. Proof of Part 1 of Theorem 1.1.

Equation (13)]. Recall that the Cap Equation is a relationship between V and C which Z^w needs to satisfy in order to be homomorphic with respect to disc unzip. The equation is shown on the left side of Figure 9.

Now rearrange the cap equation by grouping all the cap terms on the right side, reverse the first strand and connect the two strands as shown in Figure 9. Note that Fact 2.2 implies that $\mathcal{A}^{sw}$ of a single strands capped on both ends is isomorphic to $\mathcal{A}^{sw}(\uparrow)$, hence we can ignore the cap at one end of the strand, as in the figure. To see that the right side of the resulting equation is indeed just $C^2 \in \mathcal{A}^{sw}(\uparrow)$, we need to note two facts. First, recall from [WKO2, Section 4.5.3] that the CW relation implies that C is even (i.e. consists of even wheels only). Hence, $S_1(C) = C$. Second, note that for any arrow diagram D, the diagram $S_1(u(D))$ vanishes when placed on a U-shaped strand, esentially by the defitition of u.

Next, we claim that $Z^w(K) = u(C) \cdot p^2(\beta^w) \in \mathcal{A}^{sw}(\uparrow_2)$. To see this, recall that B^w "multiplied" on the bottom by a vertex, punctured twice, unzipped, multiplied again by a cap, and then disc unzipped. The value of the vertex that was attached is 1 by Lemma 2.6. the two punctures result in $p^2(\beta^w)$, which can be interpreted as an element of $\mathcal{A}^{sw}(\uparrow_2)$ by Lemma 2.4. Finally, $u(C)$ comes from the attached cap which was then unzipped. On the other hand, computing $Z^w(K)$ directly from V and C, we have two caps multiplied by a vertex in $\mathcal{A}^{sw}(\uparrow_2)$, as shown in Figure 10, and these two results are equal by the homomorphicity of Z^w and its compatibility with Z^u.

Now reverse strand 1, connect the two strands and cap the top, as shown in Figure 10. By the cap equation and our previous observations, $u(C)$ cancels on the left hand side, and on the right we get C^4. Hence, we have computed C from β^w. Now plugging the value of C into the starting equation of Figure 10, we multiply by C^{-1} at the bottom of each strand to find V, completing the proof. □

3.3. proof of part (2).

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