Report on the paper "Finite type invariants for w-knotted objects II", by D. Bar-Natan and Z. Dancso

0.1. **The context.** In the 90's, a theory of finite type invariants (Vassiliev invariants) of knots and links in $\mathbb{R}^3$ was developed. In this theory, invariants take their values in spaces of diagrams. One of its main results is the construction of universal Vassiliev invariants by integration of the Knizhnik-Zamolodchikov (KZ) connection (Kontsevich integral). It was later realized that this invariant can be constructed algebraically using a Drinfeld associator. This construction was later refined to an assignment (Drinfeld associators)$\to$(homomorphic expansions of signed knotted trivalent graphs (sKTGs)), the word homomorphic referring to compatibility with possible operations on sKTGs, such as unzipping (works by Murakami-Ohtsuki (1997), Cheptea-Le (2007), Dancso (2010)).

Associators play an important role in the solution of several algebraic problems, such as quantization of Lie bialgebras into quantum enveloping algebras (solved by Etingof-Kazhdan (1994)) or the Kashiwara-Vergne (KV) problem of finding Lie series which produce isomorphisms between various constructions attached to Lie algebras (solved by Alekseev-Torossian (2012) and Alekseev-Enriquez-Torossian (2010)). This rises the question of finding natural topological frameworks for these problems, with the aim of producing a topological viewpoint on their solutions.

In the case of the KV problem, this program is completed in a series of two papers, of which the paper under review is the second. In the first paper, the relevant topological objects are introduced. These are called 'w-knots/braids/links' by the authors and correspond to categories of objects studied earlier by several groups of topologists, the most useful viewpoint for the purposes of the authors being that of "movies of flying rings in $\mathbb{R}^3$". The main result of the first paper is the construction of a universal finite type invariant for these objects and its relation to the Alexander polynomial.

0.2. **Contents of the paper.** The paper starts with the introduction of basic notions for the theory to be developed, namely the notion of circuit algebras, their filtrations and associated graded structures (called 'projectivizations'), and isomorphisms between those (called 'homomorphic expansions'); this is done in Section 2. In Section 3, the authors introduce the notion of w-tangle, which appears to be closely related to the notion of w-knot, basic to the first paper. They show in particular that this notion can be defined in a simple way in terms of circuit algebras, much in the same way as tangles can be defined in terms of planar algebras due to the Reidemeister theorem. They introduce graded spaces corresponding to w-tangles and establish their relation with the Alekseev-Torossian (AT) formalism of tangential derivations (Propositions 3.14 and 3.18). They construct a homomorphic expansion of w-tangles and show its uniqueness (Section 3.5). They explain a 4-dimensional interpretation of w-tangles (Section 3.4); they exhibit a morphism from w-tangles to ribbon tubes in $\mathbb{R}^4$, which is however only known to be surjective; proving the isomorphism would be an analogue of the Reidemeister theorem in this situation.

In Section 4, the authors enrich the circuit algebra of w-tangles to a circuit algebra of "w-foams'. From the viewpoint of flying rings in $\mathbb{R}^3$, this corresponds to the introduction of new possible behaviors of the rings, such as shrinking a ring to a point, fusion of two rings, or reversal of a ring. This circuit algebra is equipped with operations, such as unzipping. The authors introduce the associated graded structures (Section 4.2). The first main result of the paper is *an equivalence between homomorphic expansions for "w-foams" and solutions of the KV equations (Thm. 4.9).* In Section 4.5, the authors introduce unoriented versions of w-foams and construct *an equivalence between homomorphic expansions for those and solutions of the KV equations with even Duflo functions (Thm. 4.11).* The author relate w-tangled foams to sKTGs by constructing a morphism from sKTGs to w-tangled foams. They go on to *construct a sequence of equivalences (horizontal chord associators)→(hom. expansions of sKTGs)→(hom. expansions of w-tangled foams) →(symmetric solutions of KV), where the last map corresponds to the latter result.*

The last result therefore gives a map (horizontal chord associators)→(symmetric solutions of KV). An algebraic (with some ingredients of braid groups) construction of such a map is the main result from (Alekseev-Enriquez-Torossian (2010)). The authors announce the identification between the two maps for a forthcoming preprint.

0.3. **Evaluation and recommendation.** After the simultaneous developments of 3-dimensional finite type invariants and associators theory, it appeared as a reasonable idea that the subjects of 3-dimensional topological invariants and algebraic problems related to associator theory should be interrelated. To the referee's knowledge, one of the first concrete constructions of a bridge between these subjects was Haviv's thesis (under the supervision of the first author). The paper [WKO1] and the paper under review develop this bridge further.

More precisely, the paper under review establishes a connection between topology and the KV problem (modulo the restriction that the topological meaning of w-foams is not fully understood, see first paragraph of 0.2). One of its main tools is the notion of circuit algebra, which is used as a bridge between topology and algebra. This tool should prove useful in the next steps contemplated by the authors.

The paper under review both consists in a substantial step in the program of connecting algebraic problems involving associators with topology, and introduces new and natural methods in this subject. I think that the level and quality of this paper should make it suitable for publication in Math. Annalen.

0.4. **Suggestions.** The quality of the redaction should be rated as good. Here are nevertheless some minor comments.

1) As a comment on terminology, I should say that the term "projectivization" refers to "making things projective" and therefore does not seem well-suited for describing a process of "making things graded". There is also a possible undesirable confusion with the the theme of the schemes Proj(A) associated to a commutative algebra A graded by nonnegative integers. I would therefore recommend replacing "projectivization" by a term more suggestive of the meaning "making things graded"; one might think of using the root "grad-" for this purpose.
2) p. 10, l. 4, by "pairing" the authors mean "involution without fixed point".
3) p. 11, Section 3.1. It would be useful to identify explicitly the objects presented here (the circuit algebras vT, wT) with objects presented in the first paper [WKO1]. It seems that this should require some work since the objects from [WKO1] were not presented in the language of circuit algebras.

Once this is done, it would be good relating explicitly the invariant Z from Thm. 3.6 with the invariants which were introduced in [WKO1].

4) p. 13, Def. 3.4, l. 41-42. The authors imply that $(\overrightarrow{4T}, TC) = (6T, TC)$. Could they give a proof of this fact?
5) p. 16, Def. 3.10, I do not understand the definition of $vT(\uparrow_n), wT(\uparrow_n)$. The formalism developed up to that point says that elements of the various $\mathcal{A}^?$ have skeleta, whereas the elements of vT, wT do not. In any case, it does not seem that the spaces $vT(\uparrow_n), wT(\uparrow_n)$ are used anywhere in the text.
6) p. 56, references, line 39 seems useless.