

**REPORT ON ‘FINITE TYPE INVARIANTS OF w -KNOTTED OBJECTS I:
 w -KNOTS AND THE ALEXANDER POLYNOMIAL’ BY D. BAR-NATAN AND
Z. DANCZO**

1. OVERVIEW OF THE PAPER

I gave a short overview of the reviewed paper ‘Finite type invariants of w -knotted objects I: w -knots and the Alexander polynomial’ by D. Bar-Natan and Z. Dancso in my first report. So let me just summarize the paper:

- The paper is about w -knotted objects, i.e. combinatorial and topological objects akin to usual knots and links.
- The paper uses an approach motivated from finite type invariants and connects in this way these w -knotted objects to questions in universal Lie theory and related topics, e.g. finite dimensional, commutative Lie bi-algebras and associated graded spaces.
- The paper is the first of a sequence of papers which study these relations in more detail. In particular, the authors demonstrate how their approach can be used to recover the (well-known) Alexander polynomial of (long) w -knots. In a sequel, the authors relate w -knotted objects to solutions of the Kashiwara-Vergne equation using a similar approach as motivated in this paper.
- The paper has a relatively long introductory part making it reasonably self-contained.

2. SUMMARY, OPINION AND POTENTIAL ISSUES

The paper ‘Finite type invariants of w -knotted objects I: w -knots and the Alexander polynomial’ is a perfect fit for ‘Algebraic & Geometric Topology’:

- The results in this paper are smart reformulations of classical construction in the theory of finite type invariants and (left aside that it is the first paper of a sequence with very good results) is useful and interesting in its own right.
- The paper is interesting for potential readers of ‘Algebraic & Geometric Topology’ because it relates topological and algebraical questions in a new way.
- It addresses an even wider audience due to its connection to universal Lie theory.
- The paper is mostly well written and easy to read.
- The paper is the first of a sequence of papers providing a new approach to solve algebraic questions by using topological reasoning. The follow-up papers contain even stronger results and the whole sequence will be potentially very influential in the future.

One potential issue should be mentioned:

- The paper is relatively long and still a lot of (easy) arguments are left to the reader (for example in form of exercises) or are only sketched.

Still: I highly recommend the paper for publication in ‘Algebraic & Geometric Topology’ if the authors are willing to address my last (small) concerns below.

3. LIST OF TYPOS, QUESTIONS AND OTHER ISSUES

I have not checked all of the mathematical details again (they have no changed since my first report). Most of the suggestions made in my first report were addressed by the authors.

Here is a new list of typos, comments and questions that I spotted while re-reading the paper.

- (1) Page 1, Abstract, third paragraph: **Suggestion:** reformulate the sentence about the Alexander polynomial (the reader is aware that the details are inside).
- (2) Page 2, table of contents: “acknowledgement” should be “acknowledgements”.
- (3) Page 2, Subsection 1.1: **Suggestion:** remove the citation to [Dr1]. The notion “quantum group” was around before Drinfeld and if you cite him, then you should cite e.g. Jimbo as well. Drinfeld is associated with quantum groups anyway – no need to cite him if you are not giving a historical account (and you do not).
- (4) Page 2, Subsection 1.1: **Suggestion:** it is dangerous to say “even more elegantly” and citing your own paper in the same sentence.
- (5) Page 3, Subsection 1.2: **Suggestion:** the footnote 4 should be slightly replaced since it could be read as Z^4 .
- (6) Page 3, Subsection 1.2, first bullet point: this would be the perfect place to put the description that I would remove from the abstract (see in the next subsection below).
- (7) Page 4, Figure 1: the word “combinatorics” is slightly too long and does not fit into the boundary box.
- (8) Page 5, Subsection 1.3: ordinary knot theory **change it to** classical knot theory.
- (9) Page 5, Subsection 1.4: **Suggestion:** it is dangerous to cite a page (although you do it with a label), since pages might vary in different versions. You have a table of context, so no need to cite pages.
- (10) Page 5, Subsection 1.4: add “Subsection” before specific citations. Moreover, what you call sections are more subsections. Furthermore, sometimes you write “Sec.” and sometimes “Section” – stick with one convention (this happens throughout).
- (11) Page 7, first bullet point: put “-” around strand i over strand $i + 1$ (as on the page before).
- (12) Page 7, Equation (4): the “and” has the wrong size.
- (13) Page 7, near the end of the page: sequence **change it to** short exact sequence. Consequently, on the next page just say splits.
- (14) Page 7, Footnote (9): remove the “-” around a group.
- (15) Page 8, Proposition 2.3: **Suggestion:** if you cite a specific result (here [Ba]), then you should cite explicitly (here [Ba, Theorems 1 and 2]).
- (16) Page 8, bottom: remove (-) around OC – this seems to be your convention now.
- (17) Page 9, after Theorem 2.6: **Suggestion:** cite explicitly.
- (18) Page 9, Footnote (12): missing end of sentence.
- (19) Page 10, begin of Subsection 2.2.3: non-Abelian **change it to** non-abelian.
- (20) Page 11, after Equation (11): **Suggestion:** cite explicitly.
- (21) Page 11, Equation (12): separate using commas.
- (22) Page 11, close to the bottom of the page: try avoid overlapping math mode.
- (23) Page 11, after Equation (13): **Question:** I do not understand your one-sided inverse. What can I do if the $n + 1$ -th strand wiggles around the diagram? In the group ring you could set it to zero, but you are talking about the group here.
- (24) Page 12, third bullet point: Thus let... **change it to** Thus, let...
- (25) Page 12, third bullet point: Abelian **change it to** abelian
- (26) Page 12, fourth bullet point: “Undercrossings Commute” **change it to** “undercrossings commute” (as before).
- (27) Page 12, Subsection 2.3: **Suggestion:** do not use obvious.
- (28) Page 13, Figure 2: **Suggestion:** a more generic example would be to flip one of the crossings/arrows in the left picture.
- (29) Page 13, after Figure (3): Abelian group **change it to** abelian group.
- (30) Page 14, below Figure (4): “Tails Commute” (TC) **change it to** “tails commute” (TC).

- (31) Page 14, close to the end of the page: **Comment:** since 1 January 2011 the official name is Aarhus and not Århus.
- (32) Page 14, close to the end of the page: **Question:** the augmentation ideal can be defined over \mathbb{Z} if one wishes to. Where do you specifically need a field? It gets slightly messy when you want to view it as a functor into algebras though.
- (33) Page 15, middle of the page: put the induced map $\text{gr } \Phi$ into an extra line.
- (34) Page 17, Remark 2.16: **Suggestion:** the YB equation is usually written using subscripts and not superscripts.
- (35) Page 18, Proof of Theorem 2.18: Follows... **change it to** This follows...
- (36) Page 18, Proof of Theorem 2.18: the \square is missing.
- (37) Page 19, Remark 2.19: do not use obvious.
- (38) Page 19, Proof of Proposition 2.20: **Suggestion:** here you use \mathbb{Q} instead of \mathbb{F} as before. Maybe it is good to say in a sentence why. Actually, I do not see where this is really necessary in the proof.
- (39) Page 20, middle of the page: ...labelling... **change it to** ...labeling... (BE vs. AE).
- (40) Page 21, end of the page: Reference missing.
- (41) Page 22, Definition 3.1: Overcrossings Commute (OC) **change it to** “overcrossings commute” (OC) – or write just OC.
- (42) Page 22, Definition 3.1: Undercrossings Commute (OC) **change it to** “undercrossings commute” (OC) – or write just UC.
- (43) Page 22, Definition and Warning 3.2: ordinary knots **change it to** usual knots.
- (44) Page 22, Definition and Warning 3.3: Abelian **change it to** abelian.
- (45) Page 22, Figure 7: **Question:** what is the $\#$ -inverse?
- (46) Page 24, Remark 3.7: **Suggestion:** cite explicitly to the result of Joyce.
- (47) Page 25, before Subsection 3.2: “overcrossings commute” **change it to** OC.
- (48) Page 25, begin of Subsection 3.2: Abelian group **change it to** abelian group.
- (49) Page 25, end of the page: **Suggestion:** the sentence “An example is given (shown) in Figure 9.” appears twice. Remove or reformulate one.
- (50) Page 25, end of the page: Rotation number Independence **change it to** “rotation number independence”.
- (51) Page 26, after Figure 10: “Tails Commute” (TC) **change it to** TC.
- (52) Page 26, end of the page: **Suggestion:** maybe \mathcal{A}^* as before?
- (53) Page 26, end of the page: do not use obvious.
- (54) Page 27, after Proposition 2.9: **Suggestion:** cite explicitly to the Milnor-Moore theorem. Alternatively, since it is “well-known”, do not cite them at all.
- (55) Page 27, begin of Subsection 3.3: ordinary knots **change it to** usual knots.
- (56) Page 27, begin of Subsection 3.3: do not use obvious.
- (57) Page 27, bottom of the page: ...compare with (15)... **change it to** ...compare with Equation (15)...
- (58) Page 28, begin of Subsection 3.4: ordinary knots **change it to** usual knots.
- (59) Page 28, Footnote 27: **Suggestion:** Here you emphasize characteristic zero. Maybe start from the beginning only with \mathbb{Q} instead of \mathbb{F} .
- (60) Page 31, Case III: **Suggestion:** the double appearance of “case III” looks weird.
- (61) Page 31, near the end of the page: **Question:** what is relation FI?
- (62) Page 32, begin of the page: do not use obvious.
- (63) Page 32, end of the proof: **Suggestion:** “but sometimes it takes up more space” – oh, come one. Your paper has 52 pages. How can a picture hurt here?
- (64) Page 32, end of the page: “feel” **change it to** “feeling”.

- (65) Page 33, begin of the page: “Commutators Commute” **change it to** “commutators commute”. This appears twice.
- (66) Page 33, begin of Subsection 3.5: [BN1, Theorem 4, Exercise 5.1] **change it to** [BN1, Theorem 4 and Exercise 5.1].
- (67) Page 33, Preliminaries: **Suggestion:** the field question again. I would suggest to say it once and for all at the first appearance (augmentation ideal).
- (68) Page 33, Preliminaries: Abelian **change it to** abelian.
- (69) Page 33, Preliminaries: $so(3)$ **change it to** $\mathfrak{so}(3)$ or \mathfrak{so}_3 . This appears three times.
- (70) Page 34, begin of the page: do not use obvious.
- (71) Page 34, end of the page: careful with the placement of Footnote 30 since it almost looks like $\{i, j, k, \dots\}^{30}$.
- (72) Page 35, first sentence: [BN1, Theorem 4, Exercise 5.1] **change it to** [BN1, Theorem 4 and Exercise 5.1].
- (73) Page 35, before Example 3.5.3: Abelian **change it to** abelian.
- (74) Page 36, after Theorem 3.25: **Suggestion:** cite explicitly the theorem of Lee.
- (75) Page 37, begin of the page: (21) **change it to** Equation (21).
- (76) Page 37, begin of the page: (22) **change it to** Equation (22).
- (77) Page 37, Remark 3.27: **Suggestion:** cite explicitly the results you refer to.
- (78) Page 37, begin of Bulk management: (24) **change it to** Equation (24).
- (79) Page 38, begin of the page: (25) **change it to** Equation (25).
- (80) Page 38, twice on this page: (22) **change it to** Equation (22).
- (81) Page 39, Figure 16: **Suggestion:** later you use r instead of red (as in this picture). You should do it consistent – so use r here as well.
- (82) Page 42, Lemma 3.35: avoid overlapping math mode.
- (83) Page 42, Lemma 3.36: commas missing in the list of equations.
- (84) Page 43, middle of the page: “a major headache” – yes, it is.
- (85) Page 44, middle of the page: formula (22) **change it to** Equation (22) or Formula (22).
- (86) Page 44, middle of the page: (22) **change it to** Equation (22) or Formula (22).
- (87) Page 44, Footnote 34: “Framing Independence” **change it to** “framing independence”.
- (88) Page 44, Footnote 34: **Problem:** you introduce FI too late – you use it before.
- (89) Page 44, Footnote 34: first Reidemeister move **change it to** RI move.
- (90) Page 45, Comments (3): ...as we shall see... **change it to** ...as we have seen...
- (91) Page 45, Comments (6): $gl(N)$ **change it to** $\mathfrak{gl}(N)$ or \mathfrak{gl}_N .
- (92) Page 45, Subsection 4.2: **Suggestion:** you created a loop with the reference to Footnote 3. You explain it here, so no need to cite Footnote 3 again.
- (93) Page 46, Name of Subsection 4.3: **Suggestion:** “up to degree 2” sounds better I think.
- (94) Page 46, end of the page: do not use obvious.

4. GENERAL REMARKS

Some general comments.

- The abstract is still long and complicated in my humble opinion. Try to shorten it – it would improve the readability. For example, an abstract is not an introduction and I would get rid of everything in between the first sentence and the paragraph that starts with “In this article...”. Moreover, cut the last paragraph of the abstract and move it into the introduction instead.
- Why do you display some names of papers in blue? For example, [FRR] is blue, but [EK] is not.

- There are a lot of older pre-prints in your list of references. Have you checked if some of them have appeared yet? For example, [BH] has appeared: T.E. Brendle and A. Hatcher, Configuration spaces of rings and wickets, *Comment. Math. Helv.* 88 (2013), no. 1, 131-162.