110704 Notes

July-04-11
10:41 AM

We now wish to identify $\mathcal{P}(\uparrow_n)$ as the Lie algebra $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, which in itself is a combination of the Lie algebras $\mathfrak{a}_n$, $\mathfrak{tder}_n$ and $\mathfrak{tr}_n$ studied by Alekseev and Torossian [AT]. AlekseevTorossi
Here are the relevant definitions:

Definition 5.8. Let $\mathfrak{a}_n$ denote the vector space with basis $x_1, \ldots, x_n$, also regarded as an Abelian Lie algebra of dimension n. As before, let $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ denote the free Lie algebra on n generators, now identified as the basis elements of $\mathfrak{a}_n$. Let $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ be the Lie algebra of derivations acting on $\mathfrak{lie}_n$, and let

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

denote the subalgebra of "tangential derivations". A tangential derivation D is determined by the a_i's for which $D(x_i) = [x_i, a_i]$, and determines them up to the ambiguity $a_i \mapsto a_i + \alpha_i x_i$, where the α_i's are scalars. Thus as vector spaces, $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_{i=1}^n \mathfrak{lie}_n$.

Definition 5.9. Let $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ be the free associative algebra "of words", and let Ass_n^+ be the degree > 0 part of Ass_n. As before, we let $\mathfrak{tr}_n = \mathrm{Ass}_n^+ / (x_{i_1} x_{i_2} \cdots x_{i_m} = x_{i_2} \cdots x_{i_m} x_{i_1})$ denote "cyclic words" or "(coloured) wheels". Ass_n, Ass_n^+, and $\mathfrak{tr}_n$ are $\mathfrak{tder}_n$-modules and there is an obvious equivariant "trace" $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n$.

1. longer arrows
2. Give maps names

Pnses **Proposition 5.10.** *There is a short exact sequence of Lie algebras*

$$0 \to \mathfrak{tr}_n \to \mathcal{P}(\uparrow_n) \to \mathfrak{a}_n \oplus \mathfrak{tder}_n \to 0.$$

Proof. We note that the first map is well-defined: due to TC, it does not matter in what order the tails of a wheel are placed on their designated strands. how is it defined?

As vector spaces, the statement is already proven: $\mathcal{P}(\uparrow_n)$ is generated by trees and wheels (with the legs fixed on n strands). When factoring out by the wheels, only trees remain. Trees have one head and many tails. All the tails commute with each other, and commuting a tail with a head on a strands costs a wheel (by $\overrightarrow{STU}$), thus in the quotient the head also commutes with the tails. Therefore, the quotient is the space of floating (colored) trees, which we have previously identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n \cong \mathfrak{a}_n \oplus \mathfrak{tder}_n$. coloured

It remains to show that the maps are Lie algebra maps as well. For the first map this is easy: the Lie algebra $\mathfrak{tr}_n$ is commutative, and is mapped to the commutative (due to TC) subalgebra of $\mathcal{P}(\uparrow_n)$ generated by wheels.

To show that the second map is a map of Lie algebras, we give two proofs, first a "hands-on" one, then a "conceptual" one.

sp

Hands-on argument. $\mathfrak{a}_n$ is the image of sinle arrows on one strand. These commute with everything in $\mathcal{P}(\uparrow_n)$, and so does $\mathfrak{a}_n$ in the direct sum.

It remains to show that the bracket of $\mathfrak{tder}_n$ works the same way as commuting trees in $\mathcal{P}(\uparrow_n)$. Let D and D' be elements of $\mathfrak{tder}_n$ represented by $(a_1, \ldots, a_n)$ and $(a'_1, \ldots, a'_n)$, meaning that $D(x_i) = [x_i, a_i]$ and $D'(x_i) = [x_i, a'_i]$ for $i = 1, \ldots, n$. Let us compute the commutator of these elements:

$$[D, D'](x_i) = (DD' - D'D)(x_i) = D[x_i, a'_i] - D'[x_i, a_i] =$$
$$= [[x_i, a_i], a'_i] + [x_i, Da'_i] - [[x_i, a'_i], a_i] - [x_i, D'a_i] =$$
$$= [x_i, Da'_i] - [x_i, D'a_i] + [x_i, [a_i, a'_i]] = [x_i, Da'_i - D'a_i + [a_i, a'_i]].$$

Here the third equality is due to the Leibniz rule of drivations, while the fourth is a Jacobi identity.

If we had named the maps, we could use them here.

Now let T and T' be two trees in $\mathcal{P}(\uparrow_n)/\mathfrak{tr}_n$, their heads on strands i and j, respectively (i may or may not equal j). Let us call the element in $\mathfrak{lie}_n$ given by forming the appropriate commutator of the colors of T's leaves a_i, and similarly a'_j for T'. In $\mathfrak{tder}_n$, T and T' are mapped to the elements D and D' detemined by $(0, ..., a_i, ..., 0)$, and $(0, ..., a'_j, ...0)$, respectively. (In each case, the $i-th$ or the $j-th$ is the only non-zero component. The commutator of these elements is given by $[D, D'](x_i) = [Da'_i - D'a_i + [a_i, a'_i], x_i]$, and $[D, D'](x_j) = [Da'_j - D'a_j + [a_j, a'_j], x_j]$. Note that unless $i = j$, $a_j = a'_i = 0$.

In $\mathcal{P}(\uparrow_n)/\mathfrak{tr}_n$, all tails commute, as well as a head of a tree with its own tails. Therefore, commuting two trees only incurs a cost when commuting a head of one tree over the tails of the other on the same strand, and the two heads over each other, if they are on the same strand.

If $i \neq j$, then commuting the head of T over the tails of T' by $\overrightarrow{STU}$ costs a sum of trees given by Da'_j, with heads on strand j, while moving the head of T' over the tails of T costs exactly $-D'a_i$, with heads on strand i, as needed.

If $i = j$, then everything happens on starnd i, and the cost is $(Da'_i - D'a_i + [a_i, a'_i])$, where the last term is what happens when the two heads cross each other.

Conceptual argument. There is an action of $\mathcal{P}(\uparrow_n)$ on $\mathfrak{lie}_n$, the following way: introduce and extra strand on the right. An element of $\mathfrak{lie}_n$ corresponds to a tree with its head on the extra strand. The commutator of an element of $\mathcal{P}(\uparrow_n)$ (considered as an element of $\mathcal{P}(\uparrow_{n+1})$ by the obvious inclusion) is again a tree with head on strand $(n+1)$, defined to be the result of the action. picture?

The tree we are acting on has only tails on the first n strands, so elements of $\mathfrak{tr}_n$, which also only have tails, act trivially. So do single (local) arrows on one strand ($\mathfrak{a}_n$). It remains to show that trees act as $\mathfrak{tder}_n$, and it's enough to check this on the generators of $\mathfrak{lie}_n$ (as the Leibnitz rule is obviously satisfied). Generators of $\mathfrak{lie}_n$ are arrows pointing from one of the first n strands, say strand i, to strand $(n+1)$. A tree with head on strand i acts on this element, according $\overrightarrow{STU}$, by forming the commutator, which is exactly the action of $\mathfrak{tder}_n$.
□

To identify $\mathcal{P}(\uparrow_n)$ as the semidirect product $\mathfrak{tr}_n \rtimes (\mathfrak{a}_n \oplus \mathfrak{tder}_n)$, it remains to show that the short exact sequence above splits. This is indeed the case, although not canonically. Let us call the inclusion and quotient maps of the short exact sequence ι and π, respectively. Two —of the many— splitting maps $u, l : \mathfrak{tder}_n \oplus \mathfrak{a}_n \to \mathcal{P}(\uparrow_n)$ are described as follows: move way up? $tder_n \oplus \mathfrak{a}_n$ is identified with $\bigoplus_{i=1}^n \mathfrak{lie}_n$, which in turn is identified with floating (coloured) trees (including arrows). A map to $\mathcal{P}(\uparrow_n)$ can be given by specifying how to place the legs on thir specified strands. A tree may have many tails but has only one head, and due to TC, only the positioning of the head matters. Let u (for *upper*) be the map placing the head of each tree above all its tails on the same strand, while l (for *lower*) places the head below all the tails. It is obvious that these are both Lie algebra maps and that $\pi \circ u$ and $\pi \circ l$ are both the identity of $\mathfrak{tder}_n \oplus \mathfrak{a}_n$. This makes $\mathcal{P}(\uparrow_n)$ a semidirect product. □ Pict?

Definition 5.11. For any $D \in \mathfrak{tder}_n$, $(l-u)D$ is in the kernel of π, therefore is in the image of ι, so $\iota^{-1}(l-u)D$ makes sense. AT call this element ~~DIVD~~ div D

Proof?

MORE NOW.
MORE.

c:sder

5.3. **The Relationship with u-Tangles.** MORE.