Some Dimensions of Arrow Diagram Spaces

By Dror Bar-Natan; with some code borrowed from a joint project with Louis Leung, Arrow_Diagrams_and_gl (N).

Pensieve Header: Same as Dimensions.nb, but on a computer with more RAM.

Program

Diagrams

SetAttributes[Diag, Orderless];
Place[[ar], {i, j}] := {Diag[ar[i, j]], Diag[ar[j, i]]};
Place[[ar, objs__], {i, rest__}] := Flatten[Table[
   Outer[Join,
      Place[[ar], {i, rest}[[k]]],
      Place[[objs], Delete[{rest}, k]]
   ],
   {k, Length[{rest}]}]];
Diagrams[k_.*ar] := Place[Table[ar, {k}], Range[2 k]];

The Circle

DiagramRotateLeft[d_Diag] := Module[
   {labels = Union @@ (List @@ Apply[List, d, {1}])},
   d /. Thread[Rule[labels, RotateLeft[labels]]]
];
DiagramRotateToMinimal[expr_] := expr /. d_Diag -> Module[
   {bd = d, rd = DiagramRotateLeft[d]},
   While[rd != d,
      bd = First[Sort[{bd, rd}]];
      rd = DiagramRotateLeft[rd];
   bd
 ];
ClosedDiagrams[dir_] := ClosedDiagrams[dir] = Union[DiagramRotateToMinimal[Diagrams[dir]]]

Relators

Place[{r : (TC | R6T), objs__}, {i, rest__}] := Flatten[Table[
   Outer[Join,
      Place[{r}, {i, rest}[[j]], {rest}[[k]]],
      Place[{objs}, Delete[{rest}, {{j}, (k)}]]
   ],
   {k, 2, Length[{rest}]], {j, 1, k - 1}]];
Place[(R6T), {i_, j_, k_}] :=
    Permutations[{i, j, k}] / (i1_, j1_, k1_) := Diag[R6T[i1, j1, k1]];
Diagrams[R6T] := Place[(R6T), {1, 2, 3}];
Diagrams[R6T + k_. * ar] /; k > 0 := Flatten[
    Place[#, Range[2 k + 3]] & / Permutations[Table[#, {k}] Append - R6T]
]
Diagrams[R6T + k_. * ar] /; k < 0 := {};
Place[(TC), {i_, j_, k_}] := Diag @@ {TC[i, j, k], TC[j, k, i], TC[k, i, j]};
Diagrams[TC] := Place[(TC), {1, 2, 3}];
Diagrams[TC + k_. * ar] /; k > 0 := Flatten[
    Place[#, Range[2 k + 3]] & / Permutations[Table[#, {k}] Append - TC]
]
Diagrams[TC + k_. * ar] /; k < 0 := {};

Framing Independence

DiagramsSansFI[specs_] := Select[Diagrams[specs],
    FreeQ[#, ar[i_, j_] /; Abs[i - j] = 1] &
];
ClosedDiagramsSansFI[specs_] := Select[ClosedDiagrams[specs],
    FreeQ[#, ar[i_, j_] /; Abs[i - j] = 1] &
]

Relations

NormalizeDiag[diag_Diag] := Module[
    {indices = Union @@ (List @@ diag /. ar -> List),
        diag /. Thread[indices -> Range[Length[indices]]]
    ];
R[Diag[lft___, R6T[i_, j_, k_], rgt___]] := (
    +NormalizeDiag[Diag[lft, ar[i, j], ar[i + 0.5, k], rgt]]
    +NormalizeDiag[Diag[lft, ar[i, j], ar[j + 0.5, k], rgt]]
    +NormalizeDiag[Diag[lft, ar[i, k], ar[j, k + 0.5], rgt]]
    -NormalizeDiag[Diag[lft, ar[i, k], ar[i + 0.5, j], rgt]]
    -NormalizeDiag[Diag[lft, ar[i, j + 0.5], ar[j, k], rgt]]
    -NormalizeDiag[Diag[lft, ar[i, k + 0.5], ar[j, k], rgt]]
); (* oops, TC is HC; needs fixing *)
R[Diag[lft___, TC[i_, j_, k_], rgt___]] := (
    +NormalizeDiag[Diag[lft, ar[i, k], ar[j, k + 0.5], rgt]]
    -NormalizeDiag[Diag[lft, ar[i, k + 0.5], ar[j, k], rgt]]
);
RSansFI[d_Diag] := (R[d] /. Diag[lft___, ar[i_, j_], rgt___] / Abs[i - j] = 1 : 0)
Dimensions

\[ \text{DimAvLong}[\text{\_}] / \text{\_} < 2 := \text{Length}[\text{Diagrams}[\text{\_}\text{ar}]]; \]
\[ \text{DimAvLong}[\text{\_}] / \text{\_} \geq 2 := \text{Module}[] \]
\[ \{\text{diags, rels, mat, rel, i},\}
\[ \text{diags} = \text{Diagrams}[\text{\_}\text{ar}]; \]
\[ \text{rels} = \text{R} / \text{\_}\text{Diagrams}[\text{\_}6\text{T} + (\text{\_} - 2) \text{ar}]; \]
\[ \text{mat} = \text{SparseArray}[] \]
\[ \text{Join} @@ \text{SparseArray}[\]
\[ \text{rel} = \text{rels}[[i]]; \]
\[ \{i, \text{Position}[\text{diags}, \text{\_}][[1, 1]]\} \rightarrow \text{Coefficient}[\text{rel}, \text{\_}] \& / @ \]
\[ \text{Cases}[\{\text{rel}, \text{\_}\text{Diag}, \text{Infinity}\}, \]
\[ \{i, \text{Length}[\text{rels}]\} \]
\[ ]; \]
\[ \text{Length}[\text{diags}] - \text{MatrixRank}[\text{mat}] \]
\];
\[ \text{DimAvrLong}[\text{\_}] / \text{\_} < 2 := \text{Length}[\text{DiagramsSansFI}[\text{\_}\text{ar}]]; \]
\[ \text{DimAvrLong}[\text{\_}] / \text{\_} \geq 2 := \text{Module}[] \]
\[ \{\text{diags, rels, \text{\text{\_}}\text{diagtoindex}, mat, rel, i},\}
\[ \text{diags} = \text{DiagramsSansFI}[\text{\_}\text{ar}]; \]
\[ \text{rels} = \text{RSansFI} / @ \text{DiagramsSansFI}[\text{\_}6\text{T} + (\text{\_} - 2) \text{ar}]; \]
\[ \text{diagtoindex} = \text{Dispatch}[@\text{Thread}[\text{Rule}[\text{diags}, \text{Range}[\text{Length}[\text{diags}]]]]]; \]
\[ \text{mat} = \text{SparseArray}[] \]
\[ \text{Join} @@ \text{Table}[\]
\[ \text{rel} = \text{rels}[[i]]; \]
\[ \{i, \text{\_}/. \text{\text{\_}}\text{diagtoindex} \rightarrow \text{Coefficient}[\text{rel}, \text{\_}] \& / @ \]
\[ \text{Cases}[\{\text{rel}, \text{\_}\text{Diag}, \text{Infinity}\}, \]
\[ \{i, \text{Length}[\text{rels}]\} \]
\[ ]; \]
\[ \text{Length}[\text{diags}] - \text{MatrixRank}[\text{mat}] \]
\];
\[ \text{DimAvClosed}[\text{\_}] / \text{\_} < 2 := \text{Length}[\text{ClosedDiagrams}[\text{\_}\text{ar}]]; \]
\[ \text{DimAvClosed}[\text{\_}] / \text{\_} \geq 2 := \text{Module}[] \]
\[ \{\text{diags, rels, mat, rel, i},\}
\[ \text{diags} = \text{ClosedDiagrams}[\text{\_}\text{ar}]; \]
\[ \text{rels} = \text{DiagramRotateToMinimal}[\text{R} / @ \text{ClosedDiagrams}[\text{\_}6\text{T} + (\text{\_} - 2) \text{ar}]]; \]
\[ \text{mat} = \text{SparseArray}[] \]
\[ \text{Join} @@ \text{Table}[\]
\[ \text{rel} = \text{rels}[[i]]; \]
\[ \{i, \text{\_}/. \text{\text{\_}}\text{diagtoindex} \rightarrow \text{Coefficient}[\text{rel}, \text{\_}] \& / @ \]
\[ \text{Cases}[\{\text{rel}, \text{\_}\text{Diag}, \text{Infinity}\}, \]
\[ \{i, \text{Length}[\text{rels}]\} \]
\[ ]; \]
\[ \text{Length}[\text{diags}] - \text{MatrixRank}[\text{mat}] \]
\[\text{DimAvrClosed}[m_] \; \text{if} \; m < 2 \; \text{:=} \; \text{Length}[\text{ClosedDiagramsSansFI}[m \text{ar}]];\]
\[\text{DimAvrClosed}[m_] \; \text{if} \; m \geq 2 \; \text{:=} \; \text{Module}[
\{\text{diags, rels, mat, rel, i}\},
\text{diags} = \text{ClosedDiagramsSansFI}[m \text{ar}];
\text{rels} = \text{Union}[
\text{DiagramRotateToMinimal}[\text{RSansFI} \; @ \; \text{ClosedDiagramsSansFI}[\text{R6T} + (m - 2) \; \text{ar}]] \; /.
\text{Diag}[\text{lft____}, \; \text{ar[i, j]}, \; \text{rgt____}] \; /; \; \text{Abs}[i - j] = 1 \; \rightarrow \; 0
];
\text{mat} = \text{SparseArray}[
\text{Join} @@ \text{Table}[
\text{rel} = \text{rels}[i];
\{i, \text{Position[diags, #]}[[1, 1]]\} \rightarrow \text{Coefficient[rel, #]} \; & \; /@
\text{Cases}[\{\text{rel}, \; \text{diag_Diag}, \; \text{Infinity}\},
\{i, \text{Length[rels]}\}]
],
\{\text{Length[rels]}, \; \text{Length[diags]}\}
];
\text{Length[diags]} - \text{MatrixRank[mat]}
];\]

\[\text{DimAwLong}[m_] \; \text{if} \; m < 2 \; \text{:=} \; \text{Length}[\text{Diagrams}[m \text{ar}]];\]
\[\text{DimAwLong}[m_] \; \text{if} \; m \geq 2 \; \text{:=} \; \text{Module}[
\{\text{diags, rels, mat, rel, i}\},
\text{diags} = \text{Diagrams}[m \text{ar}];
\text{rels} = \text{R} \; @ \; \text{Join}[\text{Diagrams}[\text{R6T} + (m - 2) \; \text{ar}], \; \text{Diagrams}[\text{TC} + (m - 2) \; \text{ar}]];\]
\text{mat} = \text{SparseArray}[
\text{Join} @@ \text{Table}[
\text{rel} = \text{rels}[i];
\{i, \text{Position[diags, #]}[[1, 1]]\} \rightarrow \text{Coefficient[rel, #]} \; & \; /@
\text{Cases}[\{\text{rel}, \; \text{diag_Diag}, \; \text{Infinity}\},
\{i, \text{Length[rels]}\}]
],
\{\text{Length[rels]}, \; \text{Length[diags]}\}
];
\text{Length[diags]} - \text{MatrixRank[mat]}
];
\textbf{From the Dimensions of the Primitives to the Total Dimensions}

\texttt{PrimitivesToAll[p_List] := Module[{x},}
\texttt{CoefficientList[Series[}
\texttt{Product[(1-x^i)^(-\text{p}[i])}, \texttt{\{i, Length[\text{p}]\}],}
\texttt{\{x, 0, Length[\text{p}]\}], x]}
\texttt{]}

\textbf{Results}

\texttt{PrimitivesToAll[{1, 1, 1, 2, 3, 5, 8}]}
\{1, 1, 2, 3, 6, 10, 19, 33\}

\texttt{Timing[DimAvLong /@ \{1, 2, 3, 4\}}
\{28.8066, \{2, 7, 27, 139\}\}

\texttt{PrimitivesToAll[{2, 4, 15, 82}]}
\{1, 2, 7, 27, 139\}

\texttt{Timing[DimAvLong[5]]}

No more memory available.
Mathematica kernel has shut down.
Try quitting other applications and then retry.

\texttt{Timing[DimAvLong /@ \{1, 2, 3, 4\}}
\{32.665, \{2, 4, 7, 12\}\}

\texttt{PrimitivesToAll[{2, 1, 1, 1, 1, 1, 1}]}
\{1, 2, 4, 7, 12, 19, 30, 45\}
PrimitivesToAll[{0, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 0, 1, 1, 2, 2, 4, 4, 7, 8, 12, 14, 21}]

Timing[DimAvrLong /@ {1, 2, 3, 4}]
{1.42078, {0, 2, 7, 42}}

Timing[DimAvrLong[5]]
{3195.44, 246}

Timing[DimAvrClosed /@ {1, 2, 3, 4}]
{2.85857, {1, 2, 5, 19}}

Timing[DimAvrClosed[5]]
{577.641, 77}

Timing[DimAwClosed /@ {1, 2, 3, 4}]
{11.2013, {1, 1, 1, 1}}

Timing[DimAwClosed[5]]

No more memory available.
Mathematica kernel has shut down.
Try quitting other applications and then retry.

Timing[DimAvrClosed /@ {1, 2, 3, 4, 5}]
{104.895, {0, 0, 1, 4, 17}}