Pensieve header: Analysis of the Kashaev Invariant..

In[ ]:=
```
A = (v u 1 u
     u 1 u 1
     1 u v u
     u 1 u 1) /. v → 2 u^2 - 1
```

Out[ ]=
```
{{-1 + 2 u^2, u, 1, u}, {u, 1, u, 1}, {1, u, -1 + 2 u^2, u}, {u, 1, u, 1}}
```

In[ ]:=
```
NullSpace[A]
```

Out[ ]=
```
{{0, -1, 0, 1}, {1, -2 u, 1, 0}}
```

In[ ]:=
```
{u, -1}.NullSpace[A]
```

Out[ ]=
```
{-1, u, -1, u}
```

In[ ]:=
```
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[ ]:=
```
MatrixSignature[A_] := Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
```

In[ ]:=
```
Kas[K_, u_] := Kas[K, u] = Module[{v, XingsByArmpits, bends, faces, p, A, is},
    v = 2 u^2 - 1;
    XingsByArmpits = List @@ PD[K] /.
      x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
    bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
    faces = bends //. p_{x__,y_} p_{y_,z__} :→ p_{x,y,z};
    A = Table[0, Length@faces, Length@faces];
    Do[is = Position[faces, #]⟦1, 1⟧ & /@ List @@ x;
     A⟦is, is⟧ += If[Head[x] === X₊,
        (v u 1 u
         u 1 u 1
         1 u v u
         u 1 u 1), -(v u 1 u
                     u 1 u 1
                     1 u v u
                     u 1 u 1)],
     {x, XingsByArmpits}];
    A];
```

In[∘]:=
```
MatrixForm[A = Kas[Knot[3, 1], u]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]//MatrixForm=

$$\begin{pmatrix} 2-4u^2 & -1 & -1 & -2u & -2u \\ -1 & 2-4u^2 & -1 & -2u & -2u \\ -1 & -1 & 2-4u^2 & -2u & -2u \\ -2u & -2u & -2u & -3 & -3 \\ -2u & -2u & -2u & -3 & -3 \end{pmatrix}$$

In[∘]:=
```
NullSpace[A] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 \end{pmatrix}$$

In[∘]:=
```
Table[NullSpace[Kas[K, u]] // MatrixForm, {K, AllKnots[{3, 8}]}]
```

Out[∘]=

$$\left\{ \begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -2u & 0 & 1 & 0 & 1 \\ -2u & -1 & 1 & 0 & 1 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2u & 0 & -2u & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & -2u & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & -2u & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & 1 & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & \frac{1}{2u} & -\frac{1}{2u} & 0 & -\frac{1}{2u} & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & -1 & -2u & 1 & 0 & 0 & 1 \\ 1 & 0 & -2u & -1 & 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & 1 & -2u & 0 & 0 & -2u & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & 0 & -\frac{1}{2u} & 1 & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & 1 & -\frac{1}{2u} & 0 & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & -2u & 0 & 0 & -2u & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -2u & 0 & -2u & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & -2u & -2u & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -2u & -2u & -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & -2u & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & -2u & -2u & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -2u & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & 1 & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & \frac{1}{2u} & -\frac{1}{2u} & 0 & -\frac{1}{2u} & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & 1 & -2u & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & -2u & -2u & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 & 1 \\ \frac{1}{2u} & -\frac{1}{2u} & \frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & \frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -2u & 1 & -1 & 0 & 0 & -2u & 0 & 1 & 1 \\ 0 & -1 & 0 & -2u & 1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & 1 & -\frac{1}{2u} & -\frac{-1+4u^2}{2u} & 1 & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & 0 & -\frac{1}{2u} & \frac{1}{2u} & 0 & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -2u & 1 & 0 & 0 & -2u & -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 & -1 & -2u & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & -1 & -2u & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & -2u & -1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & \frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -2u & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ -2u & -1 & 0 & 1 & 0 & 1 & -2u & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1+4u^2 & -2u & -2u & -2u & 1 & 0 & 1 & 0 & 1 \\ -2u & 2u & -1 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2u & 0 & 0 & -2u & -1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 & -2u & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -2u & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -2u & -2u & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2u & 0 & -2u & -2u & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & \frac{1}{2u} & 0 & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & -\frac{-1+4u^2}{2u} & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2u & 1 & -2u & 1 & -2u & 1 & -1+4u^2 & -2u & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 2u & -1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{-1+4u^2} & \frac{1}{-1+4u^2} & -\frac{2u}{-1+4u^2} & -\frac{2u}{-1+4u^2} & \frac{1}{-1+4u^2} & -\frac{2u}{-1+4u^2} & -\frac{2u}{-1+4u^2} & -\frac{1}{1-4u^2} & 0 & 1 \\ -\frac{2u}{-1+4u^2} & -\frac{2u}{-1+4u^2} & \frac{1}{-1+4u^2} & \frac{1}{-1+4u^2} & -\frac{2u}{-1+4u^2} & -\frac{1}{1-4u^2} & -\frac{1}{1-4u^2} & -\frac{2u}{-1+4u^2} & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & -\frac{-1+4u^2}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 1 & 0 & 1 \\ -\frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & \frac{1}{2u} & -\frac{1}{2u} & -\frac{1}{2u} & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2u & 0 & -2u & -2u & 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2u & 0 & -2u & -2u & 1 & 1 & 1 & 0 \end{pmatrix}\Bigg\}$$

In[∘]:=
```
Table[NullSpace[Kas[K, u]] // Length, {K, AllKnots[{3, 12}]}] // Union
```

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

General: Further output of KnotTheory::loading will be suppressed during this calculation.

Out[∘]= {2}

In[∘]:=
```
Table[NullSpace[Kas[K, u]] // Length, {K, AllKnots[{3, 13}]}] // Union
```

Out[∘]= {2}

In[∘]:=
```
Table[Simplify@NullSpace[Kas[K, u]], {K, AllKnots[{3, 13}]}] // Flatten // Union
```

KnotTheory: Loading precomputed data in KnotTheory/13A.dts.

KnotTheory: Loading precomputed data in KnotTheory/13N.dts.

Out[∘]=
$$\left\{-1, 0, 1, \frac{1}{2u} - 2u, -\frac{1}{2u}, \frac{1}{2u}, -2u, 2u, \frac{1}{1-4u^2}, \frac{2u}{1-4u^2}, 1-4u^2, -2+4u^2, \frac{1}{-1+4u^2}, -1+4u^2, 4u-8u^3, \frac{4u-8u^3}{-1+4u^2}\right\}$$

In[•]:=
```
Table[Simplify@NullSpace[Kas[K, u]], {K, AllKnots[{3, 14}]}] // Flatten // Union
```

KnotTheory: Loading precomputed data in KnotTheory/14A.dts.

KnotTheory: Loading precomputed data in KnotTheory/14N.dts.

Out[•]=

$$\left\{-1,\ 0,\ 1,\ \frac{1}{2u}-2u,\ -\frac{1}{2u},\ \frac{1}{2u},\ -2u,\ 2u,\ -\frac{1}{2u}+2u,\ \frac{1}{1-4u^2},\ \frac{2u}{1-4u^2},\ 1-4u^2,\ -2+4u^2,\ \frac{1}{-1+4u^2},\ \frac{2u}{-1+4u^2},\ -1+4u^2,\ 4u-8u^3,\ \frac{4u-8u^3}{-1+4u^2},\ -\frac{1}{2u}+6u-8u^3,\ \frac{1-12u^2+16u^4}{-1+4u^2}\right\}$$

In[•]:=
```
Table[NullSpace[∂_u Kas[K, u]] // MatrixForm, {K, AllKnots[{3, 7}]}]
```

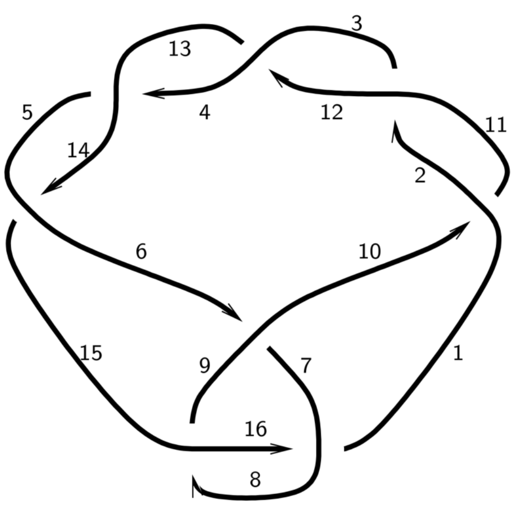
Out[•]=

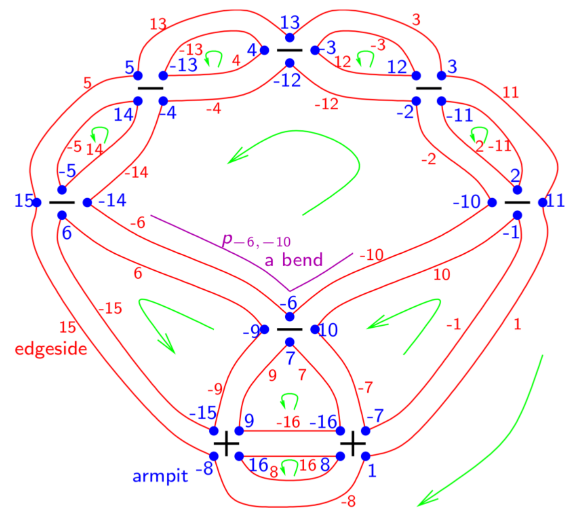
$$\left\{\begin{pmatrix}0&0&0&-1&1\end{pmatrix},\ \{\},\ \begin{pmatrix}0&0&0&0&0&-1&1\end{pmatrix},\ \begin{pmatrix}0&0&0&-1&1&0&0\\0&-1&1&0&0&0&0\end{pmatrix},\right.$$

$$\begin{pmatrix}-1&0&0&1&0&0&0&0\\-1&1&0&0&0&0&0&0\end{pmatrix},\ \{\},\ \{\},\ \begin{pmatrix}0&0&0&0&0&0&0&-1&1\end{pmatrix},\ \begin{pmatrix}0&0&0&0&0&-1&1&0&0\\-1&0&0&0&1&0&0&0&0\\-1&0&0&1&0&0&0&0&0\\-1&0&1&0&0&0&0&0&0\end{pmatrix},$$

$$\left.\begin{pmatrix}0&0&0&0&0&0&0&-1&1\\0&0&-1&0&1&0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&0&0&-1&1&0&0&0\\0&-1&0&1&0&0&0&0&0\\-1&0&1&0&0&0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&0&0&0&-1&0&0&1\\0&0&0&-1&1&0&0&0&0\end{pmatrix},\ \{\},\ \{\}\right\}$$

In[•]:=
```
PD[8₂] = PD[X[10, 1, 11, 2], X[2, 11, 3, 12], X[12, 3, 13, 4],
   X[4, 13, 5, 14], X[14, 5, 15, 6], X[8, 16, 9, 15], X[16, 8, 1, 7], X[6, 9, 7, 10]];
```





In[•]:=
```
Module[{K = 8₂, XingsByArmpits, bends},
 XingsByArmpits = List @@ PD[K] /.
   x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
 bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
 bends //. p_{x__,y_} p_{y_,z__} :→ p_{x,y,z}]
```

Out[•]=

$$p_{-13,4,-13}\,p_{-11,2,-11}\,p_{-5,14,-5}\,p_{-3,12,-3}\,p_{8,16,8}\,p_{6,-15,-9,6}$$
$$p_{9,-16,7,9}\,p_{10,-7,-1,10}\,p_{-10,-2,-12,-4,-14,-6,-10}\,p_{1,-8,15,5,13,3,11,1}$$

In[ ]:=
```
2 u NullSpace[Kas[8_2, u]] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 1-4u^2 & -1 & 2u & -1 & 0 & 2u \\ -1 & -1 & -1 & -1 & 1 & -1 & 0 & -1 & 2u & 0 \end{pmatrix}$$