Pensieve header: A unified verification testing suite for the sl_2-portfolio project, Uxi version. Continues pensieve://Projects/SL2Portfolio/nb/Verification.pdf.

Also continues pensieve://Projects/PPSA/nb/Verification.pdf and pensieve://2017-06/ and pensieve://2017-08/.

Prolog

```
In[]:= wdir = SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
<< "SL2PortfolioProgram.m"
```

```
In[]:= $p = 2; $k = 1; $U = QU;
```

```
In[]:= HL[ℰ_] := Style[ℰ, Background → Yellow];
```

DocileQ

```
In[]:= DQ /@ {ϵ^2 x y a_2, ϵ^2 x^2 y^3}
Out[]= {True, False}
```

Initialization / Utilities

In[]:= $SP_{\{\xi\to x\}}\left[\left(\xi^2+\xi+3\right)\left(x^5 e^x+7x\right)+99a\right]$

Out[]= $7+99a+21x+20e^x x^3+15e^x x^4+5e^x x^5$

In[]:= $SP_{\{\xi\to x,\eta\to y\}}\left[\left(\xi^2+\xi+3+2\xi\eta\right)\left(x^5 e^x+7x\right)+99a+e^{\delta x y}\xi\eta\right]$

Out[]= $7+99a+21x+20e^x x^3+15e^x x^4+5e^x x^5+e^{xy\delta}\delta+e^{xy\delta}xy\delta^2$

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

Verifying associativity on triples of generators:

```
In[]:= With[{bas = CU /@ {y, a, x}},
  Table[z1 ** (z2 ** z3) - (z1 ** z2) ** z3 // Simp // HL,
   {z1, bas}, {z2, bas}, {z3, bas}] ]
Out[]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying associativity on a “random” triple:

```
In[•]:= With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
          (rhs = (z1 ** z2) ** z3 // Simp) // Short,
          z1 ** (z2 ** z3) - rhs // Simp // HL
        }] // Timing
```

Out[•]= $\{1.625, \{(28 t^2 \gamma^4 + 116 t \gamma^5 \epsilon)$ CU[y, y, y, x, x] + $(4 t^3 \gamma + 8 t^2 \gamma^2 \epsilon)$ CU[y, y, a, a, a, x] + <<20>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

```
In[•]:= HL /@ DQ /@ Series[{(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ, ⅇ^(ℏ ϵ a)}, {ϵ, 0, 5}]
```

Out[•]= {True, True}

```
In[•]:= With[{bas = QU /@ {y, a, x}}, Table[{z1, z2} → Simp[z1 ** z2 - z2 ** z1], {z1, bas}, {z2, bas}] ]
```

Out[•]= {{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → γ QU[y],
{QU[y], QU[x]} → $\frac{(-1 + T)\,\text{QU}[\,]}{\hbar} - 2 T \epsilon$ QU[a] $- \gamma \epsilon \hbar$ QU[y, x]},
{{QU[a], QU[y]} → $-\gamma$ QU[y], {QU[a], QU[a]} → 0, {QU[a], QU[x]} → γ QU[x]},
{{QU[x], QU[y]} → $\frac{(1 - T)\,\text{QU}[\,]}{\hbar} + 2 T \epsilon$ QU[a] $+ \gamma \epsilon \hbar$ QU[y, x],
{QU[x], QU[a]} → $-\gamma$ QU[x], {QU[x], QU[x]} → 0}}

Verifying associativity on triples of generators:

```
In[•]:= With[{bas = QU /@ {y, a, x}},
         Table[HL[z1 ** (z2 ** z3) - (z1 ** z2) ** z3] // Simp,
          {z1, bas}, {z2, bas}, {z3, bas} ] ]
```

Out[•]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

Verifying associativity on a “random” triple (~34 secs @ $p=5, $k=2):

```
In[•]:= With[{z1 = QU[y, y, a, a, x, x], z2 = QU[y, a, x], z3 = QU[y, y, a, x]}, {
          (rhs = (z1 ** z2) ** z3 // Simp) // Short,
          HL[z1 ** (z2 ** z3) - rhs // Simp ]
        }] // Timing
```

Out[•]= $\{3.35938, \{\left(\frac{28 \gamma^4 - 56 T \gamma^4 + 28 T^2 \gamma^4}{\hbar^2} + \frac{82 \gamma^5 \epsilon - 280 T \gamma^5 \epsilon + 198 T^2 \gamma^5 \epsilon}{\hbar}\right)$ QU[y, y, y, x, x] +
<<18>> + $(1 + 8 \gamma \epsilon \hbar)$ QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}

Verifying that $\text{Lim}_{\hbar \to 0}$ QU = CU using a “random” product (~23 secs @ $p=5, $k=2):

```
In[•]:= With[{z1 = CU[y, y, a, a, x, x], z2 = CU[y, a, x], z3 = CU[y, y, a, x]}, {
          Short[lhs = z1 ** (z2 ** z3)],
          Short[rhs = (QU @@ z1) ** ((QU @@ z2) ** (QU @@ z3))],
          Expand[Limit[rhs /. U2l ⋃ {QU → CU}, ℏ → 0] - lhs] // HL
        }] // Timing
```

```
Out[•]= {10.5, {28 t^2 γ^4 CU[y, y, y, x, x] +
          116 t γ^5 ∈ CU[y, y, y, x, x] + <<44>> + CU[y, y, y, y, y, a, a, a, a, x, x, x, x],
        2 (γ^4/ℏ^2 - 2 T γ^4/ℏ^2 + T^2 γ^4/ℏ^2 + γ^5 ∈/ℏ - 2 T γ^5 ∈/ℏ + T^2 γ^5 ∈/ℏ) QU[y, y, y, x, x] +
          <<209>> + (1 + 8 γ ∈ ℏ) QU[y, y, y, y, y, a, a, a, a, x, x, x, x], 0}}
```

Verifying *σ*, *m*, *S*, and Δ.

Verifying $\sigma_{i\to j,k\to l}$:

```
In[•]:= CU@x₁ + CU@x₂ // σ₁→₃,₂→₄
```

```
Out[•]= CU[x₃] + CU[x₄]
```

Verifying relabeling using *m*:

```
In[•]:= t₁ t₃ CU[y₁, a₁, x₂] + t₁ t₁ CU[y₁, a₂, x₂] // m₁→₃
```

```
Out[•]= CU[a₂, x₂, y₃] t₃² + CU[x₂, y₃, a₃] t₃²
```

Verifying the meta-associativity of *m*:

```
In[•]:= Module[{z, u},
         Table[u = CU[z[[1]]₁, z[[2]]₂, z[[3]]₃]; HL[m₁,₃→₃@m₂,₃→₃@u == m₂,₃→₃@m₁,₂→₂@u],
          {z, Tuples[{y, a, x}, 3]}, {U, {CU, QU}}]]
```

```
Out[•]= {{True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True}, {True, True},
         {True, True}, {True, True}, {True, True}, {True, True}, {True, True}}
```

Verifying the involutivity of *S* on CU on products of triples:

```
In[•]:= With[{bas = CU /@ {y₁, a₁, x₁}},
         Table[HL@Simp[z1 ** z2 ** z3 - S₁@S₁[z1 ** z2 ** z3]],
          {z1, bas}, {z2, bas}, {z3, bas}] ]
```

```
Out[•]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
         {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

Verifying that *S* is an anti-homomorphism on CU/QU:

```
In[•]:= With[{bas = U /@ {y₁, a₁, x₁}},
         Table[HL@Simp[S₁[z1 ** z2] - S₁[z2] ** S₁[z1]],
          {z1, bas}, {z2, bas} , {U, {CU, QU}}] ]
```

```
Out[•]= {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}
```

Verifying the co-associativity of Δ:

In[∘]:=
```
Block[{bas = U /@ {y_1, a_1, x_1}},
  Table[(z1 ** z2 ** z3 // Δ_{1→1,2} // Δ_{2→2,3}) - (z1 ** z2 ** z3 // Δ_{1→1,3} // Δ_{1→1,2}) // Simp // HL,
   {z1, bas}, {z2, bas}, {z3, bas} , {U, {CU, QU}}] ]
```

Out[∘]=
```
{{{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
 {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}},
 {{{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}, {{0, 0}, {0, 0}, {0, 0}}}}
```

Verifying *S*-Δ compatibility:

```
Timing@Block[{bas = U /@ {y_1, a_1, x_1}},
  Table[z1 ** z2 ** z3 // Δ_{1→1,2} // S_i // m_{1,2→1} // Simp // HL,
   {U, {CU, QU}}, {i, 2}, {z1, bas}, {z2, bas}, {z3, bas}] ]
```

Out[∘]=
```
{{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
 {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
   {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}
```

Verifying *S*-Δ compatibility for opposite *m*, only for CU:

In[∘]:=
```
Block[{bas = CU /@ {y_1, a_1, x_1}},
  Table[z1 ** z2 ** z3 // Δ_{1→1,2} // S_i // m_{2,1→1} // Simp // HL,
   {i, 2}, {z1, bas}, {z2, bas}, {z3, bas} ]]
```

Out[∘]=
```
{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}
```

Verifying *m*-Δ compatibility:

In[∘]:=
```
Timing@Block[{bas1 = U /@ {y_1, a_1, x_1}, bas2 = U /@ {y_2, a_2, x_2}},
  Table[(z1 ** z2 ** z3 ** z4 // m_{1,2→1} // Δ_{1→1,2}) -
     (z1 ** z2 ** z3 ** z4 // Δ_{1→3,4} // Δ_{2→5,6} // m_{3,5→1} // m_{4,6→2}) // Simp // HL,
   {U, {CU, QU}}, {z1, bas1}, {z2, bas1}, {z3, bas2}, {z4, bas2} ] ]
```

Out[∘]=
```
{27.3438, {{{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
   {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}},
  {{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}, {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}},
   {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
    {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}}}}
```

Implementing θ

Verifying involutivity on CU:

In[∘]:=
```
With[{bas = CU /@ {y, a, x}},
 Table[z → Cθ[z] → HL[Cθ[Cθ[z]]], {z, bas}] ]
```

Out[∘]= {CU[y] → −CU[x] → CU[y], CU[a] → −CU[a] → CU[a], CU[x] → −CU[y] → CU[x]}

Verifying that θ is a multiplicative homomorphism on CU:

In[∘]:=
```
With[{bas = CU /@ {y, a, x}},
 Table[Cθ[z1 ** z2] - Cθ[z1] ** Cθ[z2] // HL, {z1, bas}, {z2, bas}] ]
```

Out[∘]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Verifying involutivity on QU:

In[∘]:=
```
With[{bas = QU /@ {y, a, x}},
 Table[z → Qθ[z] → HL[Simp[Qθ[Qθ[z]], PowerExpand]], {z, bas}] ]
```

Out[∘]= $\left\{QU[y] \to -\frac{QU[x]}{\sqrt{T}} - \frac{\epsilon\hbar\, QU[a, x]}{\sqrt{T}} \to QU[y],\ QU[a] \to -QU[a] \to QU[a],\ QU[x] \to \left(-\frac{1}{\sqrt{T}} + \frac{\gamma\epsilon\hbar}{\sqrt{T}}\right) QU[y] - \frac{\epsilon\hbar\, QU[y, a]}{\sqrt{T}} \to QU[x]\right\}$

Verifying that θ is a multiplicative homomorphism on QU:

In[∘]:=
```
With[{bas = QU /@ {y, a, x}},
 Table[{z1, z2} → HL[Simp[Qθ[z1 ** z2] - Qθ[z1] ** Qθ[z2]]], {z1, bas}, {z2, bas}] ]
```

Out[∘]= {{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
{{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
{{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}

Verifying that θ is a co-multiplicative morphism on CU:

In[∘]:=
```
With[{bas = CU /@ {y1, a1, x1}},
 Table[Cθ@Δ1→1,2@z - Δ1→1,2@ Cθ@z // HL, {z, bas}] ]
```

Out[∘]= {0, 0, 0}

Verifying that θ is a co-multiplicative morphism on QU: (Fails!)

In[∘]:=
```
With[{bas = QU /@ {y1, a1, x1}},
 Table[res = FullSimplify@PowerExpand[Qθ@Δ1→1,2@z - Δ1→2,1@ Qθ@z], {z, bas}] ]
```

Out[∘]= $\left\{\frac{1}{T_1\sqrt{T_2}}\left(-\epsilon\hbar\left(QU[a_1, x_2] + QU[a_2, x_2]\right) + QU[x_2]\left(-1+\sqrt{T_1}\right) + QU[x_1]\sqrt{T_1} + \sqrt{T_1}\left(\epsilon\hbar\left(QU[a_1, x_1] + QU[a_1, x_2] + QU[a_2, x_2]\right) - \left(QU[x_1] + \epsilon\hbar\, QU[a_1, x_1]\right)\sqrt{T_2}\right)\right),\ 0,\ \frac{1}{\sqrt{T_1}\sqrt{T_2}}\left(\left(\left(-1+\gamma\epsilon\hbar\right) QU[y_2] - \epsilon\hbar\left(QU[a_1, y_2] + QU[y_2, a_2]\right)\right)\left(-1+\sqrt{T_1}\right) + \left(\left(-1+\gamma\epsilon\hbar\right) QU[y_1] - \epsilon\hbar\, QU[y_1, a_1]\right)\sqrt{T_2} + \left(\left(1-\gamma\epsilon\hbar\right) QU[y_1] + \epsilon\hbar\, QU[y_1, a_1]\right) T_2\right)\right\}$

In[∘]:=
```
res /. ϵ → 0
```

Out[∘]= $\frac{-QU[y_2]\left(-1+\sqrt{T_1}\right) - QU[y_1]\sqrt{T_2} + QU[y_1]\, T_2}{\sqrt{T_1}\sqrt{T_2}}$

The Asymmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Docility of A𝔻$f:

In[∘]:=
```
HL@DQ@Block[{$p = 4}, Collect[SS@A𝔻$f /. ω → a₁, ϵ]]
```

Out[∘]= True

Scaling behaviour of A𝔻$f:

In[∘]:=
```
HL@Simplify[A𝔻$f == ((A𝔻$f /. γ → 1) /. {ϵ → γ ϵ, a → γ^-1 a, ω → γ^-1 ω})]
```

Out[∘]= True

In[∘]:=
```
HL@FullSimplify[
  A𝔻$f == ((A𝔻$f /. γ → 1) /. {ℏ → γ^2 ℏ, ϵ → ϵ / γ, a → a / γ, t → γ^-2 t, ω → γ^-3 ω})]
```

Out[∘]= True

In[•]:=
```
Block[{$p = 4, $k = 3}, {res = A𝔻[y_QU], res /. ϵ → 0}]
```

Out[•]=

$$
\begin{aligned}
\Big\{ & \left(\frac{2}{3}\gamma^2\epsilon^2\hbar^2+\frac{1}{2}t\gamma^2\epsilon^2\hbar^3-\frac{5}{6}\gamma^3\epsilon^3\hbar^3+\frac{1}{5}t^2\gamma^2\epsilon^2\hbar^4-\frac{13}{20}t\gamma^3\epsilon^3\hbar^4\right)\text{CU}[y]+ \\
& \Big(1+\frac{t\hbar}{2}-\gamma\epsilon\hbar+\frac{t^2\hbar^2}{6}-\frac{7}{12}t\gamma\epsilon\hbar^2+\frac{1}{2}\gamma^2\epsilon^2\hbar^2+\frac{t^3\hbar^3}{24}-\frac{5}{24}t^2\gamma\epsilon\hbar^3+\frac{1}{3}t\gamma^2\epsilon^2\hbar^3- \\
& \quad \frac{1}{6}\gamma^3\epsilon^3\hbar^3+\frac{t^4\hbar^4}{120}-\frac{13}{240}t^3\gamma\epsilon\hbar^4+\frac{91}{720}t^2\gamma^2\epsilon^2\hbar^4-\frac{89}{720}t\gamma^3\epsilon^3\hbar^4\Big)\text{CU}[y]+ \\
& \left(-\frac{1}{12}t\gamma\epsilon\hbar^2-\frac{1}{24}t^2\gamma\epsilon\hbar^3-\frac{1}{80}t^3\gamma\epsilon\hbar^4-\frac{1}{720}t^2\gamma^2\epsilon^2\hbar^4+\frac{1}{720}t\gamma^3\epsilon^3\hbar^4\right)\text{CU}[y]+ \\
& \left(\frac{1}{3}\gamma^3\epsilon^3\hbar^3+\frac{4}{15}t\gamma^3\epsilon^3\hbar^4\right)\text{CU}[y]+ \\
& \Big(\gamma\epsilon\hbar+\frac{2}{3}t\gamma\epsilon\hbar^2-\frac{7}{6}\gamma^2\epsilon^2\hbar^2+\frac{1}{4}t^2\gamma\epsilon\hbar^3-\frac{5}{6}t\gamma^2\epsilon^2\hbar^3+\frac{2}{3}\gamma^3\epsilon^3\hbar^3+\frac{1}{15}t^3\gamma\epsilon\hbar^4- \\
& \quad \frac{13}{40}t^2\gamma^2\epsilon^2\hbar^4+\frac{91}{180}t\gamma^3\epsilon^3\hbar^4\Big)\text{CU}[y]+\left(-\frac{2}{3}\gamma^2\epsilon^3\hbar^3-\frac{8}{15}t\gamma^2\epsilon^3\hbar^4\right)\text{CU}[y,a]+ \\
& \Big(-\epsilon\hbar-\frac{2}{3}t\epsilon\hbar^2+\frac{7}{6}\gamma\epsilon^2\hbar^2-\frac{1}{4}t^2\epsilon\hbar^3+\frac{5}{6}t\gamma\epsilon^2\hbar^3-\frac{2}{3}\gamma^2\epsilon^3\hbar^3-\frac{1}{15}t^3\epsilon\hbar^4+ \\
& \quad \frac{13}{40}t^2\gamma\epsilon^2\hbar^4-\frac{91}{180}t\gamma^2\epsilon^3\hbar^4\Big)\text{CU}[y,a]+\left(-\frac{1}{3}\gamma^2\epsilon^3\hbar^3-\frac{4}{15}t\gamma^2\epsilon^3\hbar^4\right)\text{CU}[y,a]+ \\
& \left(\frac{1}{6}\gamma\epsilon^2\hbar^2+\frac{1}{12}t\gamma\epsilon^2\hbar^3+\frac{1}{40}t^2\gamma\epsilon^2\hbar^4+\frac{1}{360}t\gamma^2\epsilon^3\hbar^4\right)\text{CU}[y,a]+ \\
& \left(\frac{1}{12}t\gamma\epsilon^2\hbar^3+\frac{1}{20}t^2\gamma\epsilon^2\hbar^4+\frac{1}{360}t\gamma^2\epsilon^3\hbar^4\right)\text{CU}[y,a]+ \\
& \left(-\frac{4}{3}\gamma\epsilon^2\hbar^2-t\gamma\epsilon^2\hbar^3+\frac{5}{3}\gamma^2\epsilon^3\hbar^3-\frac{2}{5}t^2\gamma\epsilon^2\hbar^4+\frac{13}{10}t\gamma^2\epsilon^3\hbar^4\right)\text{CU}[y,a]- \\
& \frac{1}{20}t\gamma\epsilon^3\hbar^4\,\text{CU}[y,a,a]+\left(\frac{2\epsilon^2\hbar^2}{3}+\frac{1}{2}t\epsilon^2\hbar^3-\frac{5}{6}\gamma\epsilon^3\hbar^3+\frac{1}{5}t^2\epsilon^2\hbar^4-\frac{13}{20}t\gamma\epsilon^3\hbar^4\right)\text{CU}[y,a,a]+ \\
& \left(-\frac{1}{6}\gamma\epsilon^3\hbar^3-\frac{1}{10}t\gamma\epsilon^3\hbar^4\right)\text{CU}[y,a,a]+\left(\frac{1}{3}\gamma\epsilon^3\hbar^3+\frac{4}{15}t\gamma\epsilon^3\hbar^4\right)\text{CU}[y,a,a]+ \\
& \left(\frac{2}{3}\gamma\epsilon^3\hbar^3+\frac{8}{15}t\gamma\epsilon^3\hbar^4\right)\text{CU}[y,a,a]-\frac{1}{180}t\gamma^2\epsilon^2\hbar^4\,\text{CU}[y,y,x]+\frac{2}{45}\gamma^3\epsilon^3\hbar^4\,\text{CU}[y,y,x]+ \\
& \left(\frac{1}{12}\gamma\epsilon\hbar^2+\frac{1}{24}t\gamma\epsilon\hbar^3+\frac{1}{80}t^2\gamma\epsilon\hbar^4+\frac{1}{720}t\gamma^2\epsilon^2\hbar^4-\frac{1}{720}\gamma^3\epsilon^3\hbar^4\right)\text{CU}[y,y,x]+ \\
& \left(\frac{1}{12}\gamma^2\epsilon^2\hbar^3+\frac{1}{20}t\gamma^2\epsilon^2\hbar^4+\frac{1}{360}\gamma^3\epsilon^3\hbar^4\right)\text{CU}[y,y,x]+\left(-\frac{1}{3}\epsilon^3\hbar^3-\frac{4}{15}t\epsilon^3\hbar^4\right)\text{CU}[y,a,a,a]- \\
& \frac{4}{45}\gamma^2\epsilon^3\hbar^4\,\text{CU}[y,y,a,x]+\left(-\frac{1}{12}\gamma\epsilon^2\hbar^3-\frac{1}{20}t\gamma\epsilon^2\hbar^4-\frac{1}{360}\gamma^2\epsilon^3\hbar^4\right)\text{CU}[y,y,a,x]+ \\
& \frac{1}{20}\gamma\epsilon^3\hbar^4\,\text{CU}[y,y,a,a,x]+\frac{1}{360}\gamma^2\epsilon^2\hbar^4\,\text{CU}[y,y,y,x,x], \\
& \left(1+\frac{t\hbar}{2}+\frac{t^2\hbar^2}{6}+\frac{t^3\hbar^3}{24}+\frac{t^4\hbar^4}{120}\right)\text{CU}[y]\Big\}
\end{aligned}
$$

Verifying that the asymmetric dequantizator is a homomorphism:

In[*]:=
```
With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL[SimpT[A𝔻[z1 ** z2] - A𝔻[z1] ** A𝔻[z2]]], {z1, bas}, {z2, bas}]]
```

Out[*]=
```
{{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
 {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
 {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The Symmetric Dequantizator

Following pensieve://People/VanDerVeen/Dequant1.pdf.

Verify agreement with the formulas in pensieve://People/VanDerVeen/Dequant1.pdf:

In[*]:=
$$\left\{S\mathbb{D}\$P = \frac{\mathrm{Cosh}\left[\hbar\left(\frac{\epsilon-t}{2}+\epsilon\, a\right)\right]-\mathrm{Cosh}\left[\hbar\sqrt{\frac{t^2+\epsilon^2}{4}+\epsilon\,\varpi}\right]}{\hbar\,\mathrm{Sinh}\left[\frac{-\epsilon\,\hbar}{2}\right]\left(\varpi-\epsilon\, a^2+(t-\epsilon)\, a+t/2\right)},\right.$$

```
 Simplify[S𝔻$P == (S𝔻$P /. {a → -a - 1, t → -t})] // HL,
 PowerExpand@Simplify[(S𝔻$P /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}) ==
    S𝔻$g (S𝔻$g /. {a → -a - γ, t → -t})] // HL,
 S𝔻$Q = Simplify[S𝔻$P /. {a → c - 1/2}],
 Simplify[S𝔻$Q == (S𝔻$Q /. {c → -c, t → -t})] // HL,
 FullSimplify[S𝔻$g == FullSimplify[
    √S𝔻$Q /. c → a + 1/2 /. {ℏ → γ^2 ℏ, ϵ → ϵ/γ, a → a/γ, t → γ^-2 t, ϖ → γ^-3 ϖ}]] // HL,
 HL@DQ@Block[{$p = 4}, Collect[SS@S𝔻$g /. ω → a₁, ϵ]],
 HL@DQ@Block[{$p = 4}, Collect[SS@S𝔻$f /. ω → a₁, ϵ]]
}
```

Out[*]=
$$\left\{-\left(\left(\left(\mathrm{Cosh}\left[\left(a\,\epsilon+\frac{1}{2}(-t+\epsilon)\right)\hbar\right]-\mathrm{Cosh}\left[\sqrt{\frac{1}{4}\left(t^2+\epsilon^2\right)+\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(\frac{t}{2}+a\,(t-\epsilon)-a^2\,\epsilon+\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\right.$$
$$\left.-\left(\left(4\left(\mathrm{Cosh}\left[\frac{1}{2}(t-2\,c\,\epsilon)\,\hbar\right]-\mathrm{Cosh}\left[\frac{1}{2}\sqrt{t^2+\epsilon^2+4\,\epsilon\,\varpi}\;\hbar\right]\right)\mathrm{Csch}\left[\frac{\epsilon\,\hbar}{2}\right]\right)\Big/\left(\left(4\,c\,t+\epsilon-4\,c^2\,\epsilon+4\,\varpi\right)\hbar\right)\right),\ \mathrm{True},\ \mathrm{True},\ \mathrm{True},\ \mathrm{True}\right\}$$

Verifying the θ-symmetry:

In[*]:=
```
Table[HL@SimpT[Cθ[S𝔻[z]] == S𝔻[Qθ[z]]], {z, QU /@ {y, a, x}}]
```

Out[*]=
```
{True, True, True}
```

Verifying that the symmetric dequantizator is a homomorphism:

```
In[]:= With[{bas = QU /@ {y, a, x}},
  Table[{z1, z2} → HL@SimpT[S𝔻[z1 ** z2] - S𝔻[z1] ** S𝔻[z2]], {z1, bas}, {z2, bas}]]
Out[]= {{{QU[y], QU[y]} → 0, {QU[y], QU[a]} → 0, {QU[y], QU[x]} → 0},
  {{QU[a], QU[y]} → 0, {QU[a], QU[a]} → 0, {QU[a], QU[x]} → 0},
  {{QU[x], QU[y]} → 0, {QU[x], QU[a]} → 0, {QU[x], QU[x]} → 0}}
```

The representation ρ

Verifying that ρ represents CU and QU:

```
In[]:= Table[HL[SS[ρ[z1 ** z2] == ρ[z1].ρ[z2]] /. e^k_· /; k > $k → 0],
  {U, {CU, QU}}, {z1, U /@ {y, a, x}}, {z2, U /@ {y, a, x}}]
Out[]= {{{True, True, True}, {True, True, True}, {True, True, True}},
  {{True, True, True}, {True, True, True}, {True, True, True}}}
```

Commuting $e^{\alpha a}$ with $e^{\xi x}$:

```
In[]:= Table[HL[ρ[e^(ξ U@x)].ρ[e^(α U@a)] == ρ[e^(α U@a)].ρ[e^(e^(-γ α) ξ U@x)]], {U, {CU, QU}}]
Out[]= {True, True}
```

tSW

```
In[]:= HL@Simp[𝕆$U[{y3, a3, x3}3, tSW_xy,1,2→3 /. {ξ1 → ℏ ξ1, η2 → ℏ η2} /. U2l] -
    SS[𝕆$U[{x3, y3}3, SS[e^(ℏ (ξ1 x3+η2 y3))]] /. U2l]]
Out[]= 0

In[]:= HL@Simp[
   𝕆$U[{y1, a1, x1}1, tSW_xa,1,1→1 /. {ξ1 → ℏ ξ1, α1 → ℏ α1}] - SS[𝕆$U[{x1, a1}1, SS[e^(ℏ (ξ1 x1+α1 a1))]]]]
Out[]= 0

In[]:= HL@Simp[
   𝕆$U[{y1, a1, x1}1, tSW_ay,1,1→1 /. {η1 → ℏ η1, α1 → ℏ α1}] - SS[𝕆$U[{a1, y1}1, SS[e^(ℏ (η1 y1+α1 a1))]]]]
Out[]= 0
```

R in QU.

```
In[]:= Table[Together@SeriesCoefficient[e_q,5[x], {x, 0, n}], {n, 0, 5}]
```

Out[]= $\left\{1, 1, \frac{1}{1+q}, \frac{1}{(1+q)(1+q+q^2)}, \frac{1}{(1+q)^2(1+q^2)(1+q+q^2)}, 1\Big/\left((1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)\right)\right\}$

```
In[]:= Table[HL@FunctionExpand[QFactorial[n, q] SeriesCoefficient[e_q,5[x], {x, 0, n}]], {n, 0, 5}]
Out[]= {1, 1, 1, 1, 1, 1}
```

```
In[]:= QU[R3,4] // Short
```

Out[]//Short= $\text{QU}[\,] + \frac{\epsilon\,\hbar\,\text{QU}[a_3, a_4]}{\gamma} + \hbar\,\text{QU}[y_3, x_4] + \frac{\epsilon \ll 1\gg \ll 1\gg \text{QU}[\ll 1\gg]}{\gamma} + \frac{1}{2}\ll 1\gg \ll 1\gg - \frac{\ll 1\gg}{\gamma} - \frac{\epsilon\,\hbar^2\,\text{QU}[\ll 1\gg]\,t_3}{\gamma^2} - \frac{\hbar^2\,\text{QU}[y_3, a_4, x_4]\,t_3}{\gamma} + \frac{\hbar^2\,\text{QU}[a_4, a_4]\,t_3^2}{2\,\gamma^2}$

Verifying R2 (~2 secs @ $p=4, $k=2):

```
In[]:= QU[R1,2 ** R^-1 1,2] // Simp // HL // Timing
```

Out[]= {0.078125, QU[]}

Verifying R3 (~156 secs @ $p=4, $k=2):

```
In[]:= {Short[lhs = QU[R1,2 ** R1,3 ** R2,3]], HL@SimpT[lhs - QU[R2,3 ** R1,3 ** R1,2]]} // Timing
```

Out[]= $\left\{0.15625, \left\{\text{QU}[\,] + \frac{\epsilon\,\hbar\,\text{QU}[a_1, a_2]}{\gamma} + \frac{\epsilon\,\hbar\,\text{QU}[a_1, a_3]}{\gamma} + \ll 73\gg + 2\,\epsilon\,\hbar^2\,\text{QU}[y_1, a_2, x_3]\,T_2 + \text{QU}[y_1, x_3]\,(\hbar - \hbar\,T_2), 0\right\}\right\}$

Exponentials as needed.

In[∘]:=
```
Block[{$p = 2, $k = 2}, TableForm[StringSplit[
    "y|a|x|Cθ@y_CU|Cθ@a_CU|Cθ@x_CU|Qθ@y_QU|Qθ@a_QU|Qθ@x_QU|A𝔻@y_QU|A𝔻@a_QU|A𝔻@x_QU|S𝔻@y_QU|S𝔻@a_QU|S𝔻
      @x_QU|S@y_CU|S@a_CU|S@x_CU|S@y_QU|S@a_QU|S@x_QU|Δ@y_CU|Δ@a_CU|Δ@x_CU|Δ@y_QU|Δ@a_QU|Δ@x_QU",
    "|"] /. s_String ↦
    {s, Normal@Simplify@Series[ToExpression[s] /. CU | QU → Times, {ϵ, 0, $k}]}]]
```

Out[∘]//TableForm=

y	y
a	a
x	x
$C\theta@y_{CU}$	$-x$
$C\theta@a_{CU}$	$-a$
$C\theta@x_{CU}$	$-y$
$Q\theta@y_{QU}$	$-\frac{x}{\sqrt{T}}-\frac{a\,x\,\epsilon\,\hbar}{\sqrt{T}}-\frac{a^2\,x\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$Q\theta@a_{QU}$	$-a$
$Q\theta@x_{QU}$	$-\frac{y}{\sqrt{T}}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{\sqrt{T}}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\sqrt{T}}$
$A\mathbb{D}@y_{QU}$	$\frac{2}{3}a^2\,y\,\epsilon^2\,\hbar^2+\frac{1}{6}y\left(6+3\,t\,\hbar+t^2\,\hbar^2\right)+\frac{1}{12}y\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-4\,a\left(3+2\,t\,\hbar\right)\right)$
$A\mathbb{D}@a_{QU}$	a
$A\mathbb{D}@x_{QU}$	x
$S\mathbb{D}@y_{QU}$	$y+\frac{1}{48}t^2\,y\,\hbar^2+\frac{1}{24}y\left(-2\,a\,t+x\,y\,\gamma\right)\epsilon\,\hbar^2+\frac{1}{12}a^2\,y\,\epsilon^2\,\hbar^2$
$S\mathbb{D}@a_{QU}$	a
$S\mathbb{D}@x_{QU}$	$\frac{7}{12}a^2\,x\,\epsilon^2\,\hbar^2+x\left(1+\frac{t\,\hbar}{2}+\frac{7\,t^2\,\hbar^2}{48}\right)+\frac{1}{24}x\,\epsilon\,\hbar\left(x\,y\,\gamma\,\hbar-2\,a\left(12+7\,t\,\hbar\right)\right)$
$S@y_{CU}$	$-y$
$S@a_{CU}$	$-a$
$S@x_{CU}$	$-x$
$S@y_{QU}$	$-\frac{y}{T}+\frac{y\,(-a+\gamma)\,\epsilon\,\hbar}{T}-\frac{y\,(a-\gamma)^2\,\epsilon^2\,\hbar^2}{2\,T}$
$S@a_{QU}$	$-a$
$S@x_{QU}$	$-x-a\,x\,\epsilon\,\hbar-\frac{1}{2}a^2\,x\,\epsilon^2\,\hbar^2$
$\Delta@y_{CU}$	y_1+y_2
$\Delta@a_{CU}$	a_1+a_2
$\Delta@x_{CU}$	x_1+x_2
$\Delta@y_{QU}$	$y_1+T_1\,y_2-\epsilon\,\hbar\,a_1\,T_1\,y_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,T_1\,y_2$
$\Delta@a_{QU}$	a_1+a_2
$\Delta@x_{QU}$	$x_1+x_2-\epsilon\,\hbar\,a_1\,x_2+\frac{1}{2}\epsilon^2\,\hbar^2\,a_1^2\,x_2$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S1[QU[y1]] /. QU → Times,
   exps = Exp_{QU1,$k}[η, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S1@O_QU[{y1}1, SS[e^(ħ η y1)]] - O_QU[{y1, a1, x1}1, (exps /. η → ħ η)]]
  }]
```

Out[∘]=
$$\left\{4.67188, \left\{a_1\left(-\frac{\epsilon\hbar}{T_1}+\frac{\gamma\epsilon^2\hbar^2}{T_1}\right)y_1+\left(-\frac{1}{T_1}+\frac{\gamma\epsilon\hbar}{T_1}-\frac{\gamma^2\epsilon^2\hbar^2}{2T_1}\right)y_1-\frac{\epsilon^2\hbar^2a_1^2y_1}{2T_1},\right.\right.$$
$$\mathbb{E}\left[0,-\frac{\eta y_1}{T_1},1+\frac{\left(2\gamma\eta\hbar T_1y_1-2\eta\hbar a_1T_1y_1-\gamma\eta^2\hbar y_1^2\right)\epsilon}{2T_1^2}+\left(-\frac{\gamma^2\eta\hbar^2y_1}{2T_1}+\frac{\gamma\eta\hbar^2a_1y_1}{T_1}-\frac{\eta\hbar^2a_1^2y_1}{2T_1}+\frac{7\gamma^2\eta^2\hbar^2y_1^2}{4T_1^2}-\frac{2\gamma\eta^2\hbar^2a_1y_1^2}{T_1^2}+\right.\right.$$
$$\left.\left.\left.\left.\frac{\eta^2\hbar^2a_1^2y_1^2}{2T_1^2}-\frac{\gamma^2\eta^3\hbar^2y_1^3}{T_1^3}+\frac{\gamma\eta^3\hbar^2a_1y_1^3}{2T_1^3}+\frac{\gamma^2\eta^4\hbar^2y_1^4}{8T_1^4}\right)\epsilon^2+O[\epsilon]^3\right],0\right\}\right\}$$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S1[QU[a1]] /. QU → Times,
   exps = Exp_{QU1,$k}[α, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S1@O_QU[{a1}1, SS[e^(ħ α a1)]] - O_QU[{y1, a1, x1}1, exps /. α → ħ α]]
  }]
```

Out[∘]=
$$\{2.20313, \{-a_1, \mathbb{E}[-\alpha a_1, 0, 1+O[\epsilon]^3], 0\}\}$$

In[∘]:=
```
Timing@Block[{$p = 4, $k = 2}, {
   s = S1[QU[x1]] /. QU → Times,
   exps = Exp_{QU1,$k}[ξ, s], (* Warning: wrong unless $p≥$k+1! *)
   HL@Simp[S1@O_QU[{x1}1, SS[e^(ħ ξ x1)]] - O_QU[{y1, a1, x1}1, (exps /. ξ → ħ ξ)]]
  }]
```

Out[∘]=
$$\left\{1.51563, \left\{-x_1-\epsilon\hbar a_1x_1-\frac{1}{2}\epsilon^2\hbar^2a_1^2x_1,\right.\right.$$
$$\mathbb{E}\left[0,-\xi x_1,1+\left(-\xi\hbar a_1x_1-\frac{1}{2}\gamma\xi^2\hbar x_1^2\right)\epsilon+\left(-\frac{1}{2}\xi\hbar^2a_1^2x_1+\frac{1}{4}\gamma^2\xi^2\hbar^2x_1^2-\gamma\xi^2\hbar^2a_1x_1^2+\right.\right.$$
$$\left.\left.\left.\left.\frac{1}{2}\xi^2\hbar^2a_1^2x_1^2-\frac{1}{2}\gamma^2\xi^3\hbar^2x_1^3+\frac{1}{2}\gamma\xi^3\hbar^2a_1x_1^3+\frac{1}{8}\gamma^2\xi^4\hbar^2x_1^4\right)\epsilon^2+O[\epsilon]^3\right],0\right\}\right\}$$

$S(e^{\eta y} e^{\alpha a} e^{\xi x})$

```
In[∘]:= Timing@Block[{$p = 3, $k = 1}, {
    (sexp = m_{3,2,1→1}[Exp_{QU_1,$k}[η, S_1[QU[y_1]] /. QU → Times] Exp_{QU_2,$k}[α, S_2[QU[a_2]] /. QU → Times]
        Exp_{QU_3,$k}[ξ, S_3[QU[x_3]] /. QU → Times]]) /. u_1 ⧴ u,
    HL@SimpT[𝕆_QU[{y_1, a_1, x_1}_1, sexp /. U2l /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ}] -
      S_1@𝕆_QU[{y_1, a_1, x_1}_1, SS[e^{ℏ (η y_1+α a_1+ξ x_1)}]]]
  }]
```

Out[∘]= $\{1.95313, \{\mathbb{E}[-a\,\alpha, \frac{\mathcal{A}\,\eta\,\xi - T\,\mathcal{A}\,\eta\,\xi - y\,\mathcal{A}\,\eta\,\hbar - T\,x\,\mathcal{A}\,\xi\,\hbar}{T\,\hbar},$

$1 + \frac{1}{4\,T^2\,\hbar}\,(-3\,\mathcal{A}^2\,\gamma\,\eta^2\,\xi^2 + 4\,T\,\mathcal{A}^2\,\gamma\,\eta^2\,\xi^2 - T^2\,\mathcal{A}^2\,\gamma\,\eta^2\,\xi^2 + 8\,a\,T\,\mathcal{A}\,\eta\,\xi\,\hbar - 4\,T\,\mathcal{A}\,\gamma\,\eta\,\xi\,\hbar + 4\,T^2\,\mathcal{A}\,\gamma\,\eta\,\xi\,\hbar +$
$6\,y\,\mathcal{A}^2\,\gamma\,\eta^2\,\xi\,\hbar - 2\,T\,y\,\mathcal{A}^2\,\gamma\,\eta^2\,\xi\,\hbar + 6\,T\,x\,\mathcal{A}^2\,\gamma\,\eta\,\xi^2\,\hbar - 2\,T^2\,x\,\mathcal{A}^2\,\gamma\,\eta\,\xi^2\,\hbar - 4\,a\,T\,y\,\mathcal{A}\,\eta\,\hbar^2 + 4\,T\,y\,\mathcal{A}\,\gamma$
$\eta\,\hbar^2 - 2\,y^2\,\mathcal{A}^2\,\gamma\,\eta^2\,\hbar^2 - 4\,a\,T^2\,x\,\mathcal{A}\,\xi\,\hbar^2 - 4\,T\,x\,y\,\mathcal{A}^2\,\gamma\,\eta\,\xi\,\hbar^2 - 2\,T^2\,x^2\,\mathcal{A}^2\,\gamma\,\xi^2\,\hbar^2)\,\epsilon + O[\epsilon]^2], 0\}\}$

$\Delta_{1\to1,2}(e^{\eta y_1} e^{\alpha a_1} e^{\xi x_1})$

```
In[∘]:= Timing@Block[{$p = 4, $k = 2}, {
    sexp = m_{1,3,5→1}@m_{2,4,6→2}@Times[ (* Warning: wrong unless $p≥$k+1! *)
        ReplacePart[1 → 0]@Exp_{QU_1,$k}[η, Δ_{1→1,2}[QU[y_1]] /. QU → Times],
        ReplacePart[2 → 0]@Exp_{QU_3,$k}[α, Δ_{3→3,4}[QU[a_3]] /. QU → Times],
        ReplacePart[1 → 0]@Exp_{QU_5,$k}[ξ, Δ_{5→5,6}[QU[x_5]] /. QU → Times]
      ] /. {η → ℏ η, α → ℏ α, ξ → ℏ ξ},
    HL@SimpT[
      𝕆_QU[{y_1, a_1, x_1}_1, {y_2, a_2, x_2}_2, sexp] - Δ_{1→1,2}@𝕆_QU[{y_1, a_1, x_1}_1, SS[e^{ℏ (η y_1+α a_1+ξ x_1)}]]]
  }]
```

Out[∘]= $\{15.3125, \{\mathbb{E}[\alpha\,\hbar\,a_1 + \alpha\,\hbar\,a_2, \xi\,\hbar\,x_1 + \xi\,\hbar\,x_2 + \eta\,\hbar\,y_1 + \eta\,\hbar\,T_1\,y_2,$

$1 + \frac{1}{2}\,(-2\,\xi\,\hbar^2\,a_1\,x_2 + \gamma\,\xi^2\,\hbar^3\,x_1\,x_2 - 2\,\eta\,\hbar^2\,a_1\,T_1\,y_2 + \gamma\,\eta^2\,\hbar^3\,T_1\,y_1\,y_2)\,\epsilon +$

$\frac{1}{24}\,(12\,\xi\,\hbar^3\,a_1^2\,x_2 + 6\,\gamma^2\,\xi^2\,\hbar^4\,x_1\,x_2 - 12\,\gamma\,\xi^2\,\hbar^4\,a_1\,x_1\,x_2 + 4\,\gamma^2\,\xi^3\,\hbar^5\,x_1^2\,x_2 + 12\,\xi^2\,\hbar^4\,a_1^2\,x_2^2 +$
$4\,\gamma^2\,\xi^3\,\hbar^5\,x_1\,x_2^2 - 12\,\gamma\,\xi^3\,\hbar^5\,a_1\,x_1\,x_2^2 + 3\,\gamma^2\,\xi^4\,\hbar^6\,x_1^2\,x_2^2 + 12\,\eta\,\hbar^3\,a_1^2\,T_1\,y_2 +$
$24\,\eta\,\xi\,\hbar^4\,a_1^2\,T_1\,x_2\,y_2 - 12\,\gamma\,\eta\,\xi^2\,\hbar^5\,a_1\,T_1\,x_1\,x_2\,y_2 + 6\,\gamma^2\,\eta^2\,\hbar^4\,T_1\,y_1\,y_2 - 12\,\gamma\,\eta^2\,\hbar^4\,a_1\,T_1\,y_1\,y_2 -$
$12\,\gamma\,\eta^2\,\xi\,\hbar^5\,a_1\,T_1\,x_2\,y_1\,y_2 + 6\,\gamma^2\,\eta^2\,\xi^2\,\hbar^6\,T_1\,x_1\,x_2\,y_1\,y_2 + 4\,\gamma^2\,\eta^3\,\hbar^5\,T_1\,y_1^2\,y_2 + 12\,\eta^2\,\hbar^4\,a_1^2\,T_1^2\,y_2^2 +$
$4\,\gamma^2\,\eta^3\,\hbar^5\,T_1^2\,y_1\,y_2^2 - 12\,\gamma\,\eta^3\,\hbar^5\,a_1\,T_1^2\,y_1\,y_2^2 + 3\,\gamma^2\,\eta^4\,\hbar^6\,T_1^2\,y_1^2\,y_2^2)\,\epsilon^2 + O[\epsilon]^3], 0\}\}$

Zip and Bind

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

```
In[]:= Timing@{E0 = 𝔼[0, Sum[a_{10 i+j} x_i ξ_j, {i, 3}, {j, 3}],
       1 + ϵ Sum[f_i[x1, x2, x3] ξ_i, {i, 3}] + ϵ Sum[f_{10 i+j}[x1, x2, x3] ξ_i ξ_j, {i, 3}, {j, 3}]],
    lhs = QZip_{{ξ1,ξ2},Simplify}@E0,
    HL[lhs == QZip_{{ξ1},Simplify}@QZip_{{ξ2},Simplify}@E0]}
```

Out[]=
$\{35.0625,$
$\{\mathbb{E}[0, a_{11} x_1 \xi_1 + a_{21} x_2 \xi_1 + a_{31} x_3 \xi_1 + a_{12} x_1 \xi_2 + a_{22} x_2 \xi_2 + a_{32} x_3 \xi_2 + a_{13} x_1 \xi_3 + a_{23} x_2 \xi_3 + a_{33} x_3 \xi_3, 1 + \epsilon (\xi_1 f_1[x_1, x_2, x_3] + \xi_2 f_2[x_1, x_2, x_3] + \xi_3 f_3[x_1, x_2, x_3]) + \epsilon (\xi_1^2 f_{11}[x_1, x_2, x_3] + \xi_1 \xi_2 f_{12}[x_1, x_2, x_3] + \xi_1 \xi_3 f_{13}[x_1, x_2, x_3] + \xi_1 \xi_2 f_{21}[x_1, x_2, x_3] + \xi_2^2 f_{22}[x_1, x_2, x_3] + \xi_2 \xi_3 f_{23}[x_1, x_2, x_3] + \xi_1 \xi_3 f_{31}[x_1, x_2, x_3] + \xi_2 \xi_3 f_{32}[x_1, x_2, x_3] + \xi_3^2 f_{33}[x_1, x_2, x_3])],$
$\mathbb{E}[0, \frac{\cdots 1 \cdots}{-1 + \cdots 3 \cdots + a_{22}}, \frac{\cdots 140 \cdots + \epsilon\, a_{21}^2 \cdots 1 \cdots}{(\cdots 1 \cdots)^3}], \text{True}\}\}$

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```
In[]:= Timing@{
    Eh = 𝔼[0, h Sum[a_{10 i+j} x_i ξ_j, {i, 3}, {j, 3}],
       1 + ϵ Sum[f_i[x1, x2, x3] ξ_i, {i, 3}] + ϵ Sum[f_{10 i+j}[x1, x2, x3] ξ_i ξ_j, {i, 3}, {j, 3}]],
    lhs = Normal[Eh /. 𝔼[L_, Q_, P_] :> Series[P e^{L+Q}, {h, 0, 2}]] // Zip_{{ξ1}},
    HL@
     Simplify[lhs == Normal[QZip_{{ξ1},Simplify}[Eh] /. 𝔼[L_, Q_, P_] :> Series[P e^{L+Q}, {h, 0, 2}]]]}
```

Out[]=
$\{25.3125, \{\mathbb{E}[0,$
$h (a_{11} x_1 \xi_1 + a_{21} x_2 \xi_1 + a_{31} x_3 \xi_1 + a_{12} x_1 \xi_2 + a_{22} x_2 \xi_2 + a_{32} x_3 \xi_2 + a_{13} x_1 \xi_3 + a_{23} x_2 \xi_3 + a_{33} x_3 \xi_3),$
$1 + \epsilon (\xi_1 f_1[x_1, x_2, x_3] + \xi_2 f_2[x_1, x_2, x_3] + \xi_3 f_3[x_1, x_2, x_3]) + \epsilon (\xi_1^2 f_{11}[x_1, x_2, x_3] + \xi_1 \xi_2 f_{12}[x_1, x_2, x_3] + \xi_1 \xi_3 f_{13}[x_1, x_2, x_3] + \xi_1 \xi_2 f_{21}[x_1, x_2, x_3] + \xi_2^2 f_{22}[x_1, x_2, x_3] + \xi_2 \xi_3 f_{23}[x_1, x_2, x_3] + \xi_1 \xi_3 f_{31}[x_1, x_2, x_3] + \xi_2 \xi_3 f_{32}[x_1, x_2, x_3] + \xi_3^2 f_{33}[x_1, x_2, x_3])],$
$1 + h\, a_{11} + \cdots 577 \cdots + \frac{1}{2} h^2 \epsilon\, a_{31}^2 x_3^2 f_{11}^{(4,0,0)}[0, x_2, x_3], \text{True}\}\}$

large output | show less | show more | show all | set size limit...

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single"P". Here the z's are t and α and the ζ's are τ and a.

```
In[]:= Bind_{2}[𝔼[0, ξ (x1 + x2), 1], 𝔼[0, ξ2 (x2 + x3), 1]]
```

Out[]= $\mathbb{E}[0, \xi x_1 + \xi x_2 + \xi x_3, 1]$

```
In[]:= Bind_{2}[𝔼[0, (ξ2 + ξ3) x2, 1], 𝔼[0, (ξ1 + ξ2) x, 1]]
```

Out[]= $\mathbb{E}[0, x \xi_1 + x \xi_2 + x \xi_3, 1]$

```
In[]:= Bind_{1,2}[𝔼[0, (ξ2 + ξ3) x2 + ξ1 x1, 1], 𝔼[0, (ξ1 + ξ2) x, 1]]
```

Out[]= $\mathbb{E}[0, x \xi_1 + x \xi_2 + x \xi_3, 1]$

An xay → axy → ayx –> yax ≡ xay → xya → yxa → yax test:

```
In[]:= Bind[𝔼[α₁ a₁ + τ₁ t₁, e^(γ α₁) ξ₁ x₁ + η₁ y₁, 1], {1}, 𝔼[τ₁ t₁ + α₁ a₁, ξ₁ x₁ + η₁ y₁ + ξ₁ η₁ t₁, 1]]
Out[]= Bind[𝔼[a₁ α₁ + t₁ τ₁, y₁ η₁ + e^(γ α₁) x₁ ξ₁, 1], {1}, 𝔼[a₁ α₁ + t₁ τ₁, y₁ η₁ + x₁ ξ₁ + t₁ η₁ ξ₁, 1]]
```

```
In[]:= {rxa = 𝔼[τ₁ t₁ + α₁ a₁, e^(-γ α₁) ξ₁ x₁ + η₁ y₁, 1];
 rxy = 𝔼[τ₁ t₁ + α₁ a₁, ξ₁ x₁ + η₁ y₁ - ξ₁ η₁ t₁, 1];
 ray = 𝔼[τ₁ t₁ + α₁ a₁, e^(-γ α₁) η₁ y₁ + ξ₁ x₁, 1];
 lhs = Expand /@ rxa ~ B₁ ~ rxy ~ B₁ ~ ray,
 HL[lhs == Expand /@ ray ~ B₁ ~ rxy ~ B₁ ~ rxa ]}
```

$$\text{Out}= \left\{\mathbb{E}\left[a_1\,\alpha_1 + t_1\,\tau_1,\ \frac{y_1\,\eta_1}{\mathcal{A}_1} + \frac{x_1\,\xi_1}{\mathcal{A}_1} - \frac{t_1\,\eta_1\,\xi_1}{\mathcal{A}_1},\ 1\right],\ \text{True}\right\}$$

Tensorial Representations

Associativity of tm.

```
In[]:= Table[Block[{$U = U, $k = kk},
   {lhs = tm_{1,2→2} ~ B₂ ~ tm_{2,3→1};
    {$U, $k} -> HL[lhs ≡ tm_{2,3→2} ~ B₂ ~ tm_{1,2→1}]}
  ], {U, {CU, QU}}, {kk, 0, 1}]
Out[]= {{{{CU, 0} → True}, {{CU, 1} → True}}, {{{QU, 0} → True}, {{QU, 1} → True}}}
```

```
In[]:= Block[{$U = CU, $k = 2}, Timing@{lhs = tm_{1,2→2} ~ B₂ ~ tm_{2,3→1};
    HL[lhs ≡ tm_{2,3→2} ~ B₂ ~ tm_{1,2→1}]}]
Out[]= {1.54688, {True}}
```

tS is an anti-homomorphism for tm.

```
In[]:= HL[(tS₁ tS₂) ~ B_{1,2} ~ tm_{1,2→1} ≡ tm_{2,1→1} ~ B₁ ~ tS₁]
Out[]= True
```

Testing co-associativity.

```
In[]:= HL[tΔ_{1→1,2} ~ B₂ ~ tΔ_{2→2,3} ≡ tΔ_{1→1,3} ~ B₁ ~ tΔ_{1→1,2}]
Out[]= True
```

Testing S is an anti-co-homomorphism

```
In[]:= HL[tS₁ ~ B₁ ~ tΔ_{1→1,2} ≡ tΔ_{1→2,1} ~ B_{1,2} ~ (tS₁ tS₂)]
Out[]= True
```

Testing convolution inverse:

```
In[]:= {HL[tΔ_{1→1,2} ~ B₁ ~ tS₁ ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙],
  HL[tΔ_{1→1,2} ~ B₂ ~ tS₂ ~ B_{1,2} ~ tm_{1,2→1} ≡ tη ~ B_{} ~ t𝟙]}
Out[]= {True, True}
```

Testing R2

In[$\circ$]:= $\mathrm{HL}\left[\left(\overline{\mathrm{tR}}_{1,2}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to1}\ \mathrm{tm}_{2,4\to2}\right) == \mathrm{t}\mathbb{1}\right]$

Out[$\circ$]= True

Testing quasi-triangular axioms

In[$\circ$]:= $\mathrm{HL}\left[\left(\mathrm{t}\Delta_{1\text{->}1,2}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{1,3\to1}\ \mathrm{tm}_{2,4\to2}\right) \equiv \left(\mathrm{t}\Delta_{1\to2,1}\ \mathrm{tR}_{3,4}\right) \sim \mathrm{B}_{1,2,3,4} \sim \left(\mathrm{tm}_{3,1\to1}\ \mathrm{tm}_{4,2\to2}\right)\right]$

Out[$\circ$]= True

In[$\circ$]:= $\mathrm{HL}\left[\mathrm{tR}_{1,3} \sim \mathrm{B}_{1} \sim \mathrm{t}\Delta_{1\to1,2} \equiv \left(\mathrm{tR}_{1,4}\ \mathrm{tR}_{2,3}\right) \sim \mathrm{B}_{3,4} \sim \mathrm{tm}_{3,4\to3}\right]$

Out[$\circ$]= True

Testing R3

In[$\circ$]:= $\mathrm{HL}\left[\left(\mathrm{tR}_{2,3}\ \mathrm{tR}_{1,4}\ \mathrm{tR}_{5,6}\right) \sim \mathrm{B}_{\mathrm{Range@6}} \sim \left(\mathrm{tm}_{1,5\to1}\ \mathrm{tm}_{2,6\to2}\ \mathrm{tm}_{3,4\to3}\right) \equiv \left(\mathrm{tR}_{1,2}\ \mathrm{tR}_{5,3}\ \mathrm{tR}_{6,4}\right) \sim \mathrm{B}_{\mathrm{Range@6}} \sim \left(\mathrm{tm}_{1,5\to1}\ \mathrm{tm}_{2,6\to2}\ \mathrm{tm}_{3,4\to3}\right)\right]$

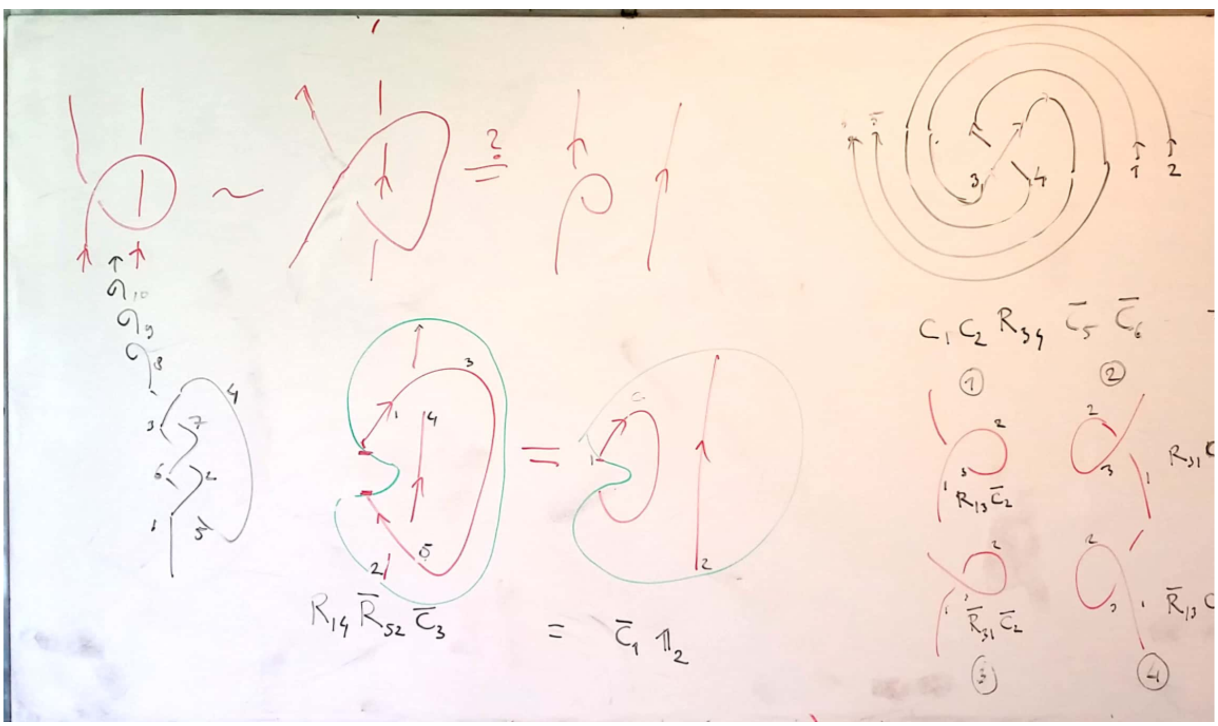
Out[$\circ$]= True

tC is the counterclockwise spinner; $\overline{\mathrm{tC}}$ is its inverse:

In[$\circ$]:= $\mathrm{Block}\left[\{\$k = 1\},\ \mathrm{HL}\left[\left(\mathrm{tC}_{1}\ \overline{\mathrm{tC}}_{2}\right) \sim \mathrm{B}_{1,2} \sim \mathrm{tm}_{1,2\to1} \equiv \mathrm{t}\mathbb{1}\right]\right]$

Out[$\circ$]= True

The 180 419 blackboard :



Cyclic R2 as on the 180419 blackboard:

In[∘]:= $$\text{Block}\left[\{\$k=2\},\ \text{HL}\left[\left(tR_{1,4}\,\overline{tR}_{5,2}\,\overline{tC}_3\right)\sim B_{\{1,3,2,4\}}\sim\left(tm_{1,3\to1}\,tm_{2,4\to2}\right)\sim B_{1,5}\sim tm_{1,5\to1}\equiv\overline{tC}_1\right]\right]\ //\ \text{Timing}$$

Out[∘]= {31.4688, True}

Swirl relation as on the 180419 blackboard:

In[∘]:= $$\text{Block}\Big[\{\$k=1\},\ \text{HL}\Big[tR_{1,2}\equiv\left(tC_1\,tC_2\,tR_{3,4}\,\overline{tC}_5\,\overline{tC}_6\right)\sim B_{1,2,3,4}\sim\left(tm_{1,3\to1}\,tm_{2,4\to2}\right)\sim B_{1,2,5,6}\sim\left(tm_{1,5\to1}\,tm_{2,6\to2}\right)\Big]\Big]\ //\ \text{Timing}$$

Out[∘]= {1.42188, True}

The Four Kinks as on the 180419 blackboard:

$$\begin{aligned}&\text{Timing@Block}\Big[\{\$k=2,K1,K2,K3,K4\},\\&\quad\text{Column@}\Big\{K1=\left(tR_{1,3}\,\overline{tC}_2\right)\sim B_{1,2}\sim tm_{1,2\to1}\sim B_{1,3}\sim tm_{1,3\to1},\ K2=\left(tR_{3,1}\,tC_2\right)\sim B_{1,2}\sim tm_{1,2\to1}\sim B_{1,3}\sim tm_{1,3\to1},\\&\qquad K3=\left(\overline{tR}_{3,1}\,\overline{tC}_2\right)\sim B_{1,2}\sim tm_{1,2\to1}\sim B_{1,3}\sim tm_{1,3\to1},\ K4=\left(\overline{tR}_{1,3}\,tC_2\right)\sim B_{1,2}\sim tm_{1,2\to1}\sim B_{1,3}\sim tm_{1,3\to1},\\&\qquad \text{HL}\ /@\ \Big\{\\&\qquad\quad K1\equiv tKink_1,\ K3\equiv\overline{tKink}_1,\\&\qquad\quad K1\equiv K2,\ K3\equiv K4,\\&\qquad\quad \left(K1\left(K3\sim B_{1,2}\sim tm_{1,2\to2}\right)\right)\sim B_{1,2}\sim tm_{1,2\to1}\equiv t\mathbb{1},\\&\qquad\quad K1\sim B_1\sim tS_1\equiv K1,\ K3\sim B_1\sim tS_1\equiv K3\Big\}\\&\quad\Big\}\Big]\end{aligned}$$

Out[∘]=

$$\mathbb{E}\Big[-\frac{\hbar a_1 t_1}{\gamma},\ \hbar x_1 y_1,\ \frac{1}{\sqrt{T_1}}+\frac{\left(4\gamma\hbar a_1+4\hbar a_1^2-\gamma^2\hbar^3x_1^2y_1^2\right)\epsilon}{4\gamma\sqrt{T_1}}+\frac{1}{288\gamma^2\sqrt{T_1}}\left(144\gamma^2\hbar^2a_1^2+288\gamma\hbar^2a_1^3+144\hbar^2a_1^4-72\gamma^3\hbar^4a_1x_1^2y_1^2-72\gamma^2\hbar^4a_1^2x_1^2y_1^2+32\gamma^4\hbar^5x_1^3y_1^3+9\gamma^4\hbar^6x_1^4y_1^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

$$\mathbb{E}\Big[-\frac{\hbar a_1 t_1}{\gamma},\ \hbar x_1 y_1,\ \frac{1}{\sqrt{T_1}}+\frac{\left(4\gamma\hbar a_1+4\hbar a_1^2-\gamma^2\hbar^3x_1^2y_1^2\right)\epsilon}{4\gamma\sqrt{T_1}}+\frac{1}{288\gamma^2\sqrt{T_1}}\left(144\gamma^2\hbar^2a_1^2+288\gamma\hbar^2a_1^3+144\hbar^2a_1^4-72\gamma^3\hbar^4a_1x_1^2y_1^2-72\gamma^2\hbar^4a_1^2x_1^2y_1^2+32\gamma^4\hbar^5x_1^3y_1^3+9\gamma^4\hbar^6x_1^4y_1^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

$$\mathbb{E}\Big[\frac{\hbar a_1 t_1}{\gamma},\ -\frac{\hbar x_1 y_1}{T_1},\ \sqrt{T_1}+\frac{\left(-4\gamma\hbar a_1T_1^2-4\hbar a_1^2T_1^2-8\gamma\hbar^2a_1T_1x_1y_1-3\gamma^2\hbar^3x_1^2y_1^2\right)\epsilon}{4\gamma T_1^{3/2}}+\frac{1}{288\gamma^2T_1^{7/2}}\left(144\gamma^2\hbar^2a_1^2T_1^4+288\gamma\hbar^2a_1^3T_1^4+144\hbar^2a_1^4T_1^4+576\gamma\hbar^3a_1^3T_1^3x_1y_1+144\gamma^4\hbar^4T_1^2x_1^2y_1^2-648\gamma^3\hbar^4a_1T_1^2x_1^2y_1^2+792\gamma^2\hbar^4a_1^2T_1^2x_1^2y_1^2-320\gamma^4\hbar^5T_1x_1^3y_1^3+432\gamma^3\hbar^5a_1T_1x_1^3y_1^3+81\gamma^4\hbar^6x_1^4y_1^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

$$\mathbb{E}\Big[\frac{\hbar a_1 t_1}{\gamma},\ -\frac{\hbar x_1 y_1}{T_1},\ \sqrt{T_1}+\frac{\left(-4\gamma\hbar a_1T_1^2-4\hbar a_1^2T_1^2-8\gamma\hbar^2a_1T_1x_1y_1-3\gamma^2\hbar^3x_1^2y_1^2\right)\epsilon}{4\gamma T_1^{3/2}}+\frac{1}{288\gamma^2T_1^{7/2}}\left(144\gamma^2\hbar^2a_1^2T_1^4+288\gamma\hbar^2a_1^3T_1^4+144\hbar^2a_1^4T_1^4+576\gamma\hbar^3a_1^3T_1^3x_1y_1+144\gamma^4\hbar^4T_1^2x_1^2y_1^2-648\gamma^3\hbar^4a_1T_1^2x_1^2y_1^2+792\gamma^2\hbar^4a_1^2T_1^2x_1^2y_1^2-320\gamma^4\hbar^5T_1x_1^3y_1^3+432\gamma^3\hbar^5a_1T_1x_1^3y_1^3+81\gamma^4\hbar^6x_1^4y_1^4\right)\epsilon^2+O[\epsilon]^3\Big]$$

{True, True, True, True, True, True, True}

Trefoil as on the 180419 blackboard:

In[∘]:= $$\mathbf{Timing}\Big[\ \mathbf{Z} = \mathbf{tR}_{1,5}\,\mathbf{tR}_{6,2}\,\mathbf{tR}_{3,7}\,\overline{\mathbf{tC}_4}\,\overline{\mathbf{tKink}_8}\,\overline{\mathbf{tKink}_9}\,\overline{\mathbf{tKink}_{10}};\ \mathbf{Do}[\mathbf{Z};\ \mathbf{Z} = \mathbf{Z} \sim \mathbf{B}_{1,k} \sim \mathbf{tm}_{1,k\to 1},\ \{k, 2, 10\}];\ \mathbf{Z}\Big]$$

Out[∘]= $$\Big\{113.813,\ \mathbb{E}\Big[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \Big(\big(-2\hbar a_1 T_1 - \gamma\hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\gamma\hbar^2 T_1 x_1 y_1 - 2\gamma\hbar^2 T_1^4 x_1 y_1\big)\,\epsilon\Big) \Big/ \big(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\big) + O[\epsilon]^2\Big]\Big\}$$

In[∘]:= $$\mathbf{Timing@Block}\Big[\{\$k = 2\},\ \mathbf{Z} = \mathbf{tR}_{1,5}\,\mathbf{tR}_{6,2}\,\mathbf{tR}_{3,7}\,\overline{\mathbf{tC}_4}\,\overline{\mathbf{tKink}_8}\,\overline{\mathbf{tKink}_9}\,\overline{\mathbf{tKink}_{10}};\ \mathbf{Do}[\mathbf{Z};\ \mathbf{Z} = \mathbf{Z} \sim \mathbf{B}_{1,k} \sim \mathbf{tm}_{1,k\to 1},\ \{k, 2, 10\}];\ \mathbf{Z}\Big]$$

Out[∘]= \$Aborted

In[∘]:= $$\mathbf{Timing@Block}\Big[\{\$k = 2\},\ \mathbf{Z} = \mathbf{tR}_{1,5}\,\mathbf{tR}_{6,2}\,\mathbf{tR}_{3,7}\,\overline{\mathbf{tC}_4}\,\overline{\mathbf{tKink}_8}\,\overline{\mathbf{tKink}_9}\,\overline{\mathbf{tKink}_{10}}\ /.\ T\_ \to T_1;\ \mathbf{Do}[\mathbf{Z} = \mathbf{Z} \sim \mathbf{B}_{1,k} \sim \mathbf{tm}_{1,k\to 1}\ /.\ T\_ \to T_1,\ \{k, 2, 10\}];\ \mathbf{Z}\Big]$$

Out[∘]= $$\Big\{160.547,\ \mathbb{E}\Big[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \Big(\big(-2\hbar a_1 T_1 - \gamma\hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\gamma\hbar^2 T_1 x_1 y_1 - 2\gamma\hbar^2 T_1^4 x_1 y_1\big)\,\epsilon\Big) \Big/ \big(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\big) + \Big(\big(4\hbar^2 a_1^2 T_1 + \gamma^2\hbar^2 T_1^2 + 8\gamma\hbar^2 a_1 T_1^2 - 4\hbar^2 a_1^2 T_1^2 - 2\gamma^2\hbar^2 T_1^3 - 28\gamma\hbar^2 a_1 T_1^3 - 20\hbar^2 a_1^2 T_1^3 + 4\gamma^2\hbar^2 T_1^4 + 60\gamma\hbar^2 a_1 T_1^4 + 56\hbar^2 a_1^2 T_1^4 - 2\gamma^2\hbar^2 T_1^5 - 80\gamma\hbar^2 a_1 T_1^5 - 80\hbar^2 a_1^2 T_1^5 + 52\gamma\hbar^2 a_1 T_1^6 + 56\hbar^2 a_1^2 T_1^6 + 6\gamma^2\hbar^2 T_1^7 - 12\gamma\hbar^2 a_1 T_1^7 - 20\hbar^2 a_1^2 T_1^7 - 11\gamma^2\hbar^2 T_1^8 - 16\gamma\hbar^2 a_1 T_1^8 - 4\hbar^2 a_1^2 T_1^8 + 4\gamma^2\hbar^2 T_1^9 + 8\gamma\hbar^2 a_1 T_1^9 + 4\hbar^2 a_1^2 T_1^9 - 4\gamma^2\hbar^3 T_1 x_1 y_1 + 8\gamma\hbar^3 a_1 T_1 x_1 y_1 + 8\gamma^2\hbar^3 T_1^2 x_1 y_1 + 8\gamma\hbar^3 a_1 T_1^2 x_1 y_1 - 48\gamma\hbar^3 a_1 T_1^3 x_1 y_1 + 4\gamma^2\hbar^3 T_1^4 x_1 y_1 + 88\gamma\hbar^3 a_1 T_1^4 x_1 y_1 + 4\gamma^2\hbar^3 T_1^5 x_1 y_1 - 80\gamma\hbar^3 a_1 T_1^5 x_1 y_1 + 24\gamma\hbar^3 a_1 T_1^6 x_1 y_1 + 8\gamma^2\hbar^3 T_1^7 x_1 y_1 + 8\gamma\hbar^3 a_1 T_1^7 x_1 y_1 - 4\gamma^2\hbar^3 T_1^8 x_1 y_1 - 16\gamma\hbar^3 a_1 T_1^8 x_1 y_1 + 6\gamma^2\hbar^4 T_1 x_1^2 y_1^2 + 6\gamma^2\hbar^4 T_1^2 x_1^2 y_1^2 - 12\gamma^2\hbar^4 T_1^3 x_1^2 y_1^2 + 30\gamma^2\hbar^4 T_1^4 x_1^2 y_1^2 - 12\gamma^2\hbar^4 T_1^5 x_1^2 y_1^2 + 6\gamma^2\hbar^4 T_1^6 x_1^2 y_1^2 + 6\gamma^2\hbar^4 T_1^7 x_1^2 y_1^2\big)\,\epsilon^2\Big) \Big/ \big(2 - 10T_1 + 30T_1^2 - 60T_1^3 + 90T_1^4 - 102T_1^5 + 90T_1^6 - 60T_1^7 + 30T_1^8 - 10T_1^9 + 2T_1^{10}\big) + O[\epsilon]^3\Big]\Big\}$$

In[∘]:= $$\mathbf{Timing@Block}\Big[\{\$k = 3\},\ \mathbf{Z} = \mathbf{tR}_{1,5}\,\mathbf{tR}_{6,2}\,\mathbf{tR}_{3,7}\,\overline{\mathbf{tC}_4}\,\overline{\mathbf{tKink}_8}\,\overline{\mathbf{tKink}_9}\,\overline{\mathbf{tKink}_{10}}\ /.\ T\_ \to T_1;\ \mathbf{Do}[\mathbf{Z} = \mathbf{Z} \sim \mathbf{B}_{1,k} \sim \mathbf{tm}_{1,k\to 1}\ /.\ T\_ \to T_1,\ \{k, 2, 10\}];\ \mathbf{Z}\Big]$$

Out[∘]= $$\Big\{2340.17,\ \mathbb{E}\Big[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \Big(\big(-2\hbar a_1 T_1 - \gamma\hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\gamma\hbar T_1^3 - 3\gamma\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\gamma\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\gamma\hbar^2 T_1 x_1 y_1 - 2\gamma\hbar^2 T_1^4 x_1 y_1\big)\,\epsilon\Big) \Big/ \big(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\big) + \Big(\big(4\hbar^2 a_1^2 T_1 + \gamma^2\hbar^2 T_1^2 + 8\gamma\hbar^2 a_1 T_1^2 - 4\hbar^2 a_1^2 T_1^2 - 2\gamma^2\hbar^2 T_1^3 - 28\gamma\hbar^2 a_1 T_1^3 - 20\hbar^2 a_1^2 T_1^3 + 4\gamma^2\hbar^2 T_1^4 + 60\gamma\hbar^2 a_1 T_1^4 + 56\hbar^2 a_1^2 T_1^4 - 2\gamma^2\hbar^2 T_1^5 - 80\gamma\hbar^2 a_1 T_1^5 - 80\hbar^2 a_1^2 T_1^5 + 52\gamma\hbar^2 a_1 T_1^6 + 56\hbar^2 a_1^2 T_1^6 + 6\gamma^2\hbar^2 T_1^7 - 12\gamma\hbar^2 a_1 T_1^7 - 20\hbar^2 a_1^2 T_1^7 - 11\gamma^2\hbar^2 T_1^8 - 16\gamma\hbar^2 a_1 T_1^8 - 4\hbar^2 a_1^2 T_1^8 + 4\gamma^2\hbar^2 T_1^9 + 8\gamma\hbar^2 a_1 T_1^9 + 4\hbar^2 a_1^2 T_1^9 - 4\gamma^2\hbar^3 T_1 x_1 y_1 + 8\gamma\hbar^3 a_1 T_1 x_1 y_1 + 8\gamma^2\hbar^3 T_1^2 x_1 y_1 + 8\gamma\hbar^3 a_1 T_1^2 x_1 y_1 - 48\gamma\hbar^3 a_1 T_1^3 x_1 y_1 + 4\gamma^2\hbar^3 T_1^4 x_1 y_1 + 88\gamma\hbar^3 a_1 T_1^4 x_1 y_1 + 4\gamma^2\hbar^3 T_1^5 x_1 y_1 - 80\gamma\hbar^3 a_1 T_1^5 x_1 y_1 + 24\gamma\hbar^3 a_1 T_1^6 x_1 y_1 + 8\gamma^2\hbar^3 T_1^7 x_1 y_1 + 8\gamma\hbar^3 a_1 T_1^7 x_1 y_1 -$$

$4\,\gamma^2\,\hbar^3\,T_1^8\,x_1\,y_1 - 16\,\gamma\,\hbar^3\,a_1\,T_1^8\,x_1\,y_1 + 6\,\gamma^2\,\hbar^4\,T_1\,x_1^2\,y_1^2 + 6\,\gamma^2\,\hbar^4\,T_1^2\,x_1^2\,y_1^2 - 12\,\gamma^2\,\hbar^4\,T_1^3\,x_1^2\,y_1^2 +$
$30\,\gamma^2\,\hbar^4\,T_1^4\,x_1^2\,y_1^2 - 12\,\gamma^2\,\hbar^4\,T_1^5\,x_1^2\,y_1^2 + 6\,\gamma^2\,\hbar^4\,T_1^6\,x_1^2\,y_1^2 + 6\,\gamma^2\,\hbar^4\,T_1^7\,x_1^2\,y_1^2\big)\,\epsilon^2\big)\,\big/$
$\big(2 - 10\,T_1 + 30\,T_1^2 - 60\,T_1^3 + 90\,T_1^4 - 102\,T_1^5 + 90\,T_1^6 - 60\,T_1^7 + 30\,T_1^8 - 10\,T_1^9 + 2\,T_1^{10}\big) +$
$\big(\big(-32\,\hbar^3\,a_1^3\,T_1^4 - 4\,\gamma^3\,\hbar^3\,T_1^5 - 48\,\gamma^2\,\hbar^3\,a_1\,T_1^5 - 192\,\gamma\,\hbar^3\,a_1^2\,T_1^5 - 32\,\hbar^3\,a_1^3\,T_1^5 + 120\,\gamma^2\,\hbar^3\,a_1\,T_1^6 +$
$912\,\gamma\,\hbar^3\,a_1^2\,T_1^6 + 896\,\hbar^3\,a_1^3\,T_1^6 + 12\,\gamma^3\,\hbar^3\,T_1^7 - 216\,\gamma^2\,\hbar^3\,a_1\,T_1^7 - 2352\,\gamma\,\hbar^3\,a_1^2\,T_1^7 - 2656\,\hbar^3\,a_1^3\,T_1^7 -$
$88\,\gamma^3\,\hbar^3\,T_1^8 - 24\,\gamma^2\,\hbar^3\,a_1\,T_1^8 + 3888\,\gamma\,\hbar^3\,a_1^2\,T_1^8 + 4224\,\hbar^3\,a_1^3\,T_1^8 + 172\,\gamma^3\,\hbar^3\,T_1^9 + 672\,\gamma^2\,\hbar^3\,a_1\,T_1^9 -$
$2592\,\gamma\,\hbar^3\,a_1^2\,T_1^9 - 3360\,\hbar^3\,a_1^3\,T_1^9 - 304\,\gamma^3\,\hbar^3\,T_1^{10} - 2016\,\gamma^2\,\hbar^3\,a_1\,T_1^{10} - 2016\,\gamma\,\hbar^3\,a_1^2\,T_1^{10} +$
$268\,\gamma^3\,\hbar^3\,T_1^{11} + 4224\,\gamma^2\,\hbar^3\,a_1\,T_1^{11} + 7488\,\gamma\,\hbar^3\,a_1^2\,T_1^{11} + 3360\,\hbar^3\,a_1^3\,T_1^{11} - 400\,\gamma^3\,\hbar^3\,T_1^{12} -$
$4872\,\gamma^2\,\hbar^3\,a_1\,T_1^{12} - 8784\,\gamma\,\hbar^3\,a_1^2\,T_1^{12} - 4224\,\hbar^3\,a_1^3\,T_1^{12} + 532\,\gamma^3\,\hbar^3\,T_1^{13} + 3480\,\gamma^2\,\hbar^3\,a_1\,T_1^{13} +$
$5616\,\gamma\,\hbar^3\,a_1^2\,T_1^{13} + 2656\,\hbar^3\,a_1^3\,T_1^{13} - 104\,\gamma^3\,\hbar^3\,T_1^{14} - 984\,\gamma^2\,\hbar^3\,a_1\,T_1^{14} - 1776\,\gamma\,\hbar^3\,a_1^2\,T_1^{14} -$
$896\,\hbar^3\,a_1^3\,T_1^{14} - 116\,\gamma^3\,\hbar^3\,T_1^{15} - 240\,\gamma^2\,\hbar^3\,a_1\,T_1^{15} - 96\,\gamma\,\hbar^3\,a_1^2\,T_1^{15} + 32\,\hbar^3\,a_1^3\,T_1^{15} + 32\,\gamma^3\,\hbar^3\,T_1^{16} +$
$96\,\gamma^2\,\hbar^3\,a_1\,T_1^{16} + 96\,\gamma\,\hbar^3\,a_1^2\,T_1^{16} + 32\,\hbar^3\,a_1^3\,T_1^{16} + 12\,\hbar^4\,a_1^3\,T_1^3\,x_1\,y_1 - 32\,\gamma^3\,\hbar^4\,T_1^4\,x_1\,y_1 +$
$96\,\gamma^2\,\hbar^4\,a_1\,T_1^4\,x_1\,y_1 - 96\,\gamma\,\hbar^4\,a_1^2\,T_1^4\,x_1\,y_1 - 72\,\hbar^4\,a_1^3\,T_1^4\,x_1\,y_1 + 80\,\gamma^3\,\hbar^4\,T_1^5\,x_1\,y_1 -$
$96\,\gamma^2\,\hbar^4\,a_1\,T_1^5\,x_1\,y_1 - 480\,\gamma\,\hbar^4\,a_1^2\,T_1^5\,x_1\,y_1 + 252\,\hbar^4\,a_1^3\,T_1^5\,x_1\,y_1 + 184\,\gamma^3\,\hbar^4\,T_1^6\,x_1\,y_1 -$
$1344\,\gamma^2\,\hbar^4\,a_1\,T_1^6\,x_1\,y_1 + 2784\,\gamma\,\hbar^4\,a_1^2\,T_1^6\,x_1\,y_1 - 600\,\hbar^4\,a_1^3\,T_1^6\,x_1\,y_1 - 336\,\gamma^3\,\hbar^4\,T_1^7\,x_1\,y_1 +$
$2592\,\gamma^2\,\hbar^4\,a_1\,T_1^7\,x_1\,y_1 - 5184\,\gamma\,\hbar^4\,a_1^2\,T_1^7\,x_1\,y_1 + 1080\,\hbar^4\,a_1^3\,T_1^7\,x_1\,y_1 - 24\,\gamma^3\,\hbar^4\,T_1^8\,x_1\,y_1 -$
$3168\,\gamma^2\,\hbar^4\,a_1\,T_1^8\,x_1\,y_1 + 4320\,\gamma\,\hbar^4\,a_1^2\,T_1^8\,x_1\,y_1 - 1512\,\hbar^4\,a_1^3\,T_1^8\,x_1\,y_1 - 120\,\gamma^3\,\hbar^4\,T_1^9\,x_1\,y_1 +$
$2304\,\gamma^2\,\hbar^4\,a_1\,T_1^9\,x_1\,y_1 + 2592\,\gamma\,\hbar^4\,a_1^2\,T_1^9\,x_1\,y_1 + 1692\,\hbar^4\,a_1^3\,T_1^9\,x_1\,y_1 - 120\,\gamma^3\,\hbar^4\,T_1^{10}\,x_1\,y_1 -$
$2016\,\gamma^2\,\hbar^4\,a_1\,T_1^{10}\,x_1\,y_1 - 10\,080\,\gamma\,\hbar^4\,a_1^2\,T_1^{10}\,x_1\,y_1 - 1512\,\hbar^4\,a_1^3\,T_1^{10}\,x_1\,y_1 - 24\,\gamma^3\,\hbar^4\,T_1^{11}\,x_1\,y_1 +$
$2592\,\gamma^2\,\hbar^4\,a_1\,T_1^{11}\,x_1\,y_1 + 12\,672\,\gamma\,\hbar^4\,a_1^2\,T_1^{11}\,x_1\,y_1 + 1080\,\hbar^4\,a_1^3\,T_1^{11}\,x_1\,y_1 - 336\,\gamma^3\,\hbar^4\,T_1^{12}\,x_1\,y_1 -$
$1728\,\gamma^2\,\hbar^4\,a_1\,T_1^{12}\,x_1\,y_1 - 8352\,\gamma\,\hbar^4\,a_1^2\,T_1^{12}\,x_1\,y_1 - 600\,\hbar^4\,a_1^3\,T_1^{12}\,x_1\,y_1 + 184\,\gamma^3\,\hbar^4\,T_1^{13}\,x_1\,y_1 +$
$672\,\gamma^2\,\hbar^4\,a_1\,T_1^{13}\,x_1\,y_1 + 2784\,\gamma\,\hbar^4\,a_1^2\,T_1^{13}\,x_1\,y_1 + 252\,\hbar^4\,a_1^3\,T_1^{13}\,x_1\,y_1 + 80\,\gamma^3\,\hbar^4\,T_1^{14}\,x_1\,y_1 +$
$480\,\gamma^2\,\hbar^4\,a_1\,T_1^{14}\,x_1\,y_1 + 96\,\gamma\,\hbar^4\,a_1^2\,T_1^{14}\,x_1\,y_1 - 72\,\hbar^4\,a_1^3\,T_1^{14}\,x_1\,y_1 - 32\,\gamma^3\,\hbar^4\,T_1^{15}\,x_1\,y_1 -$
$192\,\gamma^2\,\hbar^4\,a_1\,T_1^{15}\,x_1\,y_1 - 384\,\gamma\,\hbar^4\,a_1^2\,T_1^{15}\,x_1\,y_1 + 12\,\hbar^4\,a_1^3\,T_1^{15}\,x_1\,y_1 - 6\,\gamma^3\,\hbar^5\,T_1^2\,x_1^2\,y_1^2 +$
$18\,\gamma^2\,\hbar^5\,a_1\,T_1^2\,x_1^2\,y_1^2 - 18\,\gamma\,\hbar^5\,a_1^2\,T_1^2\,x_1^2\,y_1^2 + 36\,\gamma^3\,\hbar^5\,T_1^3\,x_1^2\,y_1^2 - 108\,\gamma^2\,\hbar^5\,a_1\,T_1^3\,x_1^2\,y_1^2 +$
$108\,\gamma\,\hbar^5\,a_1^2\,T_1^3\,x_1^2\,y_1^2 + 18\,\gamma^3\,\hbar^5\,T_1^4\,x_1^2\,y_1^2 + 234\,\gamma^2\,\hbar^5\,a_1\,T_1^4\,x_1^2\,y_1^2 - 378\,\gamma\,\hbar^5\,a_1^2\,T_1^4\,x_1^2\,y_1^2 +$
$588\,\gamma^3\,\hbar^5\,T_1^5\,x_1^2\,y_1^2 - 1620\,\gamma^2\,\hbar^5\,a_1\,T_1^5\,x_1^2\,y_1^2 + 900\,\gamma\,\hbar^5\,a_1^2\,T_1^5\,x_1^2\,y_1^2 - 2340\,\gamma^3\,\hbar^5\,T_1^6\,x_1^2\,y_1^2 +$
$4068\,\gamma^2\,\hbar^5\,a_1\,T_1^6\,x_1^2\,y_1^2 - 1620\,\gamma\,\hbar^5\,a_1^2\,T_1^6\,x_1^2\,y_1^2 + 3132\,\gamma^3\,\hbar^5\,T_1^7\,x_1^2\,y_1^2 - 6012\,\gamma^2\,\hbar^5\,a_1\,T_1^7\,x_1^2\,y_1^2 +$
$2268\,\gamma\,\hbar^5\,a_1^2\,T_1^7\,x_1^2\,y_1^2 - 2574\,\gamma^3\,\hbar^5\,T_1^8\,x_1^2\,y_1^2 + 4122\,\gamma^2\,\hbar^5\,a_1\,T_1^8\,x_1^2\,y_1^2 - 2538\,\gamma\,\hbar^5\,a_1^2\,T_1^8\,x_1^2\,y_1^2 -$
$900\,\gamma^3\,\hbar^5\,T_1^9\,x_1^2\,y_1^2 + 1332\,\gamma^2\,\hbar^5\,a_1\,T_1^9\,x_1^2\,y_1^2 + 2268\,\gamma\,\hbar^5\,a_1^2\,T_1^9\,x_1^2\,y_1^2 + 2052\,\gamma^3\,\hbar^5\,T_1^{10}\,x_1^2\,y_1^2 -$
$5436\,\gamma^2\,\hbar^5\,a_1\,T_1^{10}\,x_1^2\,y_1^2 - 1620\,\gamma\,\hbar^5\,a_1^2\,T_1^{10}\,x_1^2\,y_1^2 - 2508\,\gamma^3\,\hbar^5\,T_1^{11}\,x_1^2\,y_1^2 + 5724\,\gamma^2\,\hbar^5\,a_1\,T_1^{11}\,x_1^2\,y_1^2 +$
$900\,\gamma\,\hbar^5\,a_1^2\,T_1^{11}\,x_1^2\,y_1^2 + 666\,\gamma^3\,\hbar^5\,T_1^{12}\,x_1^2\,y_1^2 - 2358\,\gamma^2\,\hbar^5\,a_1\,T_1^{12}\,x_1^2\,y_1^2 - 378\,\gamma\,\hbar^5\,a_1^2\,T_1^{12}\,x_1^2\,y_1^2 -$
$252\,\gamma^3\,\hbar^5\,T_1^{13}\,x_1^2\,y_1^2 + 324\,\gamma^2\,\hbar^5\,a_1\,T_1^{13}\,x_1^2\,y_1^2 + 108\,\gamma\,\hbar^5\,a_1^2\,T_1^{13}\,x_1^2\,y_1^2 - 150\,\gamma^3\,\hbar^5\,T_1^{14}\,x_1^2\,y_1^2 +$
$450\,\gamma^2\,\hbar^5\,a_1\,T_1^{14}\,x_1^2\,y_1^2 - 18\,\gamma\,\hbar^5\,a_1^2\,T_1^{14}\,x_1^2\,y_1^2 - 12\,\gamma^3\,\hbar^6\,T_1\,x_1^3\,y_1^3 + 12\,\gamma^2\,\hbar^6\,a_1\,T_1\,x_1^3\,y_1^3 +$
$72\,\gamma^3\,\hbar^6\,T_1^2\,x_1^3\,y_1^3 - 72\,\gamma^2\,\hbar^6\,a_1\,T_1^2\,x_1^3\,y_1^3 - 252\,\gamma^3\,\hbar^6\,T_1^3\,x_1^3\,y_1^3 + 252\,\gamma^2\,\hbar^6\,a_1\,T_1^3\,x_1^3\,y_1^3 +$
$504\,\gamma^3\,\hbar^6\,T_1^4\,x_1^3\,y_1^3 - 600\,\gamma^2\,\hbar^6\,a_1\,T_1^4\,x_1^3\,y_1^3 - 1368\,\gamma^3\,\hbar^6\,T_1^5\,x_1^3\,y_1^3 + 1080\,\gamma^2\,\hbar^6\,a_1\,T_1^5\,x_1^3\,y_1^3 +$
$2088\,\gamma^3\,\hbar^6\,T_1^6\,x_1^3\,y_1^3 - 1512\,\gamma^2\,\hbar^6\,a_1\,T_1^6\,x_1^3\,y_1^3 - 2844\,\gamma^3\,\hbar^6\,T_1^7\,x_1^3\,y_1^3 + 1692\,\gamma^2\,\hbar^6\,a_1\,T_1^7\,x_1^3\,y_1^3 +$
$1800\,\gamma^3\,\hbar^6\,T_1^8\,x_1^3\,y_1^3 - 1512\,\gamma^2\,\hbar^6\,a_1\,T_1^8\,x_1^3\,y_1^3 - 792\,\gamma^3\,\hbar^6\,T_1^9\,x_1^3\,y_1^3 + 1080\,\gamma^2\,\hbar^6\,a_1\,T_1^9\,x_1^3\,y_1^3 -$
$552\,\gamma^3\,\hbar^6\,T_1^{10}\,x_1^3\,y_1^3 - 600\,\gamma^2\,\hbar^6\,a_1\,T_1^{10}\,x_1^3\,y_1^3 + 324\,\gamma^3\,\hbar^6\,T_1^{11}\,x_1^3\,y_1^3 + 252\,\gamma^2\,\hbar^6\,a_1\,T_1^{11}\,x_1^3\,y_1^3 -$
$216\,\gamma^3\,\hbar^6\,T_1^{12}\,x_1^3\,y_1^3 - 72\,\gamma^2\,\hbar^6\,a_1\,T_1^{12}\,x_1^3\,y_1^3 - 108\,\gamma^3\,\hbar^6\,T_1^{13}\,x_1^3\,y_1^3 + 12\,\gamma^2\,\hbar^6\,a_1\,T_1^{13}\,x_1^3\,y_1^3 -$
$3\,\gamma^3\,\hbar^7\,x_1^4\,y_1^4 + 18\,\gamma^3\,\hbar^7\,T_1\,x_1^4\,y_1^4 - 63\,\gamma^3\,\hbar^7\,T_1^2\,x_1^4\,y_1^4 + 150\,\gamma^3\,\hbar^7\,T_1^3\,x_1^4\,y_1^4 - 270\,\gamma^3\,\hbar^7\,T_1^4\,x_1^4\,y_1^4 +$
$378\,\gamma^3\,\hbar^7\,T_1^5\,x_1^4\,y_1^4 - 423\,\gamma^3\,\hbar^7\,T_1^6\,x_1^4\,y_1^4 + 378\,\gamma^3\,\hbar^7\,T_1^7\,x_1^4\,y_1^4 - 270\,\gamma^3\,\hbar^7\,T_1^8\,x_1^4\,y_1^4 +$
$150\,\gamma^3\,\hbar^7\,T_1^9\,x_1^4\,y_1^4 - 63\,\gamma^3\,\hbar^7\,T_1^{10}\,x_1^4\,y_1^4 + 18\,\gamma^3\,\hbar^7\,T_1^{11}\,x_1^4\,y_1^4 - 3\,\gamma^3\,\hbar^7\,T_1^{12}\,x_1^4\,y_1^4\big)\,\epsilon^3\big)\,\big/$
$\big(24\,T_1^3 - 168\,T_1^4 + 672\,T_1^5 - 1848\,T_1^6 + 3864\,T_1^7 - 6384\,T_1^8 + 8568\,T_1^9 - 9432\,T_1^{10} + 8568\,T_1^{11} -$
$6384\,T_1^{12} + 3864\,T_1^{13} - 1848\,T_1^{14} + 672\,T_1^{15} - 168\,T_1^{16} + 24\,T_1^{17}\big) + O[\epsilon]^4\big]\big\}$

Alternative Algorithms

In[∘]:=
```
Block[{$U = CU, $k = 2}, {
  λ_{alt,$k}[$U],
  HL@Simplify[Normal@λ_{alt,$k}[$U] == Normal@Last[tSW_{xy,1,1→1}] /. ν_-1 :→ ν]
 }]
```

Out[∘]=
$$\left\{1+\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)\epsilon+\frac{1}{2}\left(\left(2\,a\,\eta\,\xi-y\,\gamma\,\eta^2\,\xi-x\,\gamma\,\eta\,\xi^2+\frac{1}{2}\,t\,\gamma\,\eta^2\,\xi^2\right)^2+2\left(-a\,\gamma\,\eta^2\,\xi^2+y\,\gamma^2\,\eta^3\,\xi^2+x\,\gamma^2\,\eta^2\,\xi^3-\frac{1}{3}\,t\,\gamma^2\,\eta^3\,\xi^3\right)\right)\epsilon^2+O[\epsilon]^3,\ \text{True}\right\}$$

Genus Computations

In[∘]:=
```
Block[{$k = 0},
 (tΔ_{1→1,2} tΔ_{2→3,4}) ~B_{2,4}~ (tS_2 tS_4) // m_{1,4→1} // m_{1,2→1}
]
```

Out[∘]=
$$\mathbb{E}\Big[-a_1\,\alpha_2+a_3\,\alpha_2-t_1\,\tau_2+t_3\,\tau_2,\ \frac{1}{\hbar\,T_1}\,e^{-\gamma\,\alpha_1}\,\big(e^{\gamma\,\alpha_1}\,\hbar\,T_1\,y_1\,\eta_1-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,T_1\,y_1\,\eta_1-e^{\gamma\,\alpha_2}\,\hbar\,T_3\,y_1\,\eta_2+e^{\gamma\,\alpha_1}\,\hbar\,T_1\,y_3\,\eta_2-e^{2\,\gamma\,\alpha_1}\,\hbar\,T_1\,x_1\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,T_1\,x_1\,\xi_1+e^{2\,\gamma\,\alpha_1}\,T_1\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1}\,T_1^2\,\eta_1\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1^2\,\eta_1\,\xi_1-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_3\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,T_3\,\eta_2\,\xi_1-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,T_1\,x_1\,\xi_2+e^{\gamma\,\alpha_1}\,\hbar\,T_1\,x_3\,\xi_2+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_2-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1^2\,\eta_1\,\xi_2+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_3\,\eta_2\,\xi_2-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,T_3\,\eta_2\,\xi_2\big),\ 1+O[\epsilon]^1\Big]$$

In[∘]:=
```
Block[{$k = 0},
 (tΔ_{1→1,2} tΔ_{2→3,4}) ~B_{2,4}~ (tS_2 tS_4) // m_{1,4→1} // m_{1,2→1} // m_{1,3→1}
]
```

Out[∘]=
$$\mathbb{E}\Big[0,\ \frac{1}{\hbar}\,e^{-\gamma\,\alpha_1-\gamma\,\alpha_2}\,\big(e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_1-e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_1-e^{2\,\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_2+e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_2-e^{2\,\gamma\,\alpha_1}\,\hbar\,x_1\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_1+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_2\,\xi_1-e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_1+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_2-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_2+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_1\,\xi_2-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_2+e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_2-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_2-e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_2+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_2\big),\ 1+O[\epsilon]^1\Big]$$

In[∘]:=
$$\text{Expand}\Big[e^{-\gamma\,\alpha_1-\gamma\,\alpha_2}\,\big(e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_1-e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_1-e^{2\,\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_2+e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_2-e^{2\,\gamma\,\alpha_1}\,\hbar\,x_1\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_1+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_1-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_2\,\xi_1-e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_1+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_1+e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_2-e^{2\,\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_2+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_1\,\xi_2-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_2+e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_2-e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,\eta_2\,\xi_2-e^{\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_2+e^{2\,\gamma\,\alpha_1+2\,\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_2\big)\Big]$$

Out[∘]=
$$\hbar\,y_1\,\eta_1-e^{\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_1+e^{\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_2-e^{-\gamma\,\alpha_1+\gamma\,\alpha_2}\,\hbar\,y_1\,\eta_2+e^{\gamma\,\alpha_1}\,\hbar\,x_1\,\xi_1-e^{\gamma\,\alpha_1-\gamma\,\alpha_2}\,\hbar\,x_1\,\xi_1+e^{\gamma\,\alpha_1}\,\eta_1\,\xi_1-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_1\,\xi_1-e^{\gamma\,\alpha_1}\,T_1\,\eta_1\,\xi_1+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_1-e^{\gamma\,\alpha_1}\,\eta_2\,\xi_1-e^{\gamma\,\alpha_2}\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_2\,\xi_1+e^{\gamma\,\alpha_1}\,T_1\,\eta_2\,\xi_1+e^{\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_1+\hbar\,x_1\,\xi_2-e^{\gamma\,\alpha_1}\,\hbar\,x_1\,\xi_2+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_1\,\xi_2-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_1\,\xi_2+e^{\gamma\,\alpha_2}\,\eta_2\,\xi_2-e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,\eta_2\,\xi_2-e^{\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_2+e^{\gamma\,\alpha_1+\gamma\,\alpha_2}\,T_1\,\eta_2\,\xi_2$$