Case II: new edge $a \rightarrow b$ with (a) new, (b) old vertex, (c) any other old vertex.

Suppose $a \rightarrow \cdots \rightarrow c$ in join term, then in fact $a \rightarrow b \rightarrow \cdots \rightarrow c$, hence $b \rightarrow \cdots \rightarrow c$ in old join term, hence $b \rightarrow \cdots \rightarrow c$ in all other old terms, hence $a \rightarrow b \rightarrow \cdots \rightarrow c$ in all new terms.

Case III: new edge $a \rightarrow b$ with (a,b) old vertices, (c) any other old vertex.

Let (a),(b) old vertices in components of (a),(b) resp.

If $a \rightarrow \cdots \rightarrow c$ in join term, then in fact $c \rightarrow a \rightarrow b \rightarrow \cdots \rightarrow d$, hence $c \rightarrow a \rightarrow b$ and $b \rightarrow \cdots \rightarrow d$ in old join, hence $c \rightarrow a$ and $b \rightarrow \cdots \rightarrow d$ in other old terms, so $c \rightarrow a \rightarrow b \rightarrow \cdots \rightarrow d$ also.

Loops are $0$ oriented loops:

Get shorter oriented loop, use induction and

Unoriented loops:

Get shorter loops, use induction and

Chain Graphs are Linearly Independent

- le, all reductions of forests to chain graphs give same result, use induction on size of defect,
- True for forests of defect 1 as $3!$ reduction,
- Suppose true for defect $< p$.

Let $F$ be forest w/ Defect $p$, $> p$ reductions.

- If $a,b$ do not overlap, have common reduction $c$.
- Defect $A,B \leq p$ so apply induction.

- If $a,b$ overlap, find $a',a'' b',b''$ leading.
- Apply induction.

Overlaps: 3 types (by inspection)

One checks "by hand" that all ways of reducing give same result.

Koszulness: "A quotient of an exterior algebra with quadratic Gröbner basis (an exterior algebra) is Koszul (Yaovinsky).

Order on AS monomials: order generators numerically: $a_1 < a_2 < \cdots < a_2 < a_3 < \cdots$
- Order factors in monomial in increasing order.
- Order monomials length-lexicographically.
- (ignore signs, $0 < x \neq 0$).
- Multiplicative: $a \cdot b < b \cdot a$ if $a \neq 0$.
- Relations are equivalent to relations with maximal terms $b_k: a_i \leq a_j, k < j$.

Constancy of Dimension

\[ \text{all } S_n^{(m/k)} \text{ are all } S_n^{(m/k)} \text{ legal monomials} \]

\[ \text{Let } S_n^{(m/k)} \text{ be legal monomials of length } k. \]

\[ S_n^{(m/k)} \text{ generates } A_n^{(k)} \text{ by multiplicativity.} \]

\[ \dim S_n^{(m/k)} = L(n,m-k) = \dim A_n^{(k)} \text{ by BASIS} \]

Up Graphs: $\dim S_n^{(m/k)} = S(m,n-k)$

Down Graphs: $\dim S_n^{(m/k)} = 1$

Up-Down Graphs: Given collection of Up Graphs only allow Down arrows between roots of the Up Graphs.

- Counting: Divide $n$ into $k$ cycles and place Up Graphs on the cycles. Let any be root of cycle $C$. \[ \sum_{m=1}^{\infty} S(m,\ast) = S(n,k) \]

- Partition $M = m_1 \cdots m_k$ as $M_1 \cdots M_k$ and place Down Graphs on the $M_i$.

- $\dim S_n^{(m/k)} = \sum_{k=0}^{\infty} S(m,\ast) S(k,k) = L(m,n) = \dim A_n^{(m-k)}$. 

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