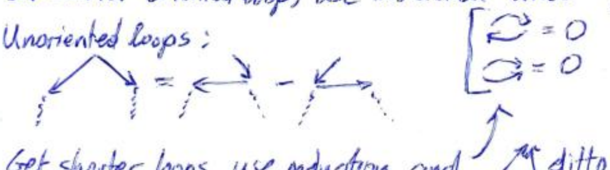
Case II: new edge $a - b$ with (a) new, (b) old vertex, (c) any other old vertex.
Suppose $a \to \dots \to c$ in join term, then in fact $a \to b \to \dots \to c$, hence $b \to \dots \to c$ in old join term, hence $b \to \dots \to c$ in all other old terms, hence $a \to b \to \dots \to c$ in all new terms.

Case III new edge $a - b$ with (a),(b) old vertices, (c) any other old vertex
Let (c),(d) old vertices in components of (a),(b) resp.
If $c \to \dots \to d$ in join term, then in fact $c \to \dots \to a \to b \to \dots \to d$, hence $c \to \dots \to a$ and $b \to \dots \to d$ in old join, hence $c \to \dots \to a$ and $b \to \dots \to d$ in other old terms, so $c \to \dots \to d$ also.

Loops are 0 Oriented loops:

Get shorter oriented loop, use induction and

Unoriented loops:

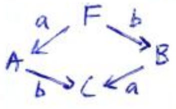


Get shorter loops, use induction and ditto.

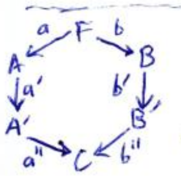
Chain Gangs are Linearly Independent

- Ie, all reductions of forests to chain gangs give same result: use induction on size of defect.
- True for forests of defect 1 as $\exists!$ reduction.
- Suppose true for defect $\leq p$.

Let F be forest w/ Defect p+1, $\geq$ reductions.



- If a,b do not overlap, have common reduction C.
- Defect A,B $\leq p$ so apply induction.



- If a,b overlap, find a', a'', b', b'' leading C
- Apply induction.

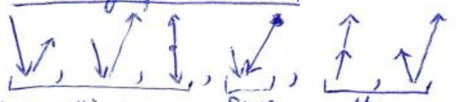
Overlaps 3 types (by inspection)

One checks "by hand" that all ways of reducing give same result.

KOSZULNESS "A quotient of an exterior algebra with quadratic Grobner basis (as exterior algebra) is Koszul (Yuzvinsky).

Order on AS monomials:
- order generators numerically: $a_{12} < a_{13} < \dots < a_{21} < a_{23} \dots$
- Order factors in monomial in increasing order.
- Order monomials length-lexicographically. (ignore signs, $0 < \alpha \neq 0$)
- Multiplicative: if $\alpha < \beta$ then $\alpha\gamma < \beta\gamma$ if $\beta\gamma \neq 0$.
- Relations are equivalent to relations with maximal terms k: all $a_{ij} \wedge a_{jk}$, $i < k < j$

Constancy of Dimension



are all length 2 legal monomials

Let $S_n^{(k)}$ be legal monomials of length k.
- $S_n^{(k)}$ generates $A_n^{!k}$ by multiplicativity.
- $\dim S_n^{(k)} = L(n, n-k) = \dim A_n^{!k}$ by BASIS Chain Gangs.

Up Graphs: $\dim S_n^{Up(k)} = s(n, n-k)$
Down Graphs: $\dim S_n^{Down(k)} = 1$.
Up-Down Graphs: Given collection of Up Graphs, only allow Down arrows between roots of the Up Graphs.
- Counting: (a) Divide $[n]$ into ℓ cycles and place Up Graphs on the cycles. Let m_i be root of cycle C_i.
- (b) Partition $M = \{m_1, \dots, m_\ell\}$ as $M_1 \sqcup \dots \sqcup M_k$ and place Down Graphs on the M_i.
- $\dim S_n^{(n-k)} = \sum_{\ell=k}^{n} s(n,\ell) S(\ell,k) = L(n,k) = \dim A_n^{!(n-k)}$