subdivide page 1: “The core material”
page 2: “Examples & interpretations"
The Pure Virtual Braid Group is Quadratic

Let $K$ be an algebra over a field $F$ with char $F = 0$, and let $I \subseteq K$ be an “augmentation ideal”; meaning $K/I = F$. 

**Definition.** Say that $K$ is quadratic if its associated graded $gr K = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a quadratic algebra. Alternatively, let $A = Q(K) = (V = I/I^n)/(ker\sigma_V : V \otimes V \to I/I^{n+1})$ be the “quadratic approximation” to $K$ (if $A$ is a functor). Then $K$ is quadratic iff the obvious map $\mu : A \to gr K$ is an isomorphism. If $G$ is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

**Why Care?** In abstract generality, $gr K$ is a simplified version of $K$ and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, $A$ is a space of “universal Lie algebraic formulas” and the “primary approach” for proving (strong) quadraticity, constructing an appropriate homomorphism $Z : K \to A$, becomes wonderful mathematics.

The Main Theorem [Lee]. $P\mu B_n$ is quadratic.

**The Overall Strategy.** Consider the “singularity tower” of $(K, I)$ (here “n” means $\otimes_n$, and $\mu$ is (always) multiplication).

$$\ldots \to I(I/I^2)^2 \to I(I/I^2)^2 \to \ldots \to I$$

We care about $im(\mu^p/I^p) = \mu^p/I^p$, so $I(I/I^2)^2$ is injective. Hence we ask:

- What’s $I(I/I^2)^2/I(I/I^2)^{2+1}$?
- How injective is this tower?

**Lemma.** $I(I/I^2)^{2+1} = (I/I^2)^{2+1} = (I/I^2) \otimes (I/I^2)$.

**Flow Chart.** Any $(K, I)$ 2-local Quadratic

$K = P\mu B_n$ by Peter [Hutchings Criterion]

Prop 1, the free case. If $J$ is an augmentation ideal in $K = F = \langle x_i \rangle$, then $F \to F/J = F$ by $x_i \mapsto [x_i]$, and define $\mu : F \to F$ by $x_i \mapsto x_i + [x_i]$. So $(K/I)_{K/I}$ is a projector.
Then $J_0 = \psi(J) = \{ w \in \mathbb{F}_e : \deg w > 0 \}$. For $J_0$ it is easy to check that $R_2 = R_0 = 0$, and hence the same is true for every $J$.

Prop 1, the general case. If $K = F/M$ and $I \subset K$, then $I = J/M$ where $J \subset F$. Then

$$J : I^p = J : I^{p-1} / \sum_{J^j} M : J^{p-j}$$

and we have

$$J : I^p \xrightarrow{M^j} J : I^{p-1}$$

onto $\sum_{J^j} M : J^{p-j}$

$$I : I^p \xrightarrow{J : I^{p-1} / \sum_{J^j} M : J^{p-j}} I : I^{p-1}$$

So

$$\ker(M^j) = \sum_{J^j} M_j^{-1} (\ker(T_{J^j})) = T_{J^j} \left( \sum_{J^j} M_j^{-1} (J^{j-1} : M : J^{p-j-1}) \right)$$

$$= T_{J^j} \left( \sum_{J^j} M_j^{-1} (M) : J^{p-j-1} \right)$$

$$= \sum_{J^j} (J^{j-1} : M : J^{p-j-1})$$

re-confirmed after typesetting.
Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is hijack page design by X.

References


\[ \text{"In the free cry, all augmentation ideals are equivalent to } \mathbb{D}. \]
Recycling:

Proof of prop 1. The first case $I_F := \langle x_1 \rangle$ is fine, since its augmentation ideal $I_F$ by $J_F := \langle w_1 F : dy v_1 \rangle$.

Using
\[
(F, I_F) \xrightarrow{x_1 + x_1} (F, J_F)
\]

For $J_F$ trivially $R_2 = 0$ and so is $R_0$, so the same holds for $J_F$.