The Pure Virtual Braid Group is Quadratic

Presented to the great algebraic masters of the Oregon School in pursuit of their wisdom and advice, in acceptance that they know all and have seen all, and in dread that we will inflict boredom upon them.

Let $K$ be an algebra over a field $F$ with char $F = 0$, and let $I \subset K$ be an “idealization ideal”; meaning $K/I = F$.

Definition. Say that $K$ is quadratic if its associated graded $\oplus_{n \geq 0} K^n/I^{n+1}$ is a quadratic algebra. Alternatively, as $A = Q(K) = (V = I/I^2)/\ker(p^*)$, $V \cong V \otimes V \rightarrow F^2$, be the “quadratic approximation” to $K$ ($Q$ is a locally functor).

Then $K$ is quadratic iff the obvious $p : A \rightarrow \bar{K}$ is an isomorphism. If $G$ is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

Why Care? In abstract generality, $gr K$ is a simplified version of $K$ and if it is quadratic it is as simple as it may be without being silly. In some concrete (somewhat generalized) knot theoretic cases, $A$ is a space of “universal Lie algebraic formulae” and the “primary approach” for proving (strong) quadracity, constructing an appropriate homomorphism $Z : K \rightarrow A$, becomes wonderful mathematics:

$\mathcal{W}$-Knots and Braids

$\mathcal{P}_{\mathcal{W}} K$ is the group

$\langle \sigma_i : 1 \leq i \neq j \leq n \rangle \bigg/ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \bigg/ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \bigg/$

of “pure virtual braids” (“braids when you look”, “dummy braids”):

The Main Theorem [L]. $P_{\mathcal{W}} K$ is quadratic.

Flow Chart.

The pair $(K, I)$ is “2-local” if $\ker \mu_p : I^{p-1} \rightarrow I^p$ is

$\sum_{j=1}^{p-1} (t^{j-1} : R_2 : I^{p-j-1})$, where $R_2 = \ker \mu_p : I^2 \rightarrow I$.

2-Injectivity. A (one-sided infinite sequence)

$\cdots \to K_{p+1} \xrightarrow{\delta_{p+1}} K_{p} \xrightarrow{\delta_{p}} K_{p-1} \to \cdots \to K_0 = K$

is “injective” if for all $p$, $\ker(\delta_p) = 0$. It is “2-injective” if its “1-reduction”:

$\cdots \to K_{p+1} \xrightarrow{\ker \delta_{p+1}} K_{p} \xrightarrow{\ker \delta_{p}} K_{p-1} \to \cdots \to K_0 = K$

is injective: i.e. if for all $p$, $\ker(\delta_p) \otimes \ker(\delta_{p+1}) = 0$. A pair $(K, I)$ is “2-injective” if

$\cdots \to I^p \xrightarrow{\mu_p = \delta_p \otimes \delta_{p+1}} I^{p-1} \to \cdots \to K$

is 2-injective, where “$\otimes$” denotes $\otimes_K$ and $\mu$ is (always) multiplication.

We enre as $\text{im}(\mu_p) = \mu_1 \circ \cdots \circ \mu_p = I^p$, so $I^p/I^{p+1} = \text{im}(\mu_p)/\text{im}(\mu_{p+1})$.

Theorem 1. If $(K, I)$ is 2-local and 2-injective, it is quadratic.