The Pure Virtual Braid Group is Quadratic

Presented to the great algebra masters of the Oregon School, in pursuit of their wisdom and advice, in acceptance that they know all and have seen all, and in dread that we will inflict boredom upon them.

Let $K$ be an algebra over a field $\mathbb{F}$ with char $\mathbb{F} = 0$, and let $I \subset K$ be an “augmentation ideal”; meaning $K/I = \mathbb{F}$.

Definition: Say that $K$ is quadratic if its associated graded $gr K = \bigoplus_{n=0}^\infty I^n/I^{n+1}$ is a quadratic algebra. Alternatively, let $A = Q(K) = (V = I/I^2)/\ker(\mu_1 : V \otimes V \to I/I^2)$ be the “quadratic approximation” to $K$ ($Q$ is a lovely functor). Then $K$ is quadratic iff the obvious $\mu : A \to gr K$ is an isomorphism.

Why Care? • In abstract generality, $gr K$ is a simplified version of $K$ and if it is quadratic it is as simple as it may be. • In some concrete (somewhat generated) knot theoretic cases, $A$ is a space of “universal Lie algebraic formulas” and the “primary approach” for proving/constructing, constructing an appropriate homomorphism $Z : K \to A$, becomes wonderful mathematician:

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\mathop{\text{graded}}$</th>
<th>$\varepsilon$-knots</th>
<th>$\gamma$-knots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\mathop{\text{Lie algebras}}$</td>
<td>$\mathop{\text{Lie algebras}}$</td>
<td>$\mathop{\text{Lie algebras}}$</td>
</tr>
<tr>
<td>$Z$</td>
<td>$\mathop{\text{associative}}$</td>
<td>$\mathop{\text{Enright-Karev quantization}}$</td>
<td>$\mathop{\text{Kashiwara-Vergne}}$</td>
</tr>
</tbody>
</table>

Abstract Generalities. $(K,I)$: an algebra and an “augmentation ideal” in it, $K = \varinjlim K/I^n$ the “I-tactic completion”. $gr I K = \bigoplus_{n=0}^\infty I^n/I^{n+1}$ has a product $\mu$, especially $\mu_1 : (C = I/I^2) \otimes (C = I/I^2) \to I/I^2$. The “I-tactic approximation” $A(K) = \mathcal{F}(\mathcal{C})/\ker(\mu_1)$ is a functor using $\mu$ on $gr K$.

The Prized Object. A “progressive $A$-expansion” $A$ homomorphic filtered $Z : K \to A$ for which $gr Z : gr K \to Z$ inverts $\mu$, (given homomorphismicity, $\sim Z$ reduces the identity on $I/I^2 = C$).

Dror’s Dream. All interesting graded objects and relations, especially those around quantum groups, arise this way.

Example. $K = \mathbb{F}[t^\pm 1]/(t^m)$, $I = (t^m)$,

$$(I/I^{m+1})^* = \{\text{invariants of type } m\} =: V_m$$

$$(t^m/t^{m+1}) = V_m/V_{m-1} = C = \{t^m, t^{m-1}, \ldots, t^1, t^0\} = \{|\rangle \}$$

$\ker(\mu_1) = \langle [t^0, t^0] = 0 = [t^2, t^1 + t^0] \rangle = \langle t^1 \rangle$ relations

$A = (t^{2n}/t^{2n+1}, \text{no horizontal chord diagram})$

$Z$ universal finite type invariant, the Kontsevich integral.

Why Prized? Since $K$ and shows it “as big” as $A$; reduces topological questions to quadratic algebra; gives life and meaning to questions in graded algebra; universalizes those more than “universal enveloping algebras” and allows for richer quotients.

Define 2-injective: $\longleftarrow$ for good SATS then for $(K,I)$.

Define 2-local $\longleftarrow$ for Thm 1

from Thm 1 $\longleftarrow$ for Thm 2.

Def. 2-injective:

$K = \{ \begin{array}{c}
\text{The set of all} \\
\text{2D projections of reality}
\end{array}$

An expansion $Z$ is a choice of a “progressive scale” algorithm.

$K/K_1 = K_1/K_2 = K_2/K_3 = \ldots = K_n/K_{n+1}$

$\ker(C/K_1 \to C/K_{n+1})$
\[ I : p+1 \to I : p \to I : p-1 \to \cdots \to k \]
is 2-injective, where

\[ \begin{array}{c}
k_{p+1} \to k_p \to k_{p-1} \to \cdots \to k \\
\end{array} \]
is 2-injective means that \( \ker d_{p+1} = \ker d_p \).

\( \iff \) its reduction

\[ \begin{array}{c}
k_{p+1} \to k_p \to k_{p-1} \to \cdots \\
\end{array} \]
is injective.

**Def.** 2-local:

\[ \ker (I : p \to I : p-1) = \bigoplus_{j=1}^{p-1} I : j-1 \cdot (\ker I : j) \cdot I : p-j-1 \]

**Thm.** 2-injective + 2-local \( \Rightarrow \) Quadratic

**Pf.**

\[ \begin{array}{c}
(I/I^2)^{\otimes p} \to I : p / \mu(I : p+1) \\
\end{array} \]

\[ \begin{array}{c}
\frac{I^p}{I^{p+1}} \cong \frac{I^p / \ker M_p}{M(I : p+1) / \ker M_p+1} = \frac{I^p}{M(I : p+1) + \ker M_p} \\
\end{array} \]

\[ \begin{array}{c}
= (I/I^2)^{\otimes p} / \ker M_p \\
\end{array} \]

Syzygy complete at: Quadraticity: 2-Injectivity
Footnotes
1. Following a homonymous paper and thesis by Peter Lee (see ???). All serious work here is his; page design by DBN.

References