The Pure Virtual Braid Group is Quadratic

Let $K$ be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F} = 0$, and let $I \subset K$ be an “augmentation ideal”; so $K/I \rightarrow \mathbb{F}$.

Definition. Say that $K$ is quadratic if its associated graded $gr-K = \bigoplus_{i=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A = q(K) = (V = I/I^2)/(\mathbb{F}R_2 = ker(\mu_2 : V \otimes V \rightarrow I^2/I^3))$ be the “quadratic approximation” to $K$ (i.e. a lovely functor). Then $K$ is quadratic iff the obvious $\mu : A \rightarrow gr-K$ is an isomorphism. If $G$ is a group, we say it is quadratic if its group ring, with its augmentation ideal, is quadratic.

The Overall Strategy. Consider the “singularity tower” of $(K, I)$ (here “$\approx$” means $\otimes K$ and $\mu$ is always multiplication):

\[ \cdots \xrightarrow{\mu_{p+1}} I^p \xrightarrow{\mu_p} I^{p+1} \rightarrow \cdots \rightarrow K \]

We care as $im(\mu_p^g = \mu_1 \circ \cdots \mu_p) = I^p$, so $I^p/I^{p+1} = im(\mu_1) \cap \cdots \cap im(\mu_p)$. Hence we ask:

- What’s $I^p/I^{p+1}$?
- How injective is this tower?

Lemma. \( I^p/I^{p+1} \approx (I/F)^{\otimes p} = V^{\otimes p} \), set $\pi : F^p \rightarrow V^{\otimes p}$.

Flow Chart.

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<th>$K = P\beta_2$</th>
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<td>Then $s$ of Peter</td>
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<td>Hutchinson's Criterion $\Rightarrow$ 2-injective</td>
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Proposition 1. The sequence $\begin{array}{c} \mathcal{R}_p = \bigoplus_{i=0}^{p-1} I^{i+1} ; R_0 \Rightarrow I^p \rightarrow I^{p+1} \rightarrow \cdots \rightarrow K \end{array}$ is exact, where $\mathcal{R}_p = \{ I \cap K \}$ is the “2-local”.

The General Case. If $J$ is an augmentation ideal in $K = F[x_1, \ldots, x_n]$ and $\mathcal{R}_p$ is a quadratic algebra, then $\mathcal{R}_p$ is quadratic.

The General Case. If $K = F/(M)$ where $M$ is a vector space of “moves” and $J \subset K$, then $J = J/(M)$ where $J \subset F$. Then $\mathcal{R}_p = J^p/(\sum J^j ; M J^j) ; F^p \rightarrow F^{p+1}$ and we have

\[ \begin{array}{c} \mathcal{R}_p = J^p/(\sum J^j ; M J^j) ; F^p \rightarrow F^{p+1} \end{array} \]

Theorem. If $\mathcal{R}_p$ is quadratic, then $\mathcal{R}_{p+1}$ is quadratic.

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Why Care? In abstract generality, $gr-K$ is a simplified version of $K$ and if it is quadratic it is as simple as it may be without being silly. In some concrete (somewhat generalized) knot theoretic cases, $A$ is a space of “universal Lie algebra formulas” and the “primary lie” for proving (strong) quadraticity, constructing an appropriate homomorphism $Z : K \rightarrow A$ becomes wonderful mathematics:

- Knots and $K$ Braid
- $v$-Knots
- $w$-Knots

- A Measured Lie algebras [BN]
- Lie bialgebras [Hav]
- Finite dimensional Lie algebras [BN]
- Z
- Etingof-Kazhdan quantization
- Kashiwara-Vergnes
- Alekseev-Torossian

2-Injectivity. A (onesided infinite) sequence $\begin{array}{c} \cdots \rightarrow K_{p+1} ; \mathcal{R}_{p+1} \rightarrow K_p \rightarrow \cdots \rightarrow K_0 = K \end{array}$ is “injective” if for all $p > 0, ker(\mu_p) = 0$. It is “2-injective” if its “1-reduction” is injective; i.e. if for all $p > 0, ker(\mu_p \circ \mu_{p+1}) = ker(\mu_{p+1})$. A pair $(K, I)$ is “2-injective” if its singularity tower is 2-injective.

Proposition 2. If $(K, I)$ is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\begin{array}{c} \cdots \rightarrow K_{p+1} ; \mathcal{R}_{p+1} \rightarrow K_p \rightarrow \cdots \rightarrow K_0 = K \end{array}$, get $\begin{array}{c} \cdots \rightarrow K_{p+1} ; \mathcal{R}_{p+1} \rightarrow K_p \rightarrow \cdots \rightarrow K_0 = K \end{array}$.

The X Lemma (inspired by [Hut]).

\[ \begin{array}{c} A_0 \rightarrow B \rightarrow C_0 \rightarrow \cdots \rightarrow A_1 \rightarrow B \rightarrow C_1 \end{array} \]

If the above diagram is Conway ($\approx$) exact, then its two diagonals have the same 2-injectivity defect. That is, $A_0 \rightarrow B \rightarrow C_0$ and $A_1 \rightarrow B \rightarrow C_1$ are exact, then $ker(\beta_1 \circ \alpha_0) \approx ker(\beta_0 \circ \alpha_1)/ker \alpha_1$.

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The Hutchinson Criterion [Hut].

The singularity tower of $(K, I)$ is 2-injective iff on the right, $ker(\mu \circ \mu) = ker(\mu)$. That is, if every “diagrammatic syzygy” is also a $F^{p+1}$ topological syzygy.

Conclusion. We need to know that $(K, I)$ is “syzygy complete” that every diagrammatic syzygy is a topological syzygy, that $ker(\mu \circ \mu) = ker(\mu)$. 

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The Pure Virtual Braid Group is Quadratic, II

Examples and Interpreations

Example.

$K = \begin{pmatrix} (K)^{p+1} \end{pmatrix}$

$V = \begin{pmatrix} (v^j v^i) = (v^j v^i) \end{pmatrix}$

$\ker \mu_2 = \begin{pmatrix} (v^j, v^i) = 0 \end{pmatrix} = \begin{pmatrix} (v^j, v^i + v^i) \end{pmatrix} = \begin{pmatrix} (4T) \end{pmatrix}$

$A = \begin{pmatrix} \text{horizontal chord diagrams mod 4T} \end{pmatrix} = \begin{pmatrix} \text{4T} \end{pmatrix}$

Z: universal finite type invariant, the Kontsevich integral.

$\mathcal{P}_B$ is the group

$\langle \sigma_{ij} : 1 \leq i \neq j \leq n \rangle$

$\sigma_{ij} = \sigma_{j,i}$

of “pure virtual braids” (”braids when you look”, “blunder braids”):

The Main Theorem [Lee]. $\mathcal{P}_B$ is quadratic.

$A_n = q(\mathcal{P}_B)$.

$V = \frac{V^n}{V^{n+1}} = \begin{pmatrix} (v^j v^i) = (v^j v^i) \end{pmatrix}$

$A_n = \begin{pmatrix} (v^j v^i) \end{pmatrix}\begin{pmatrix} (a_{ij} a_{jk}) = (a_{ij} a_{jk}) \end{pmatrix}\begin{pmatrix} (c_{ij}^k = a_{ij} a_{jk}) \end{pmatrix}$

$Y_{ijk} := 

\begin{pmatrix} Y_{ijk} : Y_{jik} : \ldots \end{pmatrix} \rightarrow \begin{pmatrix} Y_{ijk} : Y_{jik} : \ldots \end{pmatrix} \rightarrow \begin{pmatrix} Y_{ijk} : Y_{jik} : \ldots \end{pmatrix}$

Syzygy Completeness, for $\mathcal{P}_B$ means:

Is every relation between the $Y_{ijk}$'s and the $c_{ij}^k$'s also a relation between the $Y_{ijk}$'s and the $C_{ij}^k$'s?

Just for fun.

James Gillespie’s Sightline 2 (1994) is a sculpture, and (arguably) Toronto’s largest sculpture. Find it next to University of Toronto’s Hart House.