The Pure Virtual Braid Group is Quadratic!

Abstract Generality:

Let $K$ be a unital algebra over a field $F$ with char $F = 0$, and let $I \subset K$ be an “augmentation ideal”, so $K/I \cong F$.

Definition: Say that $K$ is quadratic if its associated graded $gr K = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a quadratic algebra. Alternatively, let $A = \bigoplus_{n \geq 0} I^n/I^{n+1}$ be the “augmentation ideal" of $K$ (if $K$ is always multiplicative).

We can assume $\dim_k I = \dim_k I/I^2$.

The Overall Strategy: Consider the “augmentation tower" of $K$: I (where $\text{dim}_F \cong 0$ and $I$ (always multiplication):

$\cdots \to I^{p-1} \xrightarrow{\cdot \mu_{p-1}} I^p \xrightarrow{\cdot \mu_p} I^p/I^{p+1} \to \cdots \to K$.

We see as $\text{im}(\mu_p) = \mu_p \cdots \mu_2 \mu_1 F = F$, so $I^{p+1} = \text{im}(\mu_p) \cap \text{im}(\mu_p)$. Hence we ask:

- How injective is this tower?
- What’s $\text{im}(\mu_p/I^{p+1})$?

Lemma: $I^{p+1}/I^{p+1} \cong I^{p+1}/I^p \cong V^p$.

Proof:

- Any $(A, B)$
- $K$ is quadratic
- $I^{p}/I^{p+1}$ is injective
- $V^p$ is a quadratic algebra

Proposition 1: The sequence

$\mathcal{R}_2 := \bigoplus_{j=1}^{p-1} (I^j/I^{j+1} ; \mathcal{R}_2)$

is exact, where $\mathcal{R}_2 := \ker F^2 \to I$; so $(K, I)$ is “2-local”.

The Free Case: If $I$ is an augmentation ideal in $K = F$, define $\psi : F \to P$ by $x \mapsto x + (x)$. Then $J_0 = \psi(J) = \{x \in F : \deg x \leq 0\}$. For $J_0$, it is easy to check that $\mathcal{R}_2 = \mathcal{R}_2 = 0$, and hence the same is true for every $J$.

The General Case: If $K = F/I(M)$ (where $M$ is a vector space of “moves”) and $I \subset K$, then $I = J_0/I(M)$ where $J \subset F$. Then $I^p/I^{p+1} \cong I^p/I^{p+1}$, and we have

$\mathcal{R}_2 := \bigoplus_{j=1}^{p-1} (I^j/I^{j+1} ; \mathcal{R}_2)$

is simpler than may seem! It’s an “augmentation bimodule” $\mathcal{R}_2 = I \otimes \mathcal{R}_2$, where $\psi \mapsto (\psi) \otimes \psi$ for $\psi \in I$ and $\mathcal{R}_2$ is a quadratic algebra.

$\mathcal{R}_2$ is simpler than may seem! In $\mathcal{R}_2 = I^{p+1} ; \mathcal{R}_2$ $p^{p+1}$ factors may be replaced by $I = I/I^2$.

Why Care?

- In abstract generality, $gr K$ is a simplified version of $K$ and if it is quadratic it is as simple as it may be without being silly.
- In some concrete (somewhat generalization) knot theoretic cases, $A$ is a space of “universal Lie algebraic formulas” and the “primary approach” for proving (strong) quadraticity, constructing an appropriate homomorphism $Z : K \to A$ becomes wonderful mathematics:

The X Lemmas [inspired by [Hut]].

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If the above diagram is commutative (i.e., exact), then its dual has the same “2-injectivity” defect.

- If $A_0 \to B \to C_0$ and $A_0 \to B \to C_0$ are exact, then $\ker(\beta_0 \otimes \alpha) = \ker(\beta_0 \otimes \alpha_0)$, and $\ker(\beta_0 \otimes \alpha_0) = \ker(\beta_0 \otimes \alpha_0) \cap \ker(\beta_0 \otimes \alpha_0)$. Thus, if every diagrammatic syzygy is also a “topological syzygy”.

Conclusion. We need to know that $(K, I)$ is “syzygy complete” — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\beta_0 \otimes \alpha_0) = \ker(\beta_0 \otimes \alpha_0)$.

Do I still need to put $\mathcal{R}_2$ in? Yes.
Looking at the long exact seq corresponding to the above in homology, to see that $R_2$ surjects on $R_2^1$ I need to know that clear as

$$M_2(\ker T_1^2) = \ker T_1^1$$
$$M_2(T^2,I) = I^3$$

to also know that $R_2 \to R_2^1$ we need $M_2|\ker T_2^1 \subset \ker T_1^1$ meaning $0 = \ker M_2 \cap \ker T_2^1$ meaning $0 = \ker R_2 \cap I \cdot I \cdot I$.

Claim. Under $T: I \cdot I \to V \otimes P$, $R_{ij} \to R_{ij}^1$.

Namely,

$$\ker (T^2 : I \cdot I \to I^2)$$

$$\ker (T_{ij}^2 : R_{ij} \cdot I \cdot I \to V \cdot m \otimes R_{ij} \otimes V \cdot m)$$

$$\ker (V \cdot m \to I^2 / I^3)$$

The claim is the outer statement that $\text{Coker} \ T_{ij}^\text{ind} = U_{ij}$.

It does not follow merely from the hexagon.
it does not follow

mainly from the

hexagon.

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\[ \frac{F^0}{F \circ \mu_{thd}} \cong \ker \partial_A \]

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\[ \ker F^0 = 0 \]

\[ \ker F^0 \cong U_m \]

drawn from Arv Lu.
The Pure Virtual Braid Group is Quadratic, II
Examples and Interpretations

Example:

\[
K = \begin{pmatrix}
\text{...} & \text{...} \\
\text{...} & \text{...}
\end{pmatrix}
\]

\[I = \begin{pmatrix}
\text{...} & \text{...} \\
\text{...} & \text{...}
\end{pmatrix}
\]

(goes back to [Koh])

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\[ (K/P^{i+1})^* = \{\text{invariants of type } p \} = Y_p \]

\[ (P^*/P^{i+1})^* \subset Y_p \]

\[ V = (t^1, t^2) = I \]

\[ \ker \mu_2 = \{ e^{i1}, e^{i2} \} = 0 \]

\[ \mu_1 = \{ e^{i1} + e^{i2} \} \]

\[ A = \mu_2 \]

\[ Z: \text{universal finite type invariant, the Kontsevich integral.} \]

\[ P_{B_n} \text{ is the group} \]

\[ \sigma_{ij} (1 \leq i < j \leq n ) / \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} / \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} = \sigma_{ij} \sigma_{kl} \]

of "pure virtual braids" ("braids when you look", "braid braids")

\[ \sigma_{ij} = \begin{pmatrix}
\text{...} & \text{...} \\
\text{...} & \text{...}
\end{pmatrix} \]

\[ \sigma_{24} = \begin{pmatrix}
\text{...} & \text{...} \\
\text{...} & \text{...}
\end{pmatrix} \]

The Main Theorem [Lee]. \[ P_{B_n} \text{ is quadratic.} \]

\[ A_n = \text{Attachment} \]

\[ I = \{ \text{...} \} \]

\[ V = \{ \text{...} \} \]

\[ A_2 = \{ \text{...} \} \]

\[ g_{jk} = \{ \text{...} \} \]

\[ \text{Just for fun.} \]

James Gilmore’s Sightsline #2 (1984) is a syzygy, and (surprisingly) Toronto’s largest sculpture. Find it next to University of Toronto’s Hart House.