The Pure Virtual Braid Group is Quadratic:

**Abstract Generators**

Let $K$ be a unital algebra over a field $F$ with char $F = 0$, and let $I \subset K$ be an “augmentation ideal”; set $K/I \cong F$.

**Definition**

Say that $K$ is quadratic if its associated graded $gr = \oplus_{p \geq 0} F_p$ is a quadratic algebra. Alternatively, let $A = gr(K) = (V = I/F)/I = ker(\eta_I: V \otimes V \to F/F')$ be the “quadratic approximation” to $K$ (as a $V$-valued function). Then $K$ is quadratic iff the obvious map $\eta : A \to gr K$ is an isomorphism. If $G$ is a group, say it is quadratic if its group ring is, with its usual grading.

The Overall Strategy: Consider the “singular tower” of $(K, I)$ (here “$\cdot$” means $\otimes_K$ and $\mu$ is (always) multiplication):

... $F_1 \otimes F_1 \otimes I \otimes \ldots \otimes I \longrightarrow K$.

We care only an $im(\mu) = \mu_1 : \cdots : \mu_n = P$, so $P = \mu_1^{p_1} \cdots \mu_n^{p_n}$. Hence we ask:

- How injective is this tower?
- What is $\mu_1^{p_1} \cdots \mu_n^{p_n}$?

**Lemma**

If $F/F' = V^{\oplus p}$, then $\mu_1^{p_1} \cdots \mu_n^{p_n}$.

**Flow Chart**

**Proposition 1.** The sequence:

$0 \longrightarrow I^{(p_1 + 1)} \longrightarrow F^{(p_1 + 1)} \longrightarrow \cdots \longrightarrow F^{(p_n + 1)} \longrightarrow 0$

is exact, where $F_2 := ker(\mu : F/F' \to I)$ is “2-local”.

**The Free Case**

If $I$ is an augmentation ideal in $K = F \langle x_1, x_2, x_3, \ldots \rangle$, denote $\nu = \mu_1^{p_1} \cdots \mu_n^{p_n}$. Then $\nu = \mu_i^{p_i}$ is a monic (since $\nu \in F$).

**The General Case**

If $K = F/M$ (where $M$ is a vector space of “moves”) and $I \subset K$, then $F/IM$ is a vector space of $F$ as well. Then $F/IM = \sum_{p \geq 0} F^{(p)} \cong \bigoplus_{n \geq 0} I^{(n+1)} = F/I^{(1)}$ (by induction).

**2-Diagonal**

A 2-nilpotent sequence:

$0 \longrightarrow I \longrightarrow F \longrightarrow \cdots \longrightarrow F^{(p_n + 1)} \longrightarrow K$

is “injective” if all $p > 0$, ker $d_0 = 0$. It is “2-nilpotent” if $d_0 = 0$.

**Example**

$K = \langle x_1, x_2, \ldots \rangle / \langle x_1 x_2, x_2 x_3, \ldots \rangle$

(goes back to $K(x)$)

$\eta_{(p)}(\theta_{(p)}) = (\text{invariants of type } p) \to V_p$

$(F/I^{(p+1)})^{(p)} = V_p^{(p)}$

$V = (V_0^{(p)} = 0 \to V_1^{(p)} \to V_2^{(p)} \to \cdots)$

ker $\tilde{\eta}_p = (\tilde{\eta}_p^{(p)} = 0) = (\tilde{\eta}_p^{(p)})$

James Gilmore’s Shadow {2} (1963) is a zyzyg, and (unfortunately) Toronto’s largest sculpture. Visit it next to University of Toronto’s Hart House.

The Pure Virtual Braid Group is Quadratic, II

**Example and Interpretations**

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\[(P^2/\mathcal J_{\mathcal J}^{2})^* = V_\mathcal J/\mathcal J_{\mathcal J} \quad \text{with} \quad \mathcal J = \mathcal J_{\mathcal J} \supseteq \mathcal J_{\mathcal J} = 0 \quad \text{and} \quad V_\mathcal J = \mathbb C \mathcal J \mathbb C.\]

The pure virtual braid group is quadratic. III

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\ker \rho = \pi_2(\rho_{\mathcal B}^1(M)) \quad \text{is in principle computable, and then} \quad R_2 \quad \text{follows as} \quad V_{\mathcal B}^2 = 1 = J/J(\mathcal M).\]

Syzygy Completeness, for \(Pd\mathcal B_1\), means:

\[
\mathcal R_0 = \bigoplus_{j=0}^{\infty} \rho_{\mathcal B}^j \quad \text{satisfies} \quad V^\otimes p \quad \text{for some} \quad p.\]

Is every relation between the \(\rho_{\mathcal B}^j\)'s and the \(D_{\mathcal B}^j\)'s also a relation between the \(Y_{\mathcal B}^j\)'s and the \(D_{\mathcal B}^j\)'s?
$K = \{ \mathbb{R} \} = \{ \text{The set of all } 2D \text{ projections of reality} \}$
Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently explained by him to DBN. Page design by the latter.

References


