The Pure Virtual Braid Group is Quadratic

Abstract Generalities

Let $K$ be a unital algebra over a field $F$ with char $F = 0$, and let $I \subset K$ be an "augmentation ideal": meaning $K/I = F$.

Definition. Say that $K$ is quadratic if its associated graded $gr K = \bigoplus_{p=0}^{\infty} F_p/F_{p+1}$ is a quadratic algebra. Alternatively, let $A = q(K) = (V = \langle F \rangle / (R_2 = \ker(\delta_2 : V \otimes V \rightarrow (F^2)^{\otimes 2})))$ be the "quadratic approximation" to $K$ (a is a lovely functor). Then $K$ is quadratic iff the obvious $\mu : A \rightarrow gr K$ is an isomorphism. If $G$ is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of $(K,I)$ (that $\pi$ means $\otimes_K$ and $\mu$ is (always) multiplication):

$$\ldots I^{p+1} \xrightarrow{\mu^{p+1}} I^p \xrightarrow{\mu^p} I^{p-1} \rightarrow \ldots \rightarrow K$$

We care as $im(\mu^p) = \mu_0^p \cdot \cdots \cdot \mu_{p}^p = I^p$, so $I^p/I^{p+1}$ is a direct sum of $\mu^p/I^{p+1}$. Hence we ask:

- How injective is this tower?
- What's $\mu^p/I^{p+1}$?

**Lemma.** $I^{p}/\mu^{p}(I^{p+1}) \simeq (1/I^p)^{\otimes p} = V^{\otimes_p}$.

Flow Chart.

**Proposition 1.** The sequence

$$R_p := \bigoplus_{j=1}^{p-1} (I^{j-1} : R_2 : I^{p-j-1}) \xrightarrow{\partial} \bigoplus_{j=1}^{p-1} I^{p-j} \xrightarrow{\mu} \bigoplus_{j=1}^{p-1} I^{p-j}$$

is exact, where $R_2 := \ker \mu : F^2 \rightarrow F$; $(K,I)$ is "2-local".

**The Free Case.** If $J$ is an augmentation ideal in $K = F = \langle x_1 \rangle$, denote $F \rightarrow F/J = F$ by $x_1 \mapsto x_1$ and define $\psi : F \rightarrow F$ by $x_1 \mapsto x_1 + [x_1]$. Then $J_0 = \ker \mu(J)$ is $\{w \in F : \deg w > 0\}$.

**Proposition 2.** If $(K,I)$ is 2-local and 2-injective, it is quadratic.

- The X Lemma (inspired by [Hut]).

If the above diagram is Conway ($\simeq$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $I_0 \rightarrow B \rightarrow I_1$ and $A_1 \rightarrow B \rightarrow I_1$ are exact, then $\ker(j_{1} b_0 \circ a_0)/\ker a_0 \simeq \ker(\delta_{1} a_0 \circ j_{1})/\ker a_0$.

**Conclusion.** We need to show that $(K,I)$ is "symmetric" — that every diagrammatic syzygy lifts to a $I^{p+1}$-"topological syzygy". 

James Gillespie’s Sightsline #2 (1984) is a syzygy, and (arguably) Toronto’s largest sculpture. Find it next to University of Toronto’s Hart House.
The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

Example.

\[ K = \left( \begin{array}{c} \hline 1 \\ \hline 1 \\ \hline \end{array} \right) \]

\[ I = \left( \begin{array}{c} X \\ Y \\ \end{array} \right) \]

(goes back to [Koh])

\[ (K\iota|P^{p+1})^* = (\text{invariants of type } p) = \mathcal{V}_p \]

\[ (P\iota|P^{p+1})^* = \mathcal{V}_p|\mathcal{V}_{p-1} \]

\[ V = \{ V^{ij} \} = \{ I \} \]

\[ \ker \mu_2 = \{ [i^j, \bar{i}^k] = 0 = [i^j, i^k + \bar{i}^k] \} \quad \langle 4 \text{T relations} \rangle \]

\[ A = q(K) = \left( \begin{array}{c} \text{horizontal chord diagrams mod 4T} \end{array} \right) \]

\[ Z: \text{universal finite type invariant, the Kontsevich integral.} \]

\[ \mathcal{PB}_n \text{ is the group} \]

\[ \langle \sigma_{ij} : 1 \leq i \neq j \leq n \rangle / \sigma_{ij}\sigma_{kl}\sigma_{ij} = \sigma_{kl}\sigma_{ij} \sigma_{kl} \]

\[ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \sigma_{kl} \]

of "pure virtual braids" ("braids when you look", "blunder braids");

\[ \sigma_{21} = \]

R3:

\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example.png}
\end{array} \]

The Main Theorem [Lee]. \( \mathcal{PB}_n \) is quadratic.

\[ A_n = q(\mathcal{PB}_n) \quad \text{[GPV]} \]

\[ I = \left( \begin{array}{c} \hline X \\ \hline X \\ \end{array} \right) \quad \text{with } \tilde{X} = X^{-1} = \sigma_{ij} - 1 = \tilde{X} - X \]

the "semi-virtual crossing".

\[ V = I^p \]

\[ = \{ \sigma_{ij} \} / \{ \tilde{X}, X \} \]

\[ \text{v-braids with one } \tilde{X} \]

\[ = \{ \sigma_{ij} \} / \{ \tilde{X}, X \} \]

\[ \text{with one } \tilde{X} \]

\[ \times A_n = TV \}

\[ \left\{ [a_{ij}, a_{jk}] + [a_{ij}, a_{jk}], [a_{ij}, a_{jk}] \right\} \]

\[ \sigma_{21} = \]

\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example2.png}
\end{array} \]

Figuring out \( R_2 \) and \( R_3 \).

\[ \ker \mu = \pi_2 (\mu|P^2) (M) \quad \text{in principle computable, and then } R_2 \text{ follows as } V^{\otimes 2} = (I^2)^{\otimes 2} = I^2 \]

\[ \left\{ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example3.png}
\end{array} \right. \]

\[ \text{The set of all } K \]

\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example4.png}
\end{array} \]

An expansion \( Z \) is a choice of a "progressive semi" algorithm.

\[ \ker(K/K_1 \cdots K_{n-1}) \]

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\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example5.png}
\end{array} \]

\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{example6.png}
\end{array} \]

Just for fun.