Definition. Given an ideal $I$ in a (non-commutative) ring $K$ (with product $M$) set:

$$ D_1 = I/I^2 \quad D_m = D_1^{\otimes m} \quad Q_m = I_{/I^{m+1}} $$

$$ R_2^D = \ker (D_2 \xrightarrow{m} Q_2) \subset D_2 $$

$$ R_m^D = \bigoplus D_{i-1} \otimes R_2^D \otimes D_{m-i-1} \subset \bigoplus D_m D_m $$

$$ \partial_D : R_m^D \to D_m \quad \text{the sum of the obvious inclusions.} $$

$$ A_m = \text{cok} \ker (\partial_D) . $$

(note that there is a surjection $M : A_m \to Q_m$)

$$ K_m = I^{\otimes m} \quad R_2^T = \ker (K_2 \xrightarrow{m} I^2) $$

$$ K_m = \bigoplus K_{i-1} \otimes R_2^T \otimes K_{m-i-1} \subset \bigoplus K_m K_m $$

$$ \partial_T : R_m^T \to K_m \quad \text{the sum of the obvious inclusions.} $$

Definition. $(K, I)$ is "quadratic" if $M : A_m \to Q_m$ is an isomorphism. "if" might be sufficient.

Theorem. $(K, I)$ is quadratic iff the map

$$ \ker \partial_T \to \ker \partial_D \quad \text{coming from the diagram} $$

below is a surjection:

$$ \ker \partial_T \to R_m^T \xrightarrow{2T} K_m $$
The way that this may be different than Hutchings is that Hutchings also assumes that
\[ \text{im } \mathcal{D}_T = \text{ker } (M: K_m \to K_{m-1}) \]

What means \( \text{ker } \mathcal{D} \supseteq \text{ker } \mathcal{D}_0 \)?

\[
\begin{align*}
K_{m+1} &\xrightarrow{\mathcal{D}} K_m &\xrightarrow{\mathcal{D}} K_{m-1} &\cdots &\xrightarrow{\mathcal{D}} K_0 = K \\
\text{Claim: } \text{ker } \mathcal{D} = \text{ker } \mathcal{D}_0 &\quad \text{then } \frac{M_{m} K_m}{\text{im } \mathcal{D}_{m+1}} \cong \frac{K_m}{\text{im } \mathcal{D}_{m+1} + \ker \mathcal{D}}
\end{align*}
\]

\[
\begin{align*}
0 = [\alpha] &\leftarrow [\mathcal{D} \alpha] &\leftarrow [\mathcal{D}^2 \alpha] &\leftarrow [\mathcal{D}^3 \alpha] &\cdots &\leftarrow [\mathcal{D}^m \alpha] &\leftarrow [\mathcal{D}^{m+1} \alpha] &\leftarrow [\mathcal{D}^m (\text{im } \mathcal{D}_{m+1})] &\leftarrow [\mathcal{D}^{m+1} (\text{im } \mathcal{D}_{m+1}) + \ker \mathcal{D}]
\end{align*}
\]

In our case, \( \frac{M_{m} K_m}{\text{im } \mathcal{D}_{m+1}} \cong \frac{K_m}{\text{im } \mathcal{D}_{m+1} + \ker \mathcal{D}} \) becomes \( \frac{I^n}{I^{m+1}} \cong \frac{I^n}{I^{m+1} + \ker \mathcal{D}} \).

**Question.** What is \( K_m / \mathcal{D} K_{m+1} \), i.e., \( I^{m+1} / I^m \mathcal{D}^{(m+1)} \) ?

\( (I^0 \theta I) \cdots \theta (I^2 \theta I) \leftarrow \frac{I^n}{I^{m+1} \theta I^{(m+1)}} \)

**Question.** What is \( \mathcal{R}_m^T / \mathcal{D}^T \mathcal{R}_{m+1}^T \)? First, what is \( \mathcal{D}_0^T \mathcal{R}_m^T \)?

\( \mathcal{R}_m^T = \oplus I^{(i-1)} \mathcal{D} (\text{ker } I^{(i)} \mathcal{R} \to I^2) \otimes I^{(m-i)} \)

*The literal analog of \( K_m^T \) seems more to be like...*
... \xrightarrow{d} K_{n+1} \xrightarrow{d} K_n \xrightarrow{d} K_{n-1} \xrightarrow{d} ... \xrightarrow{d} K_n \xrightarrow{d} K_{n-1} \xrightarrow{d} 0 \quad \text{(exact rows)}.}

\*

Question. Do we need \(\text{im } b = \ker b\) above? Perhaps an inclusion \(\ker b \subseteq \ker f\) is enough? No, that seems to be the wrong direction.

Question. How much is \(d_n\), as in \(K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d} K_{n-2} \xrightarrow{d} \cdots \)? used/is necessary in the above?

Ans. We need to know that \(d_n\) is a surjection. Otherwise we seem not to care about \(K_j\); I don't see that we ever use the exactness of \(K_n \xrightarrow{b} K_{n-1} \xrightarrow{d} \).

From http://katlas.math.toronto.edu/drorbn/bbs/show?shot=LeeP-110107-142826.jpg:

\[
\begin{array}{c}
\text{Big issue (?:)} \\
\text{Is \(G\) connected?}
\end{array}
\]

\[
\begin{array}{c}
\text{\(G\) might be hard to meet.}
\end{array}
\]
From papers/Fundamental:

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{K}^1_{m+1} & \rightarrow & \ker \partial|_{\mathcal{K}^1_m} & \rightarrow & \ker \partial|_{\mathcal{D}^1_m} & \rightarrow & 0 \\
\delta & & \delta & & F & & F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{K}^1_m & \rightarrow & \mathcal{K}^0_m & \rightarrow & \mathcal{D}^0_m & \rightarrow & 0 \\
\delta & & \delta & & \delta & & \delta & & (\rightarrow 0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{K}^0_{m+1}/\partial \mathcal{K}^1_{m+1} & \rightarrow & \mathcal{K}^0_m/\partial \mathcal{K}^1_m & \rightarrow & \mathcal{D}^0_m & \rightarrow & 0 \\
\delta & & \delta & & \delta & & \delta & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{K}^0_{m+1}/\partial \mathcal{K}^1_{m+1} & \rightarrow & \mathcal{K}^0_m/\partial \mathcal{K}^1_m & \rightarrow & \mathcal{D}^0_m & \rightarrow & 0
\end{array}
\]

The commutativity of \( \delta \) allows the existence of \( 2 \).