

CHECKING THE HUTCHINGS CRITERION

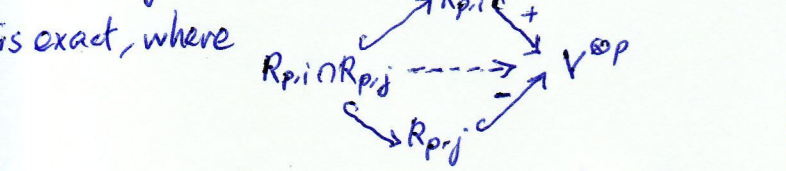
"Topological syzygies" include the "Zamolodchikov tetrahedron" which induces algebraic syzygies via:  $\{ \text{Zam. with } \sigma_{ij} \mapsto (\sigma_{ij} + 1) \}$  keeping only lowest degree in  $\sigma_{ij}$

Query: Do all algebraic syzygies arise this way?

What is  $\text{ker}(\pi\omega)$  (syzygies in  $\text{pub}_n$ )?

FACT:  $A = \text{pub}_n$  is Koszul (BEER, Lee)

HENCE:  $\bigoplus_{i < j} (R_{p,i} \cap R_{p,j}) \rightarrow \bigoplus_i R_{p,i} \rightarrow V^{\otimes p} \rightarrow A^p \rightarrow 0$



"Trivial syzygies"  $R_{p,i} \cap R_{p,j}$  if  $|i-j| > 1$

Non-trivial:  $R_{p,i} \cap R_{p,i+1} \cong V^{\otimes i} \otimes A^{i,3} \otimes V^{\otimes p-i-3}$

via:  $(\Delta_{i,1}^! \otimes \text{id}) \circ \Delta_{2,1}^! : A^{i,3*} \xrightarrow{\sim} R \otimes V \cap V \otimes R$  (\*)

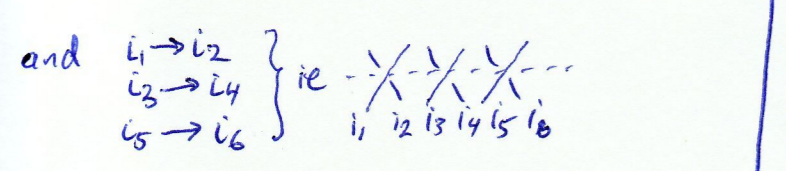
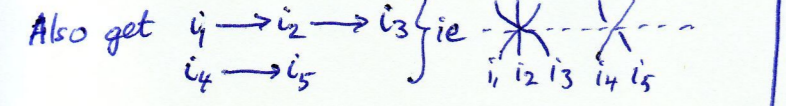
where  $\Delta_{\bullet,1}^!$  is dual to  $m^! : A^{\bullet,0} \otimes A^{1,1} \rightarrow A^{(\bullet+1)}$

BASIS  $A_n^{i,k}$  has basis indexed by unordered partitions of  $[n] = \{1, \dots, n\}$  into  $(n-k)$  ordered subsets "Lah numbers  $L(n, n-k)$ "

"Chain gangs" via  $\{ i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \} \leftrightarrow \{ a_{i_1, i_2}, a_{i_2, i_3}, \dots, a_{i_{k-1}, i_k} \}$

Ind degree 3 primarily get all chains  $i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow i_4$

These are easily calculated by (\*) and  $= F^{SYZ}(\text{Zam})$



Proof of Basis

A relations:  $[a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}] = 0$

$[a_{ij}, a_{ke}] = 0$

$A^! = \Lambda \{ a_{ij} \} \text{ mod:}$

$a_{ij} \wedge a_{ik} = a_{ij} \wedge a_{jk} - a_{ik} \wedge a_{kj}$  (V)

$a_{ik} \wedge a_{jk} = a_{ij} \wedge a_{jk} - a_{ji} \wedge a_{ik}$  (A)

$a_{ij} \wedge a_{ji} = 0$  (No loop)

Idea:  $\nabla$ : V-join, relations replace  $\nabla$ -joins by directed segments.  $\nwarrow$ : A-join

Define a "Defect" function on (AS) monomials st the maximal term in a relation is preserved by multiplication that produces no loops ("Multiplicative")

- Verify that Defect 0 monomials = Chain Gangs
- Defect 0 monomials generate  $A^!$ :
  - If a monomial's graph has a loop, it is 0.
  - Multiplicative  $\Rightarrow$  forests can be reduced to Defect 0 monomials.
- Prove that Defect 0 monomials are Indep.

Defect: A pair of vertices is "unordered" if there is no oriented sequence of edges between them.

- Defect of a tree is # unordered pairs of vertices in it.
- Defect of a forest is sum of defects of its trees.
- Clear that Defect 0 monomials  $\leftrightarrow$  Chain Gangs.
- MAX: In (V) and (A) the join terms have maximal defect.

Multiplicativity: Induction on # of vertices in relation

Show: if vertices a, b are ordered in join term, they are ordered in other terms.

Hence: adding edges either increases Defect or preserves it. Now apply "MAX"

3 Cases: Case I:  $a \rightarrow b$  new edge with a, b new vertices; clear. (connected components are same in all terms)

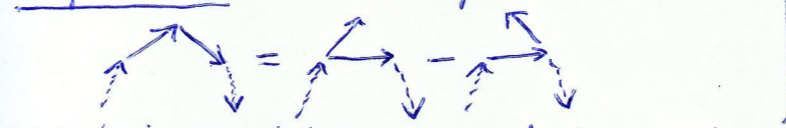
Case II: new edge  $a-b$  with (a) new, (b) old vertex, (c) any other old vertex.

Suppose  $a \rightarrow \dots \rightarrow c$  in join term, then in fact  $a \rightarrow b \rightarrow \dots \rightarrow c$ , hence  $b \rightarrow \dots \rightarrow c$  in old join term, hence  $b \rightarrow \dots \rightarrow c$  in all other old terms, hence  $a \rightarrow b \rightarrow \dots \rightarrow c$  in all new terms.

Case III: new edge  $a-b$  with (a)/(b) old vertices, (c) any other old vertex

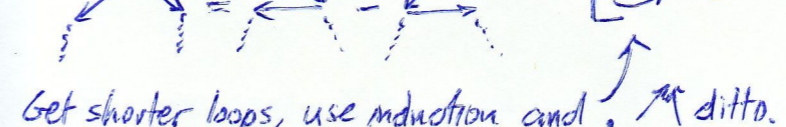
Let (d), (e) old vertices in components of (a), (b) resp. If  $c \rightarrow \dots \rightarrow d$  in join term, then in fact  $c \rightarrow \dots \rightarrow a \rightarrow b \rightarrow \dots \rightarrow d$ , hence  $c \rightarrow \dots \rightarrow a$  and  $b \rightarrow \dots \rightarrow d$  in old join, hence  $c \rightarrow \dots \rightarrow a$  and  $b \rightarrow \dots \rightarrow d$  in other old terms, so  $c \rightarrow \dots \rightarrow d$  also.

Loops are 0 Oriented loops:



Get shorter oriented loop, use induction and

Unoriented loops:

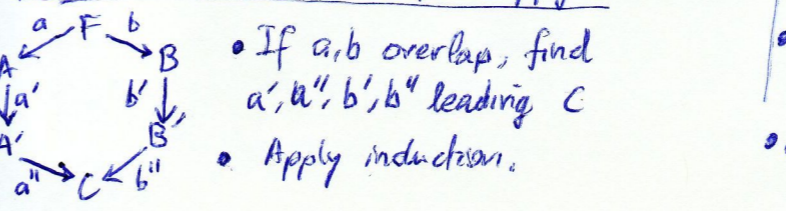
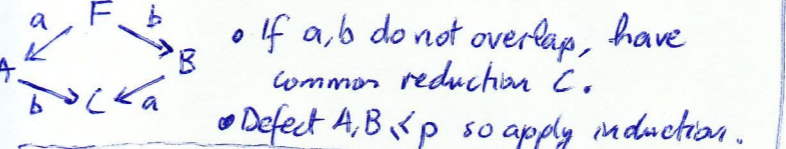


Get shorter loops, use induction and ditto.

Chain Gangs are Linearly Independent

- ie, all reductions of forests to chain gangs give same result: use induction on size of defect.
- True for forests of defect 1 as  $\exists!$  reduction.
- Suppose true for defect  $\leq p$ .

Let F be forest w/ Defect p,  $\Rightarrow$  reductions.



Overlaps 3 types (by inspection)

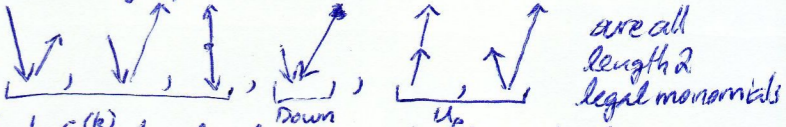


One checks "by hand" that all ways of reducing give same result.

KOSZULNESS: "A quotient of an exterior algebra with quadratic Grobner basis (as exterior algebra) is Koszul (Yuzvinsky)." Order on AS monomials: order generators numerically:  $a_{12} < a_{13} < \dots < a_{21} < a_{23} \dots$

- Order factors in monomial in increasing order.
- Order monomials length-lexicographically. (ignore signs,  $0 < \alpha \neq 0$ )
- Multiplicative: if  $\alpha < \beta$  then  $\alpha \gamma < \beta \gamma$  if  $\beta \gamma \neq 0$ .
- Relations are equivalent to relations with maximal terms  $\nwarrow$ : all  $a_{ij} \wedge a_{jk}$ ,  $i < k < j$

Constancy of Dimension



Let  $S_n^{(k)}$  be legal monomials of length k.

- $S_n^{(k)}$  generates  $A_n^{i,k}$  by multiplicativity.
- $\dim S_n^{(k)} = L(n, n-k) = \dim A_n^{i,k}$  by BASIS Chain Gangs.

Up Graphs:  $\dim S_n^{Up(k)} = s(m, n-k)$

Down Graphs:  $\dim S_n^{Down(k)} = 1$

Up-Down Graphs: Given collection of Up Graphs, only allow Down arrows between roots of the Up Graphs.

- Counting: (a) Divide  $[n]$  into  $l$  cycles and place Up Graphs on the cycles. Let  $m_i$  be root of cycle  $C_i$ .
- (b) Partition  $M = \{m_1, \dots, m_l\}$  as  $M_1, \dots, M_k$  and place Down Graphs on the  $M_i$ .
- $\dim S_n^{(n-k)} = \sum_{l=k}^n s(n, l) S(l, k) = L(n, k) = \dim A_n^{(n-k)}$