

Pensieve header: A gallery of extraction graphs (old).

Old program, from Gamma.nb.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\OU"];
```

pdf

```
In[*]:= SetAttributes[VD, Orderless]
```

pdf

```
In[*]:= Tidy[vd_VD] := Module[{ps = Union@@(List@@@vd)},
  Replace[vd, Thread[ps → Range@Length@ps], {2}]]
```

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```
In[*]:= R12Reduce1[vd_VD] := Tidy@Module[{R2s, R2}, Which[
  Length[R2s = Cases[vd, X_s_[i_, j_] → X_s_[i + 1, j + 1]] ∩ (List@@vd)] > 0,
  Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] → X_s_[i - 1, j - 1]]],
  Length[R2s = Cases[vd, X_s_[i_, j_] → X_s_[i + 1, j - 1]] ∩ (List@@vd)] > 0,
  Complement[vd, VD[R2 = First@R2s, R2 /. X_s_[i_, j_] → X_s_[i - 1, j + 1]]],
  True, DeleteCases[vd, X_[i_, j_] /; Abs[i - j] == 1]]]
```

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```
In[*]:= R12Reduce[vd_VD] := FixedPoint[R12Reduce1, vd]
```

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```
In[*]:= γ[vd_VD] := Module[{js, s1, i1, j1, s2, i2, j2},
  js = Cases[vd, X_[_, j_] → j] ∩ Cases[vd, X_[i_, _] → i - 1];
  If[Length[js] == 0, vd,
  j1 = RandomChoice[js]; i2 = j1 + 1;
  Cases[vd, X_s_[i_, j1] → (s1 = s; i1 = i)];
  Cases[vd, X_s_[i2, j_] → (s2 = s; j2 = j)];
  Tidy@Join[Complement[vd, VD[X_s1[i1, j1], X_s2[i2, j2]]],
  VD[X_s2[j1, j2], X_s1[i1, i2], X_s1s2[i1 - s1/3, j2 + s2/3], X_s1s2[i1 + s1/3, j2 - s2/3]]
  ]]
```

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```
In[*]:= Γ[vd_VD] := FixedPoint[γ, vd, 216]
```

pdf

```
In[*]:= Γ[T_] /; Head[T] != VD := Γ[VD[T]]
```

pdf

```
In[*]:= Γ̄[vd_VD] := FixedPoint[γ@*R12Reduce, vd, 216]
```

pdf

```
In[*]:= Γ̄[T_] /; Head[T] != VD := Γ̄[VD[T]]
```

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```
In[ ]:= VPB[n_, {σs___}] := VPB[n, σs];
```

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```
In[ ]:= VD /: vd1_VD ** vd2_VD := Module[{es1, es2, m2},
  es1 = Cases[vd1, EOS[i_] :=> i];
  m2 = Max[es2 = Cases[vd2, EOS[i_] :=> i]];
  Tidy[vd1 ∪ Replace[DeleteCases[vd2, _EOS],
    i_ :=> i/m2 - 1 + es1[[1 + Count[es2, e_ /; i > e]], {2}]]
]
```

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```
In[ ]:= VD[VPB[n_]] := VD@@(EOS /@ Range[n]);
VD[VPB[n_, σi,j]] := Tidy@Append[VD@@(EOS /@ Range[n]), X+1[i - 0.5, j - 0.5]];
VD[VPB[n_, σ̄i,j]] := Tidy@Append[VD@@(EOS /@ Range[n]), X-1[i - 0.5, j - 0.5]];
VD[VPB[n_, σ_, σs___]] := VD[VPB[n, σ]] ** VD[VPB[n, σs]]
```

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```
In[ ]:= VPBGenerators[n_] :=
  VPBGenerators[n] = Flatten@Table[{σi,j, σ̄i,j}, {i, n}, {j, DeleteCases[Range@n, i]}];
```

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```
In[ ]:= ProudFollowers[n_, σi,j] := ProudFollowers[n, σi,j] = Module[{p, q, s},
  Flatten[{σi,j, σj,i, σ̄j,i,
    Table[{σp,q, σq,p, σ̄p,q, σ̄q,p}, {p, {i, j}}, {q, Complement[Range[n], {i, j}]}],
    Table[{σp,q, σ̄p,q},
      {p, Complement[Range[i + 1, n], {j}]}], {q, Complement[Range[n], {i, j, p}]}]
  ]];
ProudFollowers[n_, σ̄i,j] := ProudFollowers[n, σ̄i,j] = ProudFollowers[n, σi,j] /. σi,j → σ̄i,j
```

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```
In[ ]:= ProudVPBs[n_, 0] := {VPB[n]};
ProudVPBs[n_, 1] := VPB[n, #] & /@ VPBGenerators[n];
ProudVPBs[n_, m_] /; m > 1 := Flatten[
  ProudVPBs[n, m - 1] /. VPB[n, σs___, σ_] :=> (VPB[n, σs, σ, #] & /@ ProudFollowers[n, σ])]
```

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```
In[ ]:= CountOUForms[n_, m_] := Module[{k},
  Length@Union@Flatten@Table[Γ@vpb, {k, 0, m}, {vpb, ProudVPBs[n, k]}]]
```

New Code

```
In[ ]:= AllOUs[n_, 0] := {VD@@Flatten@Table[{BT[2 i - 1], EOS[2 i]}, {i, n}]}];
AllOUs[n_, m_] /; m > 0 :=
Sort@Flatten[AllOUs[n, m - 1] /. vd_VD => Module[{BTs, EOSs, k, max0, s},
  BTs = Sort@Cases[vd, BT[i_] => i];
  EOSs = Sort@Cases[vd, EOS[i_] => i];
  max0 = Max[1, Max[Cases[vd, X_[i_, _] => i]]];
  Table[
    Tidy[Append[vd, Xs[p - 0.5, q + 0.5]]],
    {s, {-1, 1}}, {k, Length[BTs]},
    {q, BTs[[k]], EOSs[[k] - 1], {p, Select[BTs, (# >= max0) &]}
  ]]]
```

```
In[ ]:= AllOUs[3, 0]
```

```
Out[ ]:= {VD[BT[1], BT[3], BT[5], EOS[2], EOS[4], EOS[6]]}
```

```
In[ ]:= AllOUs[3, 4] // Length
```

```
Out[ ]:= 86400
```

```
In[ ]:= 4 × 3 × 4 Binomial[4, 2]
```

```
Out[ ]:= 288
```

```
In[ ]:= 23 Binomial[5, 2] 5! / 2
```

```
Out[ ]:= 4800
```

```
In[ ]:= 24 Binomial[6, 2] 6! / 2
```

```
Out[ ]:= 86400
```

```
In[ ]:= 24 Binomial[7, 3] 7! / 3!
```

```
Out[ ]:= 470400
```

```
In[ ]:= AllROUs[n_, m_] :=
  Select[AllOUs[n, m] /. vd_VD => Tidy@DeleteCases[vd, _BT], (# === R12Reduce[#] &)]
```

```
In[ ]:= AllROUs[3, 4] // Length
```

```
Out[ ]:= 41682
```

```
In[ ]:= ξ[vd_VD] := Count[Γ[vd], X[_ , _]]
```


In[]:= **Union**[**Divisors** /@ **AllROUs**[**3**, **5**]]

Out[]:= { {}, { $\sigma_{1,2}$ }, { $\sigma_{1,3}$ }, { $\sigma_{2,1}$ }, { $\sigma_{2,3}$ }, { $\sigma_{3,1}$ }, { $\sigma_{3,2}$ }, { $\bar{\sigma}_{1,2}$ }, { $\bar{\sigma}_{1,3}$ }, { $\bar{\sigma}_{2,1}$ }, { $\bar{\sigma}_{2,3}$ }, { $\bar{\sigma}_{3,1}$ }, { $\bar{\sigma}_{3,2}$ }, { $\sigma_{1,2}, \sigma_{2,3}$ }, { $\sigma_{1,2}, \sigma_{3,1}$ }, { $\sigma_{1,2}, \bar{\sigma}_{1,3}$ }, { $\sigma_{1,3}, \sigma_{2,1}$ }, { $\sigma_{1,3}, \sigma_{3,2}$ }, { $\sigma_{2,1}, \sigma_{3,2}$ }, { $\sigma_{2,1}, \bar{\sigma}_{2,3}$ }, { $\sigma_{2,3}, \sigma_{3,1}$ }, { $\sigma_{3,1}, \bar{\sigma}_{3,2}$ }, { $\bar{\sigma}_{1,2}, \sigma_{1,3}$ }, { $\bar{\sigma}_{1,2}, \bar{\sigma}_{2,3}$ }, { $\bar{\sigma}_{1,2}, \bar{\sigma}_{3,1}$ }, { $\bar{\sigma}_{1,3}, \bar{\sigma}_{2,1}$ }, { $\bar{\sigma}_{1,3}, \bar{\sigma}_{3,2}$ }, { $\bar{\sigma}_{2,1}, \sigma_{2,3}$ }, { $\bar{\sigma}_{2,1}, \bar{\sigma}_{3,2}$ }, { $\bar{\sigma}_{2,3}, \bar{\sigma}_{3,1}$ }, { $\bar{\sigma}_{3,1}, \sigma_{3,2}$ }}

In[]:= **Union**[**Divisors** /@ **AllROUs**[**3**, **6**]]

In[]:= **Union**[**Divisors** /@ **AllROUs**[**4**, **4**]]

Out[]:= { {}, { $\sigma_{1,2}$ }, { $\sigma_{1,3}$ }, { $\sigma_{1,4}$ }, { $\sigma_{2,1}$ }, { $\sigma_{2,3}$ }, { $\sigma_{2,4}$ }, { $\sigma_{3,1}$ }, { $\sigma_{3,2}$ }, { $\sigma_{3,4}$ }, { $\sigma_{4,1}$ }, { $\sigma_{4,2}$ }, { $\sigma_{4,3}$ }, { $\bar{\sigma}_{1,2}$ }, { $\bar{\sigma}_{1,3}$ }, { $\bar{\sigma}_{1,4}$ }, { $\bar{\sigma}_{2,1}$ }, { $\bar{\sigma}_{2,3}$ }, { $\bar{\sigma}_{2,4}$ }, { $\bar{\sigma}_{3,1}$ }, { $\bar{\sigma}_{3,2}$ }, { $\bar{\sigma}_{3,4}$ }, { $\bar{\sigma}_{4,1}$ }, { $\bar{\sigma}_{4,2}$ }, { $\bar{\sigma}_{4,3}$ }, { $\sigma_{1,2}, \sigma_{2,3}$ }, { $\sigma_{1,2}, \sigma_{2,4}$ }, { $\sigma_{1,2}, \sigma_{3,1}$ }, { $\sigma_{1,2}, \sigma_{3,4}$ }, { $\sigma_{1,2}, \sigma_{4,1}$ }, { $\sigma_{1,2}, \sigma_{4,2}$ }, { $\sigma_{1,2}, \sigma_{4,3}$ }, { $\sigma_{1,2}, \bar{\sigma}_{1,3}$ }, { $\sigma_{1,2}, \bar{\sigma}_{1,4}$ }, { $\sigma_{1,2}, \bar{\sigma}_{3,4}$ }, { $\sigma_{1,2}, \bar{\sigma}_{4,3}$ }, { $\sigma_{1,3}, \sigma_{2,1}$ }, { $\sigma_{1,3}, \sigma_{2,4}$ }, { $\sigma_{1,3}, \sigma_{3,2}$ }, { $\sigma_{1,3}, \sigma_{3,4}$ }, { $\sigma_{1,3}, \sigma_{4,1}$ }, { $\sigma_{1,3}, \sigma_{4,2}$ }, { $\sigma_{1,3}, \bar{\sigma}_{1,4}$ }, { $\sigma_{1,3}, \bar{\sigma}_{2,4}$ }, { $\sigma_{1,3}, \bar{\sigma}_{4,2}$ }, { $\sigma_{1,4}, \sigma_{2,1}$ }, { $\sigma_{1,4}, \sigma_{2,3}$ }, { $\sigma_{1,4}, \sigma_{3,1}$ }, { $\sigma_{1,4}, \sigma_{3,2}$ }, { $\sigma_{1,4}, \sigma_{4,2}$ }, { $\sigma_{1,4}, \sigma_{4,3}$ }, { $\sigma_{1,4}, \bar{\sigma}_{2,3}$ }, { $\sigma_{1,4}, \bar{\sigma}_{3,2}$ }, { $\sigma_{2,1}, \sigma_{3,2}$ }, { $\sigma_{2,1}, \sigma_{3,4}$ }, { $\sigma_{2,1}, \sigma_{4,2}$ }, { $\sigma_{2,1}, \sigma_{4,3}$ }, { $\sigma_{2,1}, \bar{\sigma}_{2,3}$ }, { $\sigma_{2,1}, \bar{\sigma}_{2,4}$ }, { $\sigma_{2,1}, \bar{\sigma}_{3,4}$ }, { $\sigma_{2,1}, \bar{\sigma}_{4,3}$ }, { $\sigma_{2,3}, \sigma_{3,1}$ }, { $\sigma_{2,3}, \sigma_{3,4}$ }, { $\sigma_{2,3}, \sigma_{4,1}$ }, { $\sigma_{2,3}, \sigma_{4,2}$ }, { $\sigma_{2,3}, \bar{\sigma}_{2,4}$ }, { $\sigma_{2,3}, \bar{\sigma}_{4,1}$ }, { $\sigma_{2,4}, \sigma_{3,1}$ }, { $\sigma_{2,4}, \sigma_{3,2}$ }, { $\sigma_{2,4}, \sigma_{4,1}$ }, { $\sigma_{2,4}, \sigma_{4,3}$ }, { $\sigma_{2,4}, \bar{\sigma}_{3,1}$ }, { $\sigma_{3,1}, \sigma_{4,2}$ }, { $\sigma_{3,1}, \sigma_{4,3}$ }, { $\sigma_{3,1}, \bar{\sigma}_{3,2}$ }, { $\sigma_{3,1}, \bar{\sigma}_{3,4}$ }, { $\sigma_{3,1}, \bar{\sigma}_{4,2}$ }, { $\sigma_{3,2}, \sigma_{4,1}$ }, { $\sigma_{3,2}, \sigma_{4,3}$ }, { $\sigma_{3,2}, \bar{\sigma}_{3,4}$ }, { $\sigma_{3,2}, \bar{\sigma}_{4,1}$ }, { $\sigma_{3,4}, \sigma_{4,1}$ }, { $\sigma_{3,4}, \sigma_{4,2}$ }, { $\sigma_{4,1}, \bar{\sigma}_{4,2}$ }, { $\sigma_{4,1}, \bar{\sigma}_{4,3}$ }, { $\sigma_{4,2}, \bar{\sigma}_{4,3}$ }, { $\bar{\sigma}_{1,2}, \sigma_{1,3}$ }, { $\bar{\sigma}_{1,2}, \sigma_{1,4}$ }, { $\bar{\sigma}_{1,2}, \sigma_{3,4}$ }, { $\bar{\sigma}_{1,2}, \sigma_{4,3}$ }, { $\bar{\sigma}_{1,2}, \bar{\sigma}_{2,3}$ }, { 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$\bar{\sigma}_{2,1}, \sigma_{2,3}$ }, { $\bar{\sigma}_{2,1}, \sigma_{2,4}$ }, { $\bar{\sigma}_{2,1}, \sigma_{3,4}$ }, { $\bar{\sigma}_{2,1}, \sigma_{4,3}$ }, { $\bar{\sigma}_{2,1}, \bar{\sigma}_{3,2}$ }, { $\bar{\sigma}_{2,1}, \bar{\sigma}_{3,4}$ }, { $\bar{\sigma}_{2,1}, \bar{\sigma}_{4,2}$ }, { $\bar{\sigma}_{2,1}, \bar{\sigma}_{4,3}$ }, { $\bar{\sigma}_{2,3}, \sigma_{2,4}$ }, { $\bar{\sigma}_{2,3}, \sigma_{4,1}$ }, { $\bar{\sigma}_{2,3}, \bar{\sigma}_{3,1}$ }, { $\bar{\sigma}_{2,3}, \bar{\sigma}_{3,4}$ }, { $\bar{\sigma}_{2,3}, \bar{\sigma}_{4,1}$ }, { $\bar{\sigma}_{2,3}, \bar{\sigma}_{4,2}$ }, { $\bar{\sigma}_{2,4}, \sigma_{3,1}$ }, { $\bar{\sigma}_{2,4}, \bar{\sigma}_{3,1}$ }, { $\bar{\sigma}_{2,4}, \bar{\sigma}_{3,2}$ }, { $\bar{\sigma}_{2,4}, \bar{\sigma}_{4,1}$ }, { $\bar{\sigma}_{2,4}, \bar{\sigma}_{4,3}$ }, { $\bar{\sigma}_{3,1}, \sigma_{3,2}$ }, { $\bar{\sigma}_{3,1}, \sigma_{3,4}$ }, { $\bar{\sigma}_{3,1}, \sigma_{4,2}$ }, { $\bar{\sigma}_{3,1}, \bar{\sigma}_{4,2}$ }, { $\bar{\sigma}_{3,1}, \bar{\sigma}_{4,3}$ }, { $\bar{\sigma}_{3,2}, \sigma_{3,4}$ }, { $\bar{\sigma}_{3,2}, \sigma_{4,1}$ }, { $\bar{\sigma}_{3,2}, \bar{\sigma}_{4,1}$ }, { $\bar{\sigma}_{3,2}, \bar{\sigma}_{4,3}$ }, { $\bar{\sigma}_{3,4}, \bar{\sigma}_{4,1}$ }, { $\bar{\sigma}_{3,4}, \bar{\sigma}_{4,2}$ }, { $\bar{\sigma}_{4,1}, \sigma_{4,2}$ }, { $\bar{\sigma}_{4,1}, \sigma_{4,3}$ }, { $\bar{\sigma}_{4,2}, \sigma_{4,3}$ }}

In[]:= **MinQ**[**divs_**] :=

Sort[**Sort**[**divs** /. **Thread**[**Range**@**4** → **#**]] & /@ **Permutations**[**Range**@**4**]] [**1**] === **divs**

```
In[*]:= Select[{{}, {σ1,2}, {σ1,3}, {σ1,4}, {σ2,1}, {σ2,3}, {σ2,4}, {σ3,1}, {σ3,2}, {σ3,4}, {σ4,1},
  {σ4,2}, {σ4,3}, {σ̄1,2}, {σ̄1,3}, {σ̄1,4}, {σ̄2,1}, {σ̄2,3}, {σ̄2,4}, {σ̄3,1}, {σ̄3,2}, {σ̄3,4},
  {σ̄4,1}, {σ̄4,2}, {σ̄4,3}, {σ1,2 σ2,3}, {σ1,2 σ2,4}, {σ1,2 σ3,1}, {σ1,2 σ3,4}, {σ1,2 σ4,1},
  {σ1,2 σ4,3}, {σ1,2 σ̄1,3}, {σ1,2 σ̄1,4}, {σ1,2 σ̄3,4}, {σ1,2 σ̄4,3}, {σ1,3 σ2,1}, {σ1,3 σ2,4},
  {σ1,3 σ3,2}, {σ1,3 σ3,4}, {σ1,3 σ4,1}, {σ1,3 σ4,2}, {σ1,3 σ̄1,4}, {σ1,3 σ̄2,4}, {σ1,3 σ̄4,2},
  {σ1,4 σ2,1}, {σ1,4 σ2,3}, {σ1,4 σ3,1}, {σ1,4 σ3,2}, {σ1,4 σ4,2}, {σ1,4 σ4,3}, {σ1,4 σ̄2,3},
  {σ1,4 σ̄3,2}, {σ2,1 σ3,2}, {σ2,1 σ3,4}, {σ2,1 σ4,2}, {σ2,1 σ4,3}, {σ2,1 σ̄2,3}, {σ2,1 σ̄2,4},
  {σ2,1 σ̄3,4}, {σ2,1 σ̄4,3}, {σ2,3 σ3,1}, {σ2,3 σ3,4}, {σ2,3 σ4,1}, {σ2,3 σ4,2}, {σ2,3 σ̄2,4},
  {σ2,3 σ̄4,1}, {σ2,4 σ3,1}, {σ2,4 σ3,2}, {σ2,4 σ4,1}, {σ2,4 σ4,3}, {σ2,4 σ̄3,1}, {σ3,1 σ4,2},
  {σ3,1 σ4,3}, {σ3,1 σ̄3,2}, {σ3,1 σ̄3,4}, {σ3,1 σ̄4,2}, {σ3,2 σ4,1}, {σ3,2 σ4,3}, {σ3,2 σ̄3,4},
  {σ3,2 σ̄4,1}, {σ3,4 σ4,1}, {σ3,4 σ4,2}, {σ4,1 σ̄4,2}, {σ4,1 σ̄4,3}, {σ4,2 σ̄4,3}, {σ̄1,2 σ1,3},
  {σ̄1,2 σ1,4}, {σ̄1,2 σ3,4}, {σ̄1,2 σ4,3}, {σ̄1,2 σ̄2,3}, {σ̄1,2 σ̄2,4}, {σ̄1,2 σ̄3,1}, {σ̄1,2 σ̄3,4},
  {σ̄1,2 σ̄4,1}, {σ̄1,2 σ̄4,3}, {σ̄1,3 σ1,4}, {σ̄1,3 σ2,4}, {σ̄1,3 σ4,2}, {σ̄1,3 σ̄2,1}, {σ̄1,3 σ̄2,4},
  {σ̄1,3 σ̄3,2}, {σ̄1,3 σ̄3,4}, {σ̄1,3 σ̄4,1}, {σ̄1,3 σ̄4,2}, {σ̄1,4 σ2,3}, {σ̄1,4 σ3,2}, {σ̄1,4 σ̄2,1},
  {σ̄1,4 σ̄2,3}, {σ̄1,4 σ̄3,1}, {σ̄1,4 σ̄3,2}, {σ̄1,4 σ̄4,2}, {σ̄1,4 σ̄4,3}, {σ̄2,1 σ2,3}, {σ̄2,1 σ2,4},
  {σ̄2,1 σ3,4}, {σ̄2,1 σ4,3}, {σ̄2,1 σ̄3,2}, {σ̄2,1 σ̄3,4}, {σ̄2,1 σ̄4,2}, {σ̄2,1 σ̄4,3}, {σ̄2,3 σ2,4},
  {σ̄2,3 σ4,1}, {σ̄2,3 σ̄3,1}, {σ̄2,3 σ̄3,4}, {σ̄2,3 σ̄4,1}, {σ̄2,3 σ̄4,2}, {σ̄2,4 σ3,1},
  {σ̄2,4 σ̄3,1}, {σ̄2,4 σ̄3,2}, {σ̄2,4 σ̄4,1}, {σ̄2,4 σ̄4,3}, {σ̄3,1 σ3,2}, {σ̄3,1 σ3,4},
  {σ̄3,1 σ4,2}, {σ̄3,1 σ̄4,2}, {σ̄3,1 σ̄4,3}, {σ̄3,2 σ3,4}, {σ̄3,2 σ4,1}, {σ̄3,2 σ̄4,1},
  {σ̄3,2 σ̄4,3}, {σ̄3,4 σ̄4,1}, {σ̄3,4 σ̄4,2}, {σ̄4,1 σ4,2}, {σ̄4,1 σ4,3}, {σ̄4,2 σ4,3}, MinQ]
```

```
Out[*]:= {{}, {σ1,2}, {σ̄1,2}, {σ1,2 σ2,3}, {σ1,2 σ3,4},
  {σ1,2 σ̄1,3}, {σ1,2 σ̄3,4}, {σ̄1,2 σ̄2,3}, {σ̄1,2 σ̄3,4}}
```

```
In[*]:= OUGraph[n_, m_] := Module[{gens, OUs, k, d, g, q, m1, m2},
  gens = VPBGenerators[n];
  OUs = Flatten@Table[AllROUs[n, k], {k, 0, m}];
  OURule = Dispatch@Thread[OUs → Range@Length@OUs];
  Graph[
    Range@Length@OUs,
    Union@Flatten@Table[
      m1 = Count[d, X[_], _];
      m2 = Count[q = T̄[VD[VPB[n, g]] ** d], X[_], _];
      If[m2 < m1, Labeled[(d → q) /. OURule, g], Nothing],
      {d, OUs}, {g, gens}
    ]
  ]
```

```
In[*]:= T̄[VPB[3, σ1,2 σ2,1 σ1,2 σ1,3 σ2,3]]
```

```
Out[*]:= VD[EOS[7], EOS[14], EOS[17], X1[1, 16],
  X1[2, 13], X1[3, 6], X1[4, 11], X1[8, 15], X1[9, 12], X1[10, 5]]
```

```
In[*]:= T̄[VPB[3, σ̄2,3 σ1,2 σ2,1 σ1,2 σ1,3 σ2,3]]
```

```
Out[*]:= VD[EOS[7], EOS[13], EOS[15], X1[1, 14], X1[2, 12], X1[3, 6], X1[4, 10], X1[8, 11], X1[9, 5]]
```

In[*]:= $\overline{\Gamma}[\text{VPB}[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}]]$

Out[*]:= $\text{VD}[\text{EOS}[7], \text{EOS}[14], \text{EOS}[17], X_1[1, 16], X_1[2, 13], X_1[3, 6], X_1[4, 11], X_1[8, 15], X_1[9, 12], X_1[10, 5]]]$

In[*]:= $\overline{\Gamma}[\text{VPB}[3, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}]]$

Out[*]:= $\text{VD}[\text{EOS}[9], \text{EOS}[18], \text{EOS}[25], X_{-1}[4, 20], X_{-1}[10, 19], X_{-1}[14, 21], X_1[1, 17], X_1[2, 24], X_1[3, 8], X_1[5, 15], X_1[6, 22], X_1[11, 16], X_1[12, 23], X_1[13, 7]]]$

In[*]:= $\overline{\Gamma}[\text{VPB}[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}]]$

Out[*]:= $\text{VD}[\text{EOS}[9], \text{EOS}[17], \text{EOS}[23], X_{-1}[4, 18], X_{-1}[13, 19], X_1[1, 16], X_1[2, 22], X_1[3, 8], X_1[5, 14], X_1[6, 20], X_1[10, 15], X_1[11, 21], X_1[12, 7]]]$

In[*]:= $\overline{\Gamma}[\text{VPB}[3, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}]]$

Out[*]:= $\text{VD}[\text{EOS}[9], \text{EOS}[17], \text{EOS}[23], X_{-1}[4, 18], X_{-1}[13, 19], X_1[1, 16], X_1[2, 22], X_1[3, 8], X_1[5, 14], X_1[6, 20], X_1[10, 15], X_1[11, 21], X_1[12, 7]]]$

In[*]:= $\text{chb1} = \text{VPB}[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}]; \text{chb2} = \text{VPB}[3, \overline{\sigma}_{2,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}];$
 $\{\text{chb3}, \text{chb4}\} = \{\text{chb1}, \text{chb2}\} /. \{\sigma \rightarrow \overline{\sigma}, \overline{\sigma} \rightarrow \sigma\}$

Out[*]:= $\{\text{VPB}[3, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \sigma_{2,3}], \text{VPB}[3, \sigma_{2,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{2,3}]\}$

In[*]:= $\overline{\Gamma}[\text{chb1}]$

Out[*]:= $\text{VD}[\text{EOS}[9], \text{EOS}[17], \text{EOS}[23], X_{-1}[4, 18], X_{-1}[13, 19], X_1[1, 16], X_1[2, 22], X_1[3, 8], X_1[5, 14], X_1[6, 20], X_1[10, 15], X_1[11, 21], X_1[12, 7]]]$

In[*]:= **Divisors** [$\overline{\Gamma}[\text{chb1}]$]

Out[*]:= $\{\sigma_{1,2}\}$

In[*]:= **Quotients** [$\overline{\Gamma}[\text{chb1}]$]

Out[*]:= $\{\text{VD}[\text{EOS}[4], \text{EOS}[11], \text{EOS}[15], X_{-1}[8, 12], X_1[1, 9], X_1[2, 13], X_1[5, 10], X_1[6, 14], X_1[7, 3]]\}$

In[*]:= **Divisors** /@ **Quotients** [$\overline{\Gamma}[\text{chb1}]$]

Out[*]:= $\{\{\sigma_{2,1}\}\}$

In[*]:= **Flatten** [**Quotients** /@ **Quotients** [$\overline{\Gamma}[\text{chb1}]$]]

Out[*]:= $\{\text{VD}[\text{EOS}[3], \text{EOS}[5], \text{EOS}[7], X_1[1, 4], X_1[2, 6]]\}$

In[*]:= **Divisors** /@ **Flatten** [**Quotients** /@ **Quotients** [$\overline{\Gamma}[\text{chb1}]$]]

Out[*]:= $\{\{\sigma_{1,2}\}\}$

In[*]:= **Quotients** /@ **Flatten** [**Quotients** /@ **Quotients** [$\overline{\Gamma}[\text{chb1}]$]]

Out[*]:= $\{\{\text{VD}[\text{EOS}[2], \text{EOS}[3], \text{EOS}[5], X_1[1, 4]]\}\}$

In[*]:= $\overline{\Gamma}[\text{chb2}]$

Out[*]:= $\text{VD}[\text{EOS}[7], \text{EOS}[13], \text{EOS}[15], X_1[1, 14], X_1[2, 12], X_1[3, 6], X_1[4, 10], X_1[8, 11], X_1[9, 5]]]$

In[*]:= **Divisors**[$\overline{\Gamma}$ [chb2]]

Out[*]:= $\{\sigma_{1,3}\}$

In[*]:= $\overline{\Gamma}$ [chb3]

Out[*]:= $\text{VD}[\text{EOS}[9], \text{EOS}[17], \text{EOS}[23], X_{-1}[1, 16], X_{-1}[2, 22], X_{-1}[3, 8], X_{-1}[5, 14], X_{-1}[6, 20], X_{-1}[10, 15], X_{-1}[11, 21], X_{-1}[12, 7], X_1[4, 18], X_1[13, 19]]$

In[*]:= **Divisors**[$\overline{\Gamma}$ [chb3]]

Out[*]:= $\{\overline{\sigma}_{1,2}\}$

In[*]:= **Divisors** /@ **Quotients**[$\overline{\Gamma}$ [chb3]]

Out[*]:= $\{\{\overline{\sigma}_{2,1}\}\}$

In[*]:= **Divisors** /@ **Flatten**[**Quotients** /@ **Quotients**[$\overline{\Gamma}$ [chb3]]]

Out[*]:= $\{\{\overline{\sigma}_{1,2}\}\}$

In[*]:= **Quotients** /@ **Flatten**[**Quotients** /@ **Quotients**[$\overline{\Gamma}$ [chb3]]]

Out[*]:= $\{\{\text{VD}[\text{EOS}[2], \text{EOS}[3], \text{EOS}[5], X_{-1}[1, 4]]\}\}$

In[*]:= $\overline{\Gamma}$ [chb4]

Out[*]:= $\text{VD}[\text{EOS}[7], \text{EOS}[13], \text{EOS}[15], X_{-1}[1, 14], X_{-1}[2, 12], X_{-1}[3, 6], X_{-1}[4, 10], X_{-1}[8, 11], X_{-1}[9, 5]]$

In[*]:= **Divisors**[$\overline{\Gamma}$ [chb4]]

Out[*]:= $\{\overline{\sigma}_{1,3}\}$

```
In[*]:= ExtractVPB[vd_VD] := Module[{n, ds, d},
  n = Count[vd, _EOS];
  If[Length[ds = Divisors[vd]] == 0, VPB[n],
    d = First@Sort[ds];
    q =  $\overline{\Gamma}$ [VD[VPB[n, d /. { $\sigma \rightarrow \overline{\sigma}$ ,  $\overline{\sigma} \rightarrow \sigma$ ]}]] ** vd];
    Insert[ExtractVPB[q], d, 2]
  ]];
CF[vpb_VPB] := ExtractVPB[ $\overline{\Gamma}$ [vpb]];
```

In[*]:= {chb1, **CF**[chb1]}

Out[*]:= $\{\text{VPB}[3, \sigma_{2,3}, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \overline{\sigma}_{2,3}], \text{VPB}[3, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}]\}$

In[*]:= {chb2, **CF**[chb2]}

Out[*]:= $\{\text{VPB}[3, \overline{\sigma}_{2,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}], \text{VPB}[3, \sigma_{1,3}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{1,2}]\}$

In[*]:= {chb3, **CF**[chb3]}

Out[*]:= $\{\text{VPB}[3, \overline{\sigma}_{2,3}, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \sigma_{2,3}], \text{VPB}[3, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}]\}$

In[*]:= {chb4, **CF**[chb4]}

Out[*]:= $\{\text{VPB}[3, \sigma_{2,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}, \overline{\sigma}_{1,3}, \overline{\sigma}_{2,3}], \text{VPB}[3, \overline{\sigma}_{1,3}, \overline{\sigma}_{1,2}, \overline{\sigma}_{2,1}, \overline{\sigma}_{1,2}]\}$


```
In[ ]:= Table[{vpb = Module[{n = 4, m = 12, i},
  VPB[n, Sequence @@
    Table[RandomChoice[{σ, σ̄}]_{i=RandomChoice@Range@n, RandomChoice[Complement[Range@n, {i}]}], {m}]]],
  cf = CF[vpb], Length[cf] - 1}, {30}] // MatrixForm
```

Out[]//MatrixForm=

VPB [4, σ̄ _{3,4} , σ̄ _{4,1} , σ̄ _{3,2} , σ̄ _{1,4} , σ̄ _{4,1} , σ̄ _{1,2} , σ _{1,2} , σ _{1,2} , σ _{2,4} , σ _{1,4} , σ _{3,2} , σ̄ _{3,4}]	VPB [4, σ̄ ₃
VPB [4, σ _{4,3} , σ̄ _{1,4} , σ̄ _{2,3} , σ̄ _{2,1} , σ _{2,3} , σ _{2,1} , σ _{3,1} , σ _{1,2} , σ _{3,4} , σ _{4,3} , σ _{2,1} , σ̄ _{4,3}]	VPB [
VPB [4, σ _{1,2} , σ _{3,1} , σ̄ _{2,3} , σ̄ _{1,4} , σ̄ _{3,2} , σ _{4,2} , σ _{4,3} , σ _{3,1} , σ̄ _{3,2} , σ̄ _{4,2} , σ _{2,3} , σ̄ _{2,1}]	VPB [4, σ _{1,2} , σ ₃
VPB [4, σ _{3,2} , σ̄ _{2,3} , σ _{4,3} , σ̄ _{3,1} , σ̄ _{3,1} , σ _{2,4} , σ _{3,1} , σ _{4,3} , σ _{1,3} , σ _{2,1} , σ̄ _{4,2} , σ _{3,2}]	VPB [
VPB [4, σ _{1,4} , σ _{1,3} , σ _{4,3} , σ̄ _{4,3} , σ̄ _{1,4} , σ̄ _{2,1} , σ̄ _{3,1} , σ _{1,3} , σ̄ _{4,2} , σ _{3,2} , σ̄ _{4,3} , σ _{4,1}]	VPB [
VPB [4, σ _{4,1} , σ _{2,4} , σ _{2,3} , σ̄ _{4,2} , σ̄ _{4,1} , σ̄ _{1,3} , σ̄ _{4,2} , σ _{2,3} , σ̄ _{1,4} , σ̄ _{1,3} , σ̄ _{4,1} , σ _{2,1}]	VPB [4, σ _{4,1} , σ ₂
VPB [4, σ̄ _{3,4} , σ̄ _{2,3} , σ _{2,3} , σ _{4,1} , σ̄ _{1,3} , σ _{4,1} , σ̄ _{2,4} , σ _{1,4} , σ _{1,2} , σ̄ _{4,3} , σ _{2,3} , σ _{3,2}]	VPB [4, σ _{3,1} , σ ₄
VPB [4, σ̄ _{1,4} , σ̄ _{2,3} , σ _{2,1} , σ̄ _{3,1} , σ _{3,1} , σ̄ _{3,1} , σ _{3,2} , σ̄ _{4,1} , σ _{3,1} , σ _{1,3} , σ̄ _{2,1} , σ _{3,1}]	VPB [
VPB [4, σ _{2,4} , σ̄ _{4,2} , σ _{3,2} , σ̄ _{4,2} , σ̄ _{3,2} , σ̄ _{3,4} , σ _{4,1} , σ _{1,2} , σ̄ _{3,4} , σ̄ _{1,4} , σ̄ _{3,1} , σ _{3,2}]	VPB [4, σ ₂
VPB [4, σ̄ _{4,1} , σ _{3,1} , σ̄ _{1,2} , σ̄ _{2,3} , σ _{3,2} , σ _{2,4} , σ _{3,2} , σ̄ _{2,3} , σ̄ _{4,3} , σ̄ _{1,4} , σ _{1,4} , σ̄ _{2,1}]	VPB [4, σ̄ ₃
VPB [4, σ̄ _{3,4} , σ _{4,1} , σ̄ _{3,1} , σ̄ _{1,3} , σ _{2,1} , σ̄ _{3,1} , σ̄ _{4,1} , σ̄ _{4,3} , σ _{1,3} , σ̄ _{1,3} , σ _{4,1} , σ _{3,2}]	VPB [4, σ _{3,1} , σ ₄
VPB [4, σ̄ _{2,3} , σ _{3,1} , σ̄ _{2,4} , σ _{1,3} , σ̄ _{3,2} , σ̄ _{1,4} , σ̄ _{4,3} , σ _{4,2} , σ̄ _{3,4} , σ _{3,4} , σ _{2,4} , σ̄ _{1,3}]	VPB [4, σ ₂
VPB [4, σ _{1,4} , σ̄ _{3,2} , σ _{4,3} , σ̄ _{3,1} , σ̄ _{2,3} , σ̄ _{1,2} , σ̄ _{1,4} , σ _{4,3} , σ̄ _{3,1} , σ̄ _{4,2} , σ̄ _{1,4} , σ̄ _{3,4}]	VPB [4, σ _{1,4} , σ̄ ₃
VPB [4, σ̄ _{1,2} , σ _{4,3} , σ _{4,2} , σ̄ _{4,1} , σ _{4,2} , σ̄ _{1,4} , σ̄ _{1,3} , σ̄ _{1,2} , σ̄ _{4,1} , σ̄ _{1,3} , σ _{2,4} , σ _{4,3}]	VPB [4, σ ₄
VPB [4, σ̄ _{2,4} , σ̄ _{3,4} , σ _{3,4} , σ̄ _{2,3} , σ̄ _{3,2} , σ̄ _{1,3} , σ _{3,2} , σ _{3,4} , σ _{2,1} , σ̄ _{3,2} , σ _{2,4} , σ̄ _{2,3}]	VPB [4, σ̄ _{2,4} , σ̄ _{2,3} , σ̄ ₃
VPB [4, σ _{3,2} , σ̄ _{1,2} , σ̄ _{1,3} , σ̄ _{1,2} , σ̄ _{3,2} , σ̄ _{4,2} , σ _{1,2} , σ̄ _{2,4} , σ̄ _{2,3} , σ _{2,4} , σ _{2,3} , σ _{4,1}]	VPB [4, σ ₃
VPB [4, σ _{2,3} , σ _{4,1} , σ̄ _{4,2} , σ̄ _{4,2} , σ̄ _{4,2} , σ _{2,4} , σ̄ _{3,1} , σ̄ _{1,4} , σ _{1,3} , σ̄ _{2,1} , σ̄ _{2,4} , σ _{3,4}]	VPB [4, σ ₂
VPB [4, σ̄ _{2,1} , σ _{2,4} , σ _{4,2} , σ̄ _{3,2} , σ̄ _{4,1} , σ̄ _{3,4} , σ̄ _{2,3} , σ̄ _{3,4} , σ _{3,1} , σ̄ _{1,4} , σ _{3,4} , σ̄ _{4,3}]	VPB [4, σ̄ ₂
VPB [4, σ̄ _{1,4} , σ _{3,4} , σ̄ _{2,3} , σ _{4,2} , σ̄ _{2,3} , σ _{3,4} , σ _{3,2} , σ _{3,1} , σ̄ _{1,2} , σ _{1,4} , σ̄ _{3,4} , σ̄ _{3,2}]	VPB [4, σ̄ _{1,4} , σ ₃
VPB [4, σ _{3,1} , σ _{3,1} , σ _{2,4} , σ _{3,2} , σ̄ _{3,1} , σ _{2,3} , σ̄ _{4,3} , σ _{4,1} , σ _{4,3} , σ _{2,4} , σ _{4,1} , σ _{1,4}]	VPB [4, σ ₂
VPB [4, σ _{2,3} , σ _{2,4} , σ _{3,1} , σ̄ _{1,4} , σ̄ _{3,2} , σ̄ _{1,3} , σ _{3,4} , σ̄ _{1,4} , σ̄ _{2,1} , σ _{1,2} , σ̄ _{2,4} , σ _{2,3}]	VPB [4, σ ₂
VPB [4, σ̄ _{4,1} , σ̄ _{2,4} , σ̄ _{3,2} , σ̄ _{3,4} , σ̄ _{3,2} , σ _{3,1} , σ̄ _{3,1} , σ̄ _{4,3} , σ _{1,2} , σ̄ _{2,1} , σ _{4,3} , σ̄ _{2,3}]	
VPB [4, σ _{1,4} , σ _{4,1} , σ̄ _{4,3} , σ _{1,4} , σ̄ _{4,1} , σ̄ _{2,1} , σ _{4,3} , σ _{2,3} , σ _{3,2} , σ _{2,4} , σ̄ _{2,3} , σ̄ _{3,2}]	VPB [4, σ ₁
VPB [4, σ̄ _{4,1} , σ̄ _{3,1} , σ _{1,2} , σ _{4,3} , σ _{4,1} , σ _{2,4} , σ̄ _{2,3} , σ _{3,1} , σ̄ _{3,4} , σ _{2,3} , σ _{4,2} , σ̄ _{2,1}]	VPB [4, σ _{3,2} , σ _{4,2} , σ _{1,2} , σ ₄
VPB [4, σ̄ _{4,3} , σ _{2,4} , σ _{3,2} , σ _{1,2} , σ _{1,2} , σ̄ _{2,3} , σ̄ _{3,1} , σ _{4,2} , σ̄ _{2,3} , σ̄ _{3,1} , σ̄ _{4,2} , σ _{2,3}]	VPB [4, σ̄ _{4,3} , σ ₂
VPB [4, σ _{3,1} , σ̄ _{4,3} , σ _{4,2} , σ̄ _{1,3} , σ̄ _{3,2} , σ̄ _{3,1} , σ̄ _{4,3} , σ̄ _{2,3} , σ̄ _{4,1} , σ̄ _{2,4} , σ̄ _{1,2} , σ _{2,3}]	VPB [4, σ _{3,1} , σ̄ _{4,3} , σ ₄
VPB [4, σ̄ _{2,1} , σ̄ _{3,4} , σ _{1,3} , σ̄ _{1,3} , σ̄ _{4,1} , σ̄ _{1,3} , σ _{1,3} , σ̄ _{1,4} , σ _{2,1} , σ̄ _{4,3} , σ _{3,2} , σ _{1,4}]	VPB [
VPB [4, σ _{4,3} , σ _{1,4} , σ̄ _{1,3} , σ _{2,3} , σ̄ _{1,3} , σ _{4,1} , σ _{2,1} , σ _{1,4} , σ _{2,1} , σ̄ _{1,3} , σ̄ _{1,4} , σ̄ _{1,4}]	VPB [4, σ ₄
VPB [4, σ̄ _{3,1} , σ̄ _{3,1} , σ _{2,4} , σ _{2,3} , σ _{4,2} , σ̄ _{4,1} , σ _{4,3} , σ _{4,3} , σ _{1,3} , σ̄ _{2,4} , σ̄ _{1,2} , σ _{4,1}]	VPB [4, σ ₂
VPB [4, σ̄ _{3,4} , σ̄ _{1,4} , σ̄ _{4,1} , σ _{2,4} , σ _{1,4} , σ̄ _{2,3} , σ̄ _{2,1} , σ̄ _{3,1} , σ̄ _{2,3} , σ _{1,3} , σ̄ _{1,4} , σ̄ _{4,3}]	VPB [4, σ _{3,1} , σ̄ ₁

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In[*]:= Table[{vpb = Module[{n = 3, m = 10, i},
  VPB[n, Sequence @@
    Table[RandomChoice[{σ, σ̄}]i=RandomChoice@Range@n, RandomChoice[Complement[Range@n, {i}]], {m}]]],
  cf = CF[vpb], Length[cf] - 1}, {20}] // MatrixForm
```

Out[*]//MatrixForm=

VPB [3, σ _{1,2} , σ _{2,1} , σ̄ _{2,1} , σ _{3,1} , σ̄ _{2,3} , σ̄ _{2,1} , σ _{2,1} , σ̄ _{1,3} , σ̄ _{1,3} , σ _{3,1}]	VPB [3, σ _{1,2} , σ _{3,1} , σ̄ ₂]
VPB [3, σ _{3,1} , σ _{1,3} , σ̄ _{3,1} , σ _{3,1} , σ̄ _{1,3} , σ̄ _{3,1} , σ̄ _{3,1} , σ _{2,1} , σ̄ _{2,1} , σ _{3,2}]	VPB [3, σ̄ ₃]
VPB [3, σ̄ _{3,1} , σ̄ _{1,3} , σ _{3,2} , σ̄ _{3,2} , σ̄ _{2,1} , σ̄ _{3,2} , σ _{2,3} , σ _{2,3} , σ _{1,3} , σ̄ _{2,1}]	VPB [3, σ̄ _{3,1} , σ̄ _{1,3} , σ̄ _{2,1} , σ̄ ₃]
VPB [3, σ̄ _{1,3} , σ _{1,2} , σ _{1,3} , σ _{1,2} , σ _{1,3} , σ̄ _{2,3} , σ̄ _{2,1} , σ̄ _{2,3} , σ _{2,3} , σ _{2,3}]	VPB [3, σ̄ _{1,3} , σ _{1,2} , σ _{1,3} , σ ₁]
VPB [3, σ̄ _{2,3} , σ _{2,3} , σ̄ _{3,2} , σ̄ _{3,2} , σ _{3,1} , σ̄ _{3,1} , σ _{1,2} , σ̄ _{3,1} , σ _{2,1} , σ _{2,1}]	VPB [3, σ̄ _{3,2} , σ̄ _{3,2} , σ ₁]
VPB [3, σ _{3,2} , σ̄ _{3,1} , σ̄ _{3,2} , σ _{3,2} , σ̄ _{3,2} , σ̄ _{2,1} , σ̄ _{3,1} , σ̄ _{3,1} , σ _{2,1} , σ̄ _{2,3}]	VPB [3, σ _{3,2} , σ̄ _{3,1} , σ̄ _{3,2} , σ̄ ₂]
VPB [3, σ _{1,2} , σ̄ _{1,3} , σ̄ _{3,1} , σ _{3,1} , σ _{3,2} , σ _{2,1} , σ̄ _{1,2} , σ _{2,1} , σ _{2,1} , σ̄ _{2,1}]	VPB [3, σ _{1,2} , σ _{1,2} , σ _{3,2} , σ̄ ₁]
VPB [3, σ̄ _{3,2} , σ̄ _{3,2} , σ̄ _{3,2} , σ̄ _{3,2} , σ _{2,1} , σ _{3,1} , σ _{2,1} , σ _{2,1} , σ̄ _{3,1} , σ̄ _{2,1}]	VPB [3, σ̄ _{3,2} , σ̄ _{3,2} , σ̄ _{3,2} , σ _{3,1} , σ _{2,1} , σ ₃]
VPB [3, σ̄ _{3,2} , σ̄ _{1,2} , σ _{2,1} , σ̄ _{2,3} , σ _{3,2} , σ̄ _{3,1} , σ _{2,3} , σ _{1,2} , σ̄ _{3,2} , σ̄ _{1,2}]	VPB [3, σ̄ _{3,2} , σ̄ _{1,2} , σ _{2,1} , σ̄ _{2,3} , σ ₃]
VPB [3, σ̄ _{2,1} , σ̄ _{3,1} , σ _{3,2} , σ̄ _{1,2} , σ _{1,3} , σ̄ _{2,3} , σ _{1,2} , σ̄ _{1,2} , σ _{1,3} , σ _{1,2}]	VPB [3, σ̄ _{2,1} , σ̄ _{3,1} , σ _{3,2} , σ̄ ₁]
VPB [3, σ _{3,2} , σ̄ _{3,2} , σ _{2,1} , σ̄ _{3,1} , σ _{2,3} , σ _{1,2} , σ̄ _{1,2} , σ _{2,3} , σ̄ _{1,2} , σ̄ _{3,1}]	VPB [3, σ _{2,1} , σ̄ _{3,1} , σ ₂]
VPB [3, σ̄ _{2,3} , σ̄ _{2,3} , σ̄ _{1,2} , σ _{3,1} , σ _{2,1} , σ̄ _{2,3} , σ _{3,2} , σ̄ _{1,3} , σ _{3,1} , σ̄ _{2,3}]	VPB [3, σ̄ _{2,3} , σ̄ _{2,3} , σ̄ _{1,2} , σ _{3,1} , σ ₂]
VPB [3, σ _{2,1} , σ̄ _{3,2} , σ̄ _{1,2} , σ̄ _{2,3} , σ̄ _{1,3} , σ̄ _{1,3} , σ _{2,1} , σ̄ _{2,1} , σ̄ _{1,2} , σ̄ _{3,1}]	VPB [3, σ _{2,1} , σ̄ _{3,2} , σ̄ _{1,2} , σ̄ ₂]
VPB [3, σ _{3,2} , σ _{3,2} , σ̄ _{3,1} , σ _{2,1} , σ̄ _{3,2} , σ _{1,2} , σ _{1,3} , σ̄ _{2,3} , σ̄ _{1,3} , σ _{3,2}]	VPB [3, σ _{3,2} , σ _{3,2} , σ̄ _{3,1} , σ _{2,1} , σ̄ _{3,2} , σ ₁]
VPB [3, σ _{3,2} , σ̄ _{2,1} , σ _{3,1} , σ̄ _{1,2} , σ̄ _{3,2} , σ _{1,2} , σ̄ _{2,1} , σ̄ _{1,2} , σ̄ _{1,3} , σ _{1,3}]	VPB [3, σ _{3,2} , σ̄ _{2,1} , σ _{3,1} , σ̄ ₁]
VPB [3, σ̄ _{2,3} , σ _{2,3} , σ̄ _{2,3} , σ _{3,1} , σ̄ _{2,3} , σ̄ _{2,1} , σ _{1,3} , σ̄ _{1,2} , σ̄ _{3,2} , σ̄ _{3,2}]	VPB [3, σ _{2,1} , σ _{3,1} , σ _{1,3} , σ̄ _{2,1} , σ̄ ₂]
VPB [3, σ̄ _{1,2} , σ _{3,1} , σ̄ _{3,2} , σ̄ _{2,3} , σ̄ _{3,1} , σ̄ _{1,2} , σ _{3,1} , σ _{3,2} , σ _{3,2} , σ̄ _{1,3}]	VPB [3, σ̄ _{1,2} , σ _{3,1} , σ̄ _{3,2} , σ̄ _{2,3} , σ̄ ₃]
VPB [3, σ _{1,2} , σ _{1,2} , σ _{2,1} , σ _{2,3} , σ _{1,2} , σ _{1,3} , σ _{3,2} , σ _{3,1} , σ _{1,2} , σ̄ _{3,2}]	VPB [3, σ _{1,2} , σ _{1,2} , σ _{2,1} , σ _{2,3} , σ ₁]
VPB [3, σ _{2,3} , σ̄ _{3,2} , σ̄ _{1,3} , σ̄ _{1,3} , σ _{3,1} , σ̄ _{2,1} , σ̄ _{3,1} , σ _{1,2} , σ̄ _{2,1} , σ _{1,2}]	VPB [3, σ _{2,3} , σ̄ _{3,2} , σ̄ _{1,3} , σ̄ _{1,3} , σ _{3,1} , σ̄ ₂]
VPB [3, σ̄ _{1,2} , σ _{1,3} , σ _{1,2} , σ̄ _{2,1} , σ̄ _{3,1} , σ̄ _{2,1} , σ̄ _{1,2} , σ̄ _{1,3} , σ _{3,2} , σ _{1,2}]	VPB [3, σ̄ _{1,2} , σ _{1,3} , σ _{1,2} , σ̄ _{2,1} , σ ₃]

```
In[*]:= Table[{vpb = Module[{n = 3, m = 12, i},
  VPB[n, Sequence @@
    Table[RandomChoice[{σ, σ̄}]i=RandomChoice@Range@n, RandomChoice[Complement[Range@n, {i}]], {m}]]],
  cf = CF[vpb], Length[cf] - 1}, {20}] // MatrixForm
```

Out[*]= \$Aborted

```
Table[{vpb = Module[{n = 3, m = 16, i},
  VPB[n, Sequence @@
    Table[RandomChoice[{σ, σ̄}]i=RandomChoice@Range@n, RandomChoice[Complement[Range@n, {i}]], {m}]]],
  cf = CF[vpb], Length[cf] - 1}, {20}] // MatrixForm
```

```

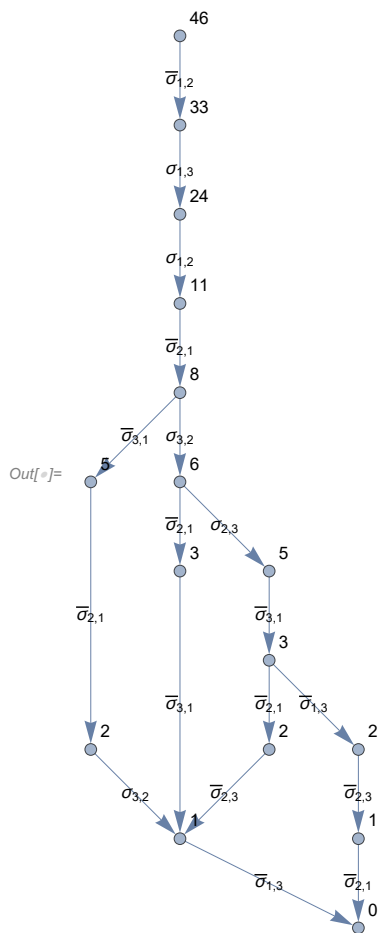
In[ ]:= ExtractionGraph[obj_] := Module[{vd, n, gs, vs, es, p, m1, m2, g, q, k},
  gs = VPBGenerators[n = Count[vd =  $\bar{\Gamma}$ [obj], _EOS]];
  vs = {vd}; es = {}; p = 0;
  While[p < Length[vs],
    m1 = Count[vd = vs[[+p]], X[_ , _]];
    Do[
      m2 = Count[q =  $\bar{\Gamma}$ [VD[VPB[n, g /. { $\sigma \rightarrow \bar{\sigma}, \bar{\sigma} \rightarrow \sigma$ ]}] ** vd], X[_ , _]];
      If[m2 < m1,
        If[! MemberQ[vs, q], AppendTo[vs, q]];
        k = Position[vs, q][[1, 1]];
        AppendTo[es, Labeled[p  $\leftrightarrow$  k, g]]
      ],
      {g, gs}
    ]
  ];
  Graph[Table[Labeled[k, Length[vs[[k]] - n], {k, p}], es]
]

```

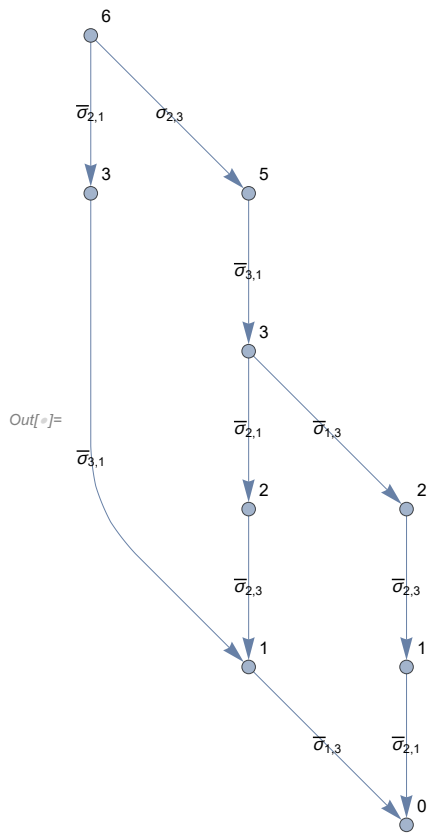
```

In[ ]:= ExtractionGraph[VPB[3,  $\bar{\sigma}_{1,2}, \sigma_{1,3}, \sigma_{1,2}, \bar{\sigma}_{2,1}, \bar{\sigma}_{3,1}, \bar{\sigma}_{2,1}, \bar{\sigma}_{1,2}, \bar{\sigma}_{1,3}, \sigma_{3,2}, \sigma_{1,2}$ ]]

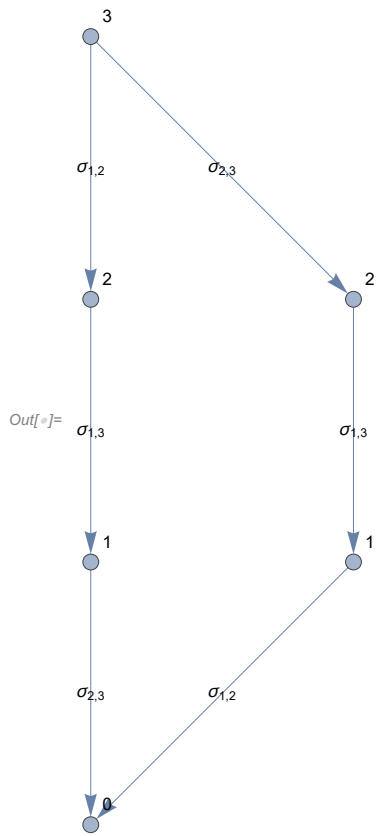
```



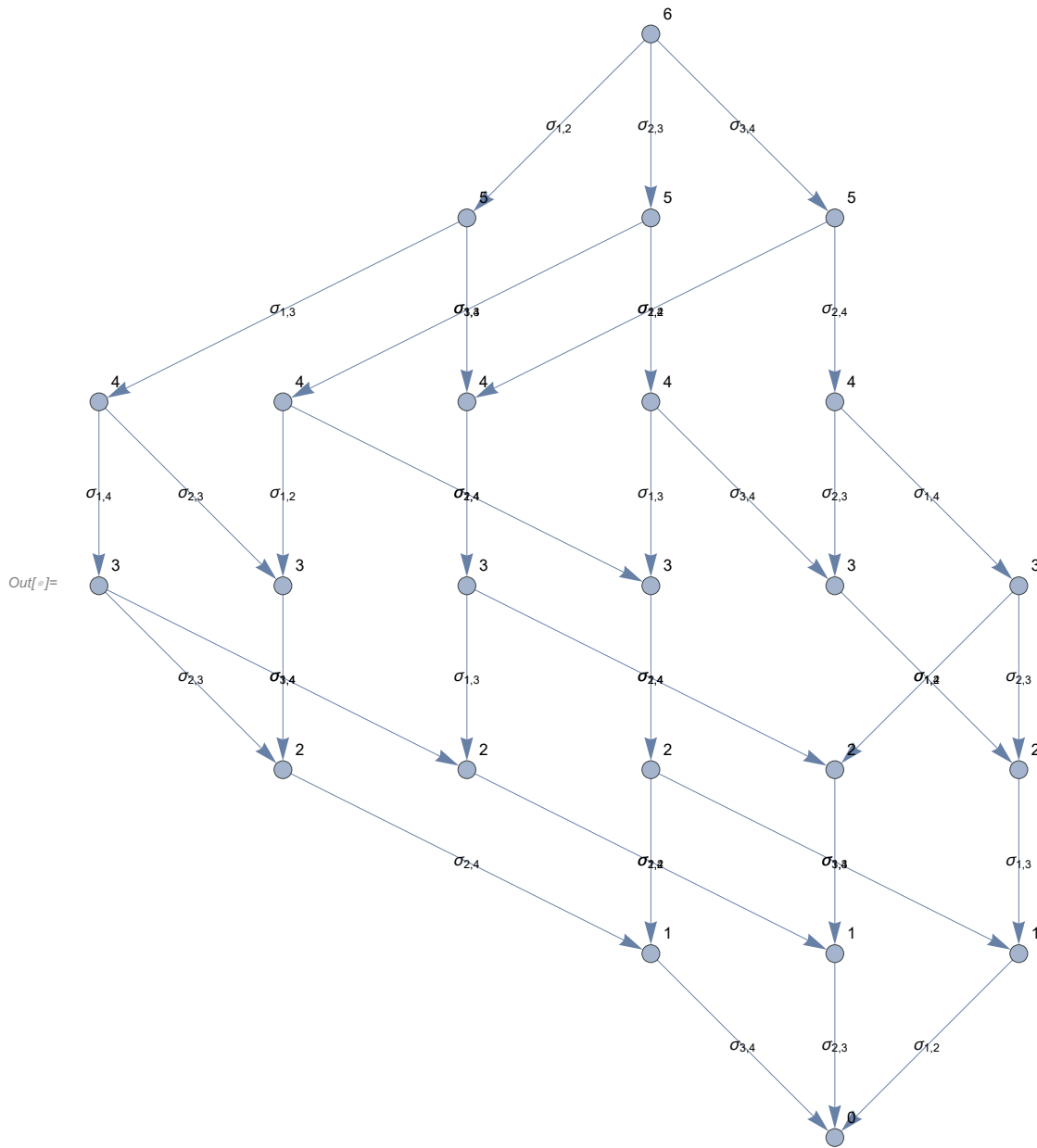
In[]:= **ExtractionGraph**[VPB[3, $\sigma_{2,3}$, $\bar{\sigma}_{3,1}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{2,3}$, $\bar{\sigma}_{1,3}$]]



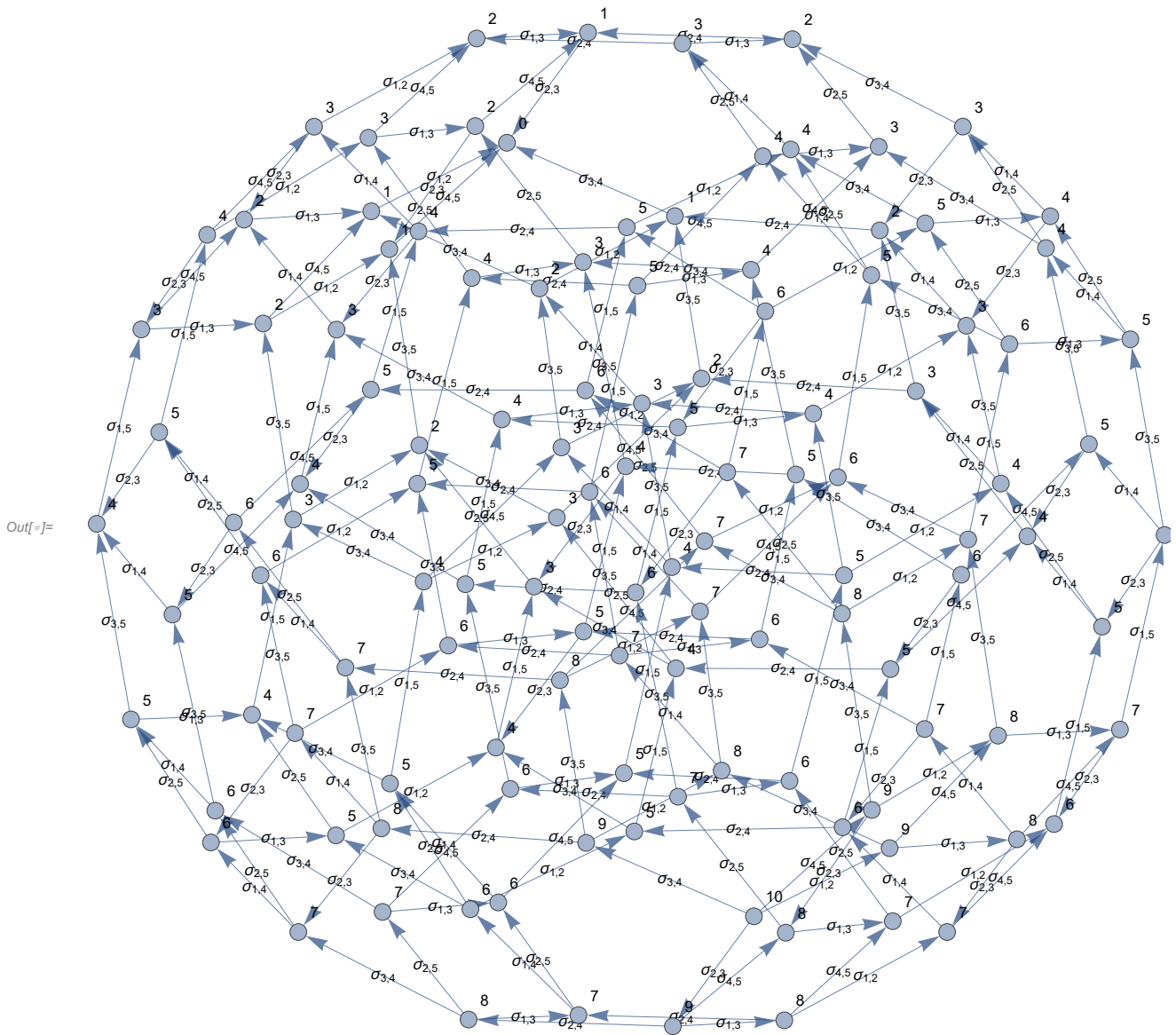
In[]:= **ExtractionGraph**[VPB[3, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{2,3}$]]



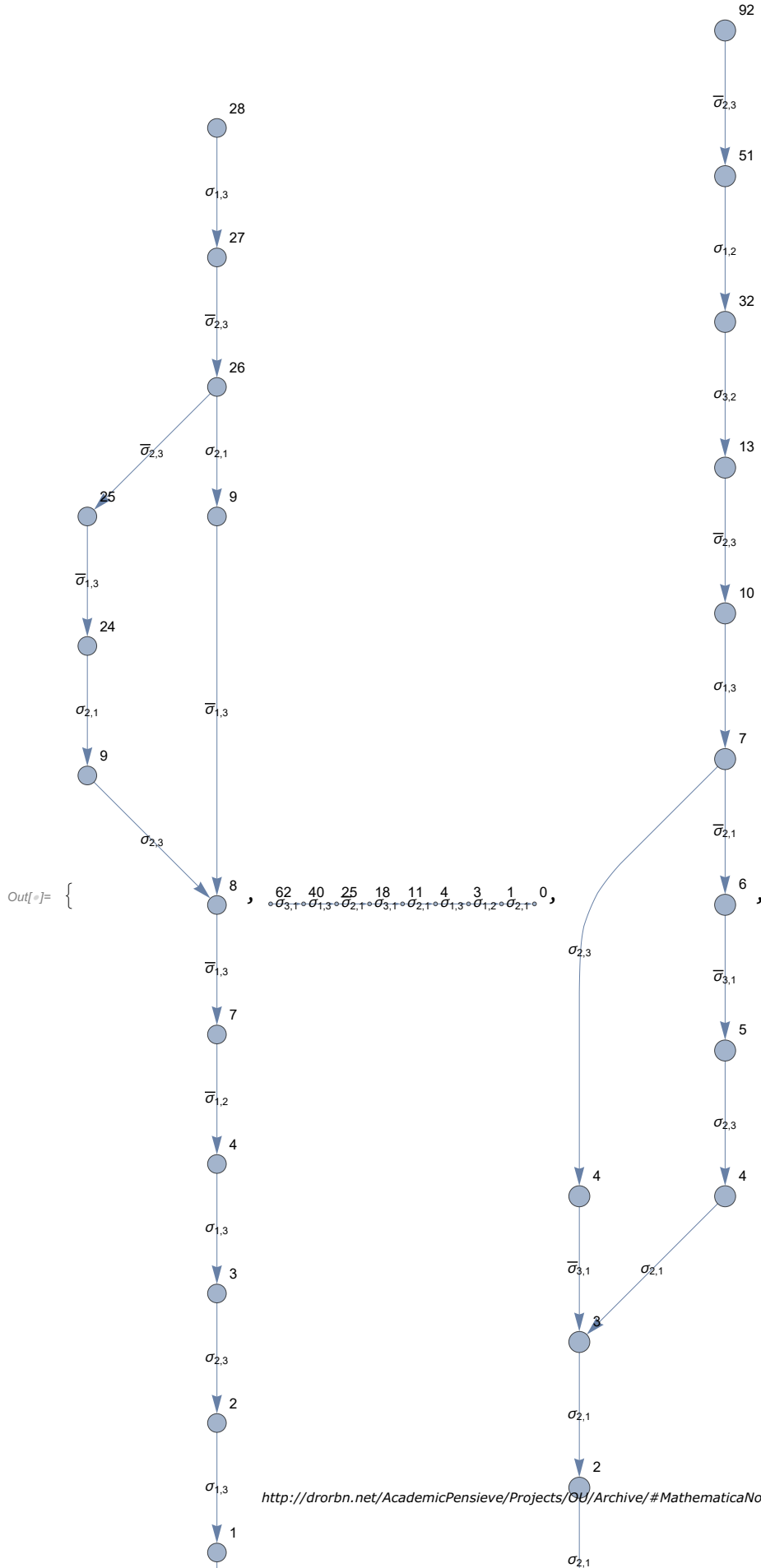
In[]:= **ExtractionGraph**[VPB[4, $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{1,4}$, $\sigma_{2,3}$, $\sigma_{2,4}$, $\sigma_{3,4}$]]

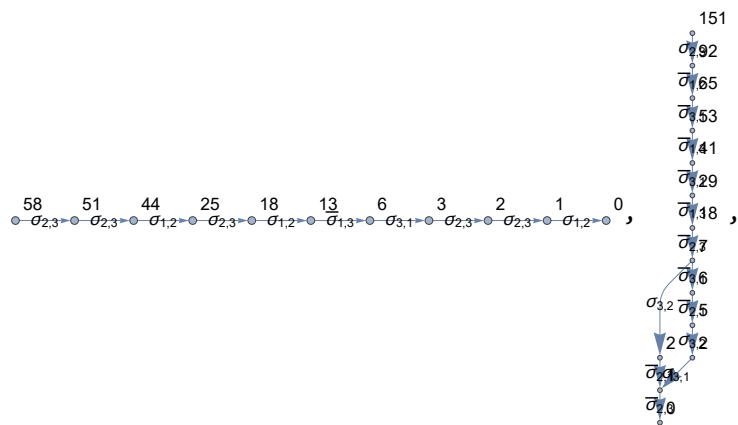


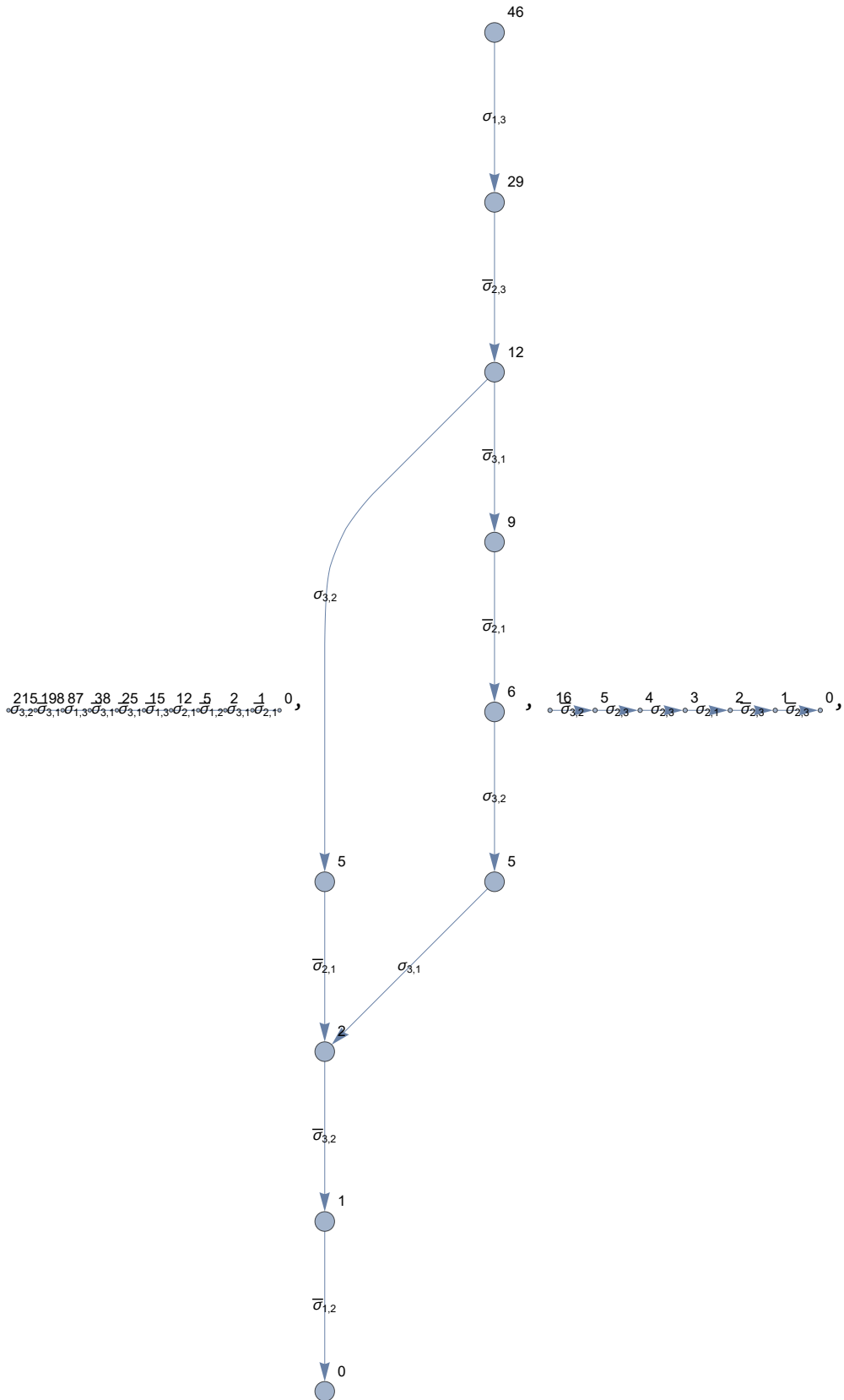
```
In[ ]:= ExtractionGraph[VPB[5,  $\sigma_{1,2}$ ,  $\sigma_{1,3}$ ,  $\sigma_{1,4}$ ,  $\sigma_{1,5}$ ,  $\sigma_{2,3}$ ,  $\sigma_{2,4}$ ,  $\sigma_{2,5}$ ,  $\sigma_{3,4}$ ,  $\sigma_{3,5}$ ,  $\sigma_{4,5}$ ]]
```

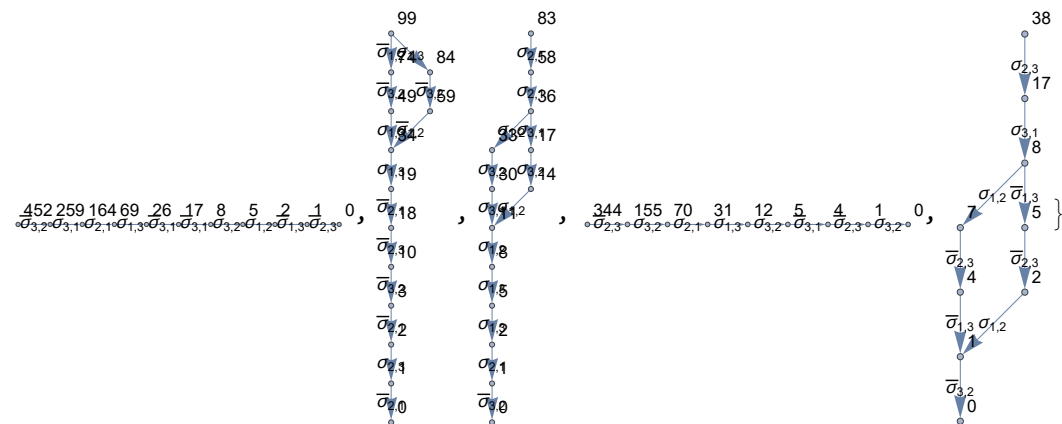
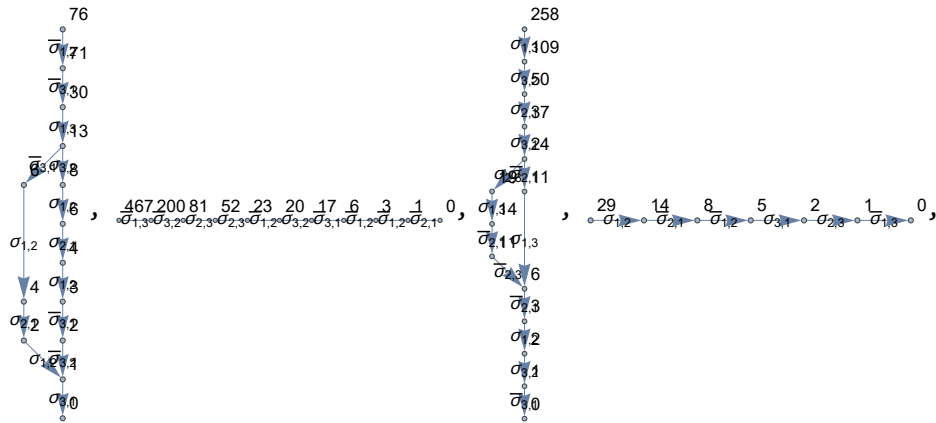


```
In[ ]:= Table[ExtractionGraph@Module[{n = 3, m = 10, i},
  VPB[n, Sequence@@Table[
    RandomChoice[{ $\sigma$ ,  $\bar{\sigma}$ }]i=RandomChoice@Range@n, RandomChoice[Complement[Range@n, {i}]]], {m}]]], {20}]
```

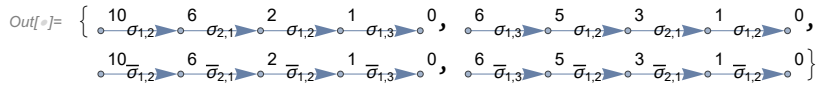




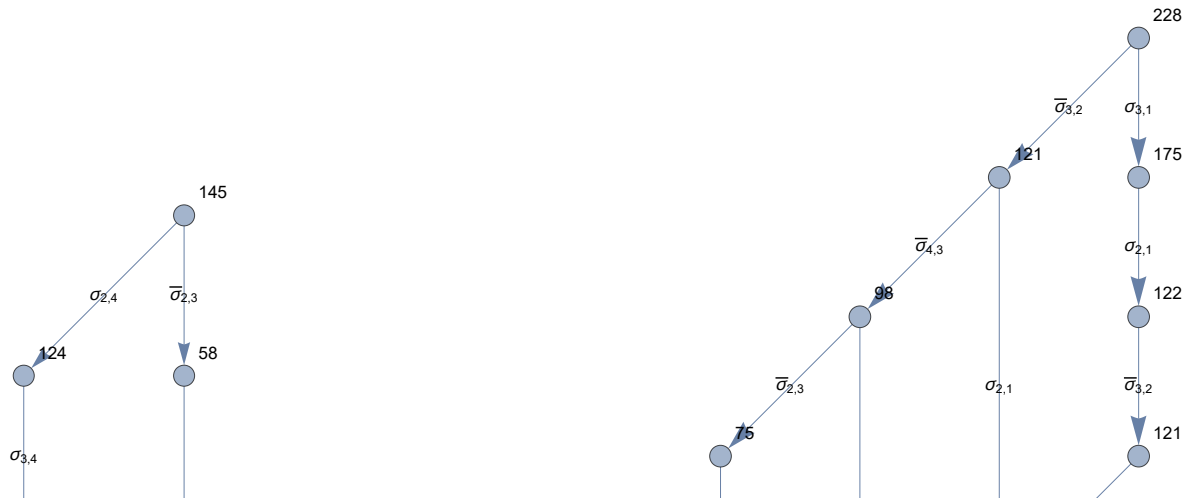


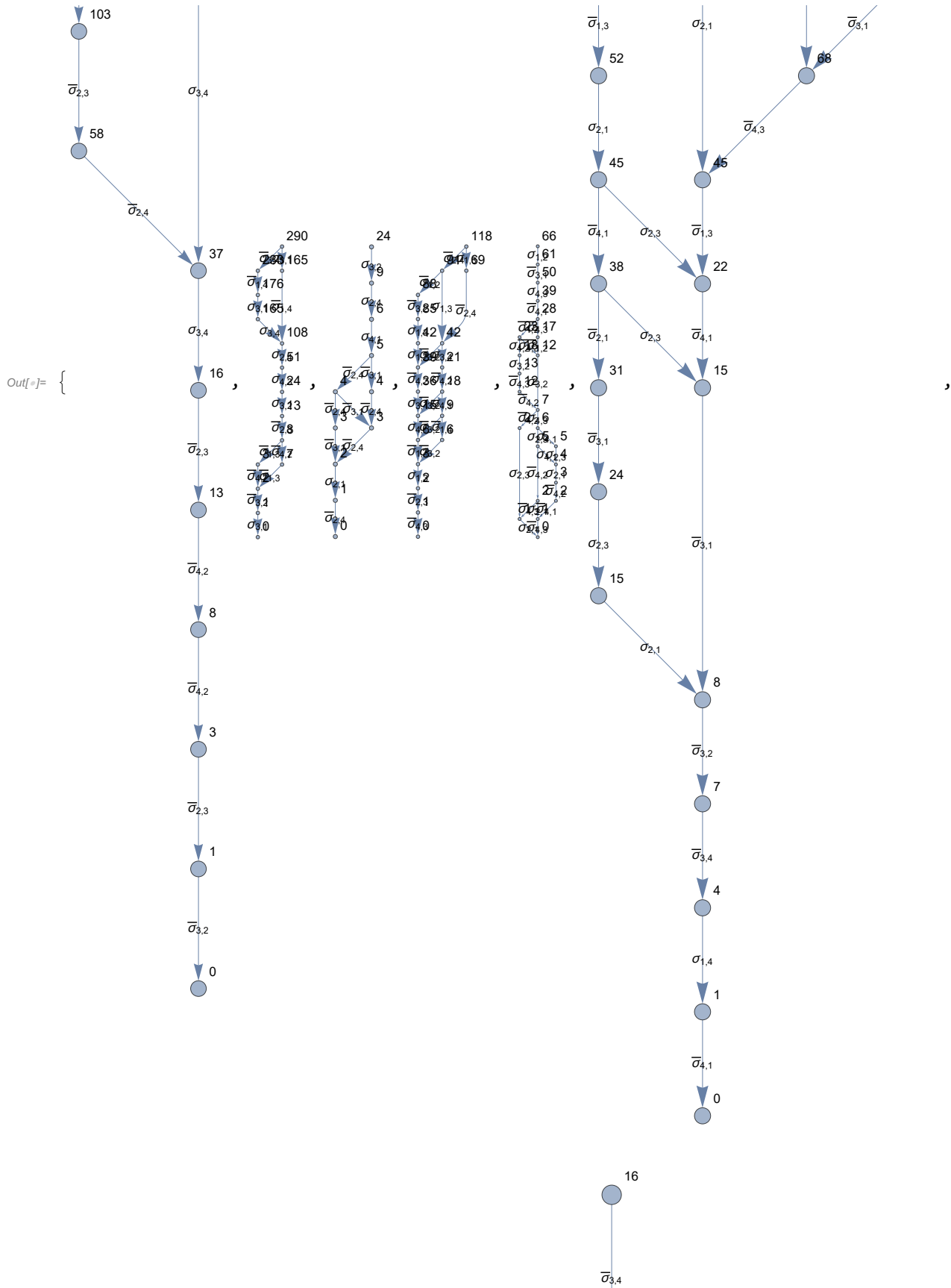


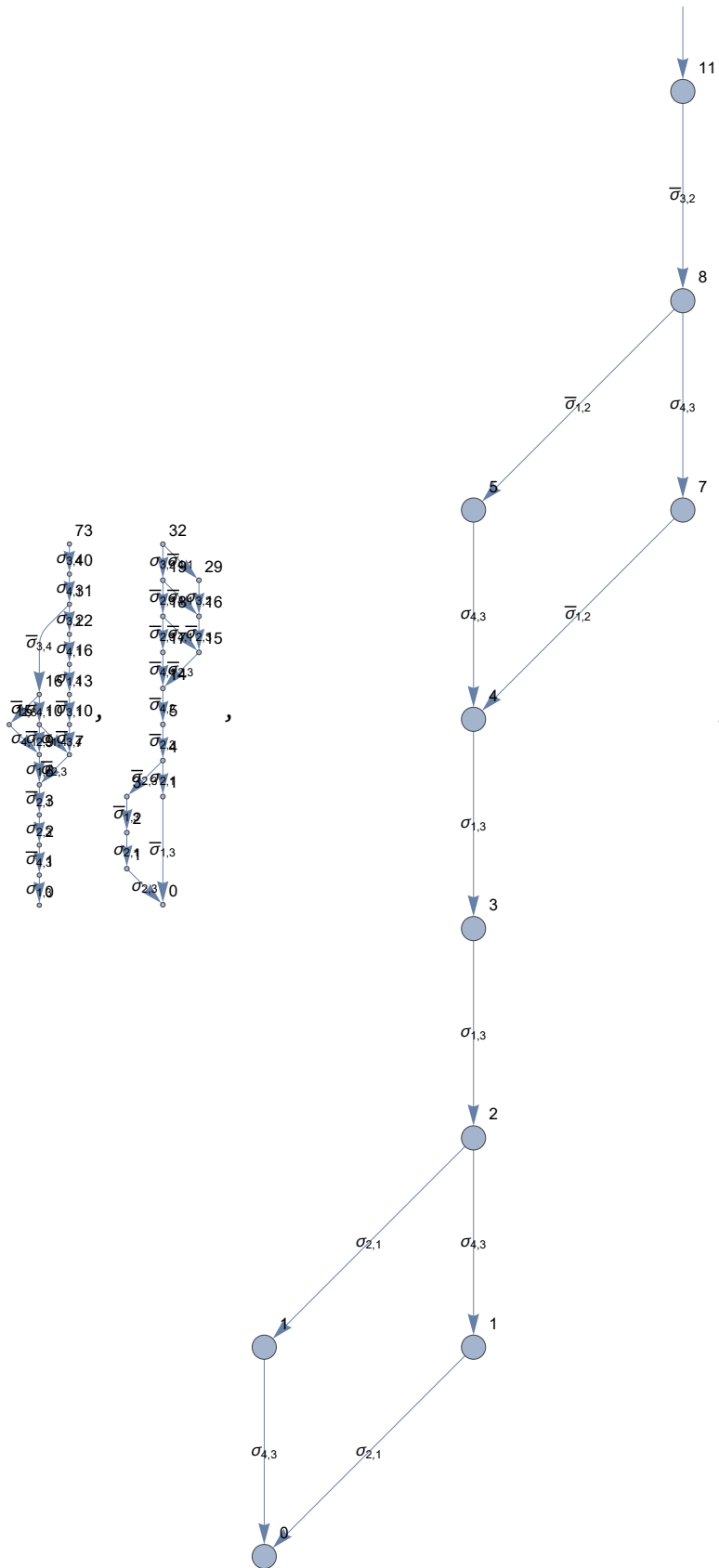
```
In[ ]:= ExtractionGraph /@ {chb1, chb2, chb3, chb4}
```



```
In[ ]:= Table [ExtractionGraph@Module [ {n = 4, m = 10, i},
VPB [n, Sequence@@ Table [
RandomChoice [ {sigma, sigma_bar} ]_{i=RandomChoice@Range@n, RandomChoice [Complement [Range@n, {i}]]}, {m}]]], {20}]
```









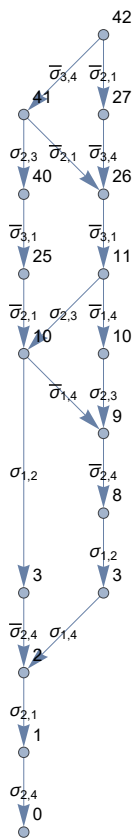
```

In[ ]:= While [True,
  vpb = Module [ { n = RandomChoice [ { 3, 4, 5, 6 } ], m = RandomChoice [ { 8, 10, 12, 16 } ], i },
    VPB [ n, Sequence @@
      Table [ RandomChoice [ { sigma, sigma_bar } ]_{i=RandomChoice@Range@n, RandomChoice [ Complement [ Range@n, { i } ] ] }, { m } ] ];
  Print [ vpb ];
  Print [ ExtractionGraph [ vpb ] ]
]

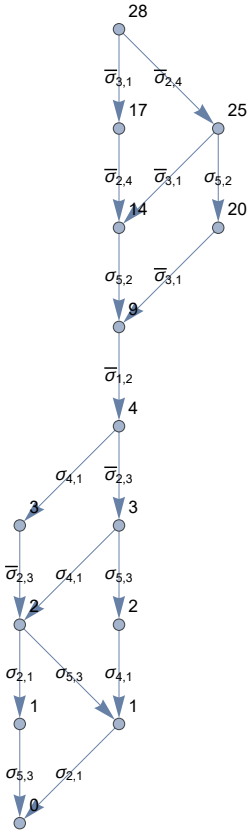
VPB [ 3, sigma_{3,2}, sigma_bar_{3,2}, sigma_{1,3}, sigma_bar_{1,3}, sigma_{3,1}, sigma_bar_{2,1}, sigma_bar_{2,1}, sigma_bar_{1,3}, sigma_{3,1}, sigma_{3,1}, sigma_bar_{3,1}, sigma_bar_{2,1} ]
sigma_bar_{3,1} -> sigma_bar_{2,1} -> sigma_bar_{2,1} -> sigma_{1,3} -> sigma_{3,1} -> sigma_bar_{3,1} -> sigma_bar_{3,1} -> sigma_bar_{2,1} -> 0

VPB [ 4, sigma_bar_{3,4}, sigma_{2,3}, sigma_bar_{3,1}, sigma_bar_{2,1}, sigma_{1,2}, sigma_bar_{2,4}, sigma_{2,1}, sigma_{2,4} ]

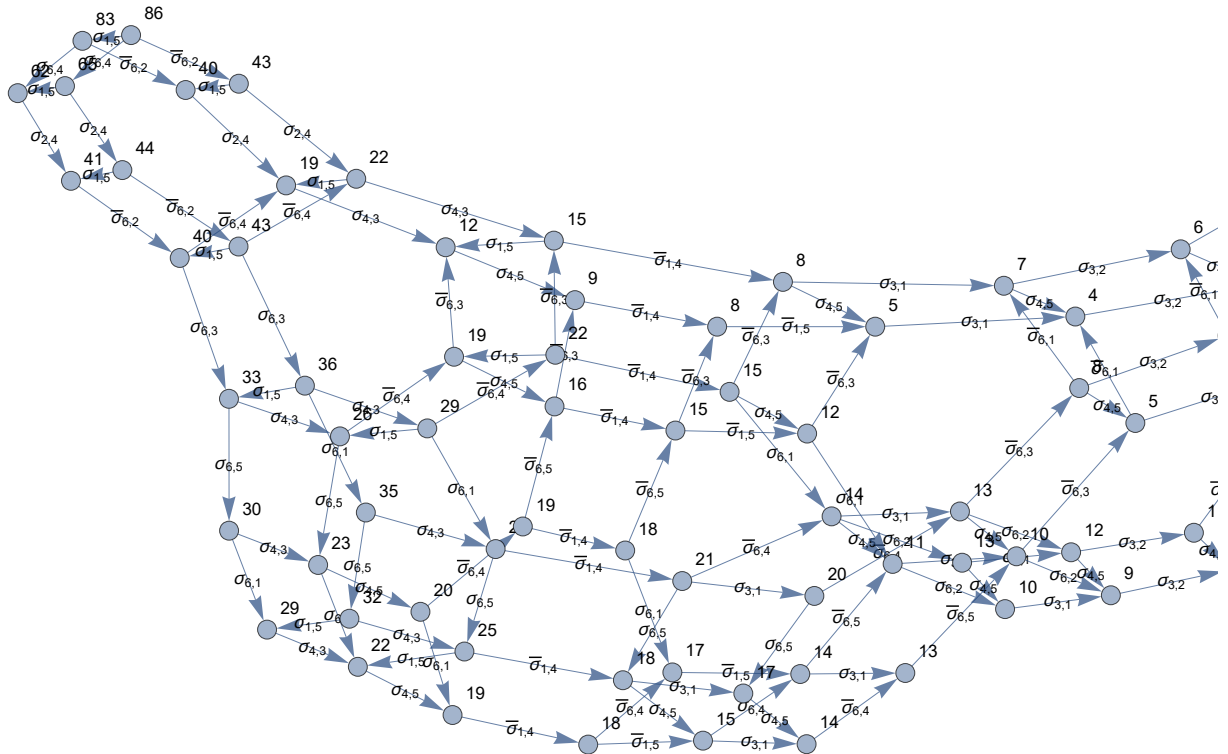
```



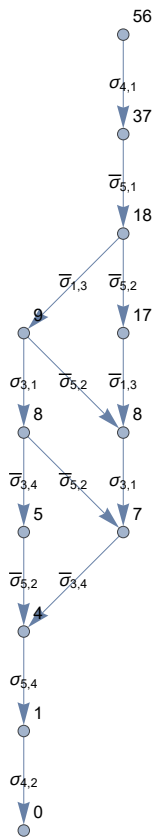
VPB [5, $\bar{\sigma}_{3,1}$, $\bar{\sigma}_{2,4}$, $\sigma_{5,2}$, $\bar{\sigma}_{1,2}$, $\sigma_{4,1}$, $\bar{\sigma}_{2,3}$, $\sigma_{2,1}$, $\sigma_{5,3}$]



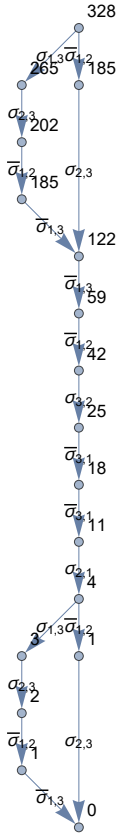
VPB [6, sigma-bar_{6,2}, sigma_{2,4}, sigma_{4,3}, sigma-bar_{1,4}, sigma_{3,1}, sigma_{4,5}, sigma_{3,2}, sigma-bar_{5,2}, sigma-bar_{3,2}, sigma-bar_{6,2}]



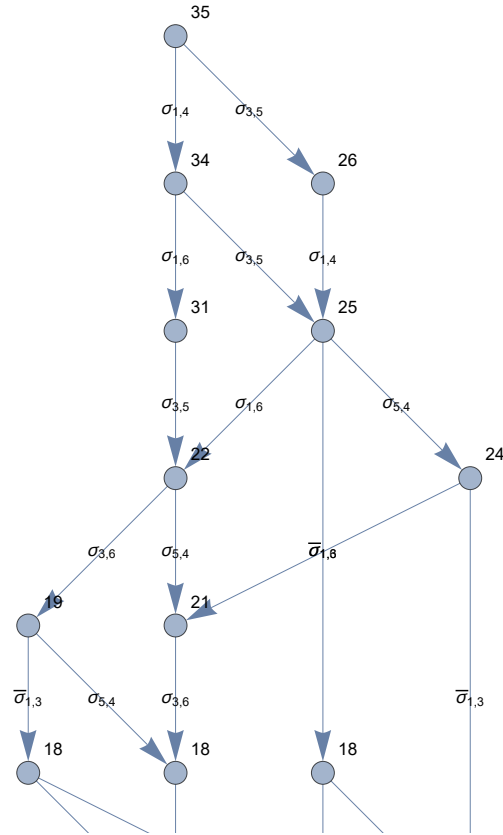
VPB [5, $\sigma_{4,1}$, $\bar{\sigma}_{5,1}$, $\bar{\sigma}_{3,4}$, $\sigma_{3,4}$, $\bar{\sigma}_{1,3}$, $\bar{\sigma}_{5,2}$, $\sigma_{3,1}$, $\bar{\sigma}_{3,4}$, $\sigma_{5,4}$, $\sigma_{4,2}$]

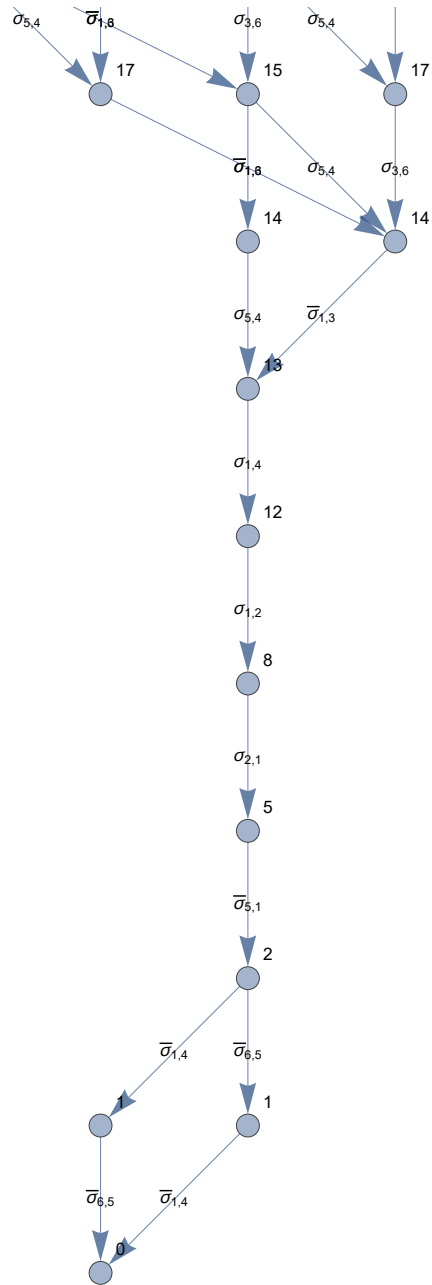


VPB [3, $\bar{\sigma}_{1,2}$, $\sigma_{2,3}$, $\bar{\sigma}_{1,3}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{2,1}$, $\sigma_{2,1}$, $\sigma_{1,2}$, $\sigma_{3,2}$, $\bar{\sigma}_{3,1}$, $\bar{\sigma}_{3,1}$, $\sigma_{2,1}$, $\bar{\sigma}_{1,2}$, $\sigma_{2,3}$, $\bar{\sigma}_{1,2}$, $\sigma_{1,2}$]

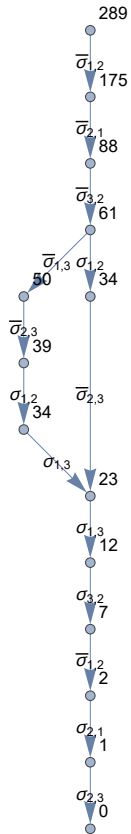


VPB [6, $\sigma_{3,5}$, $\sigma_{1,4}$, $\sigma_{5,4}$, $\bar{\sigma}_{1,3}$, $\sigma_{3,6}$, $\bar{\sigma}_{1,3}$, $\sigma_{1,4}$, $\sigma_{1,2}$, $\sigma_{2,1}$, $\bar{\sigma}_{5,1}$, $\bar{\sigma}_{6,5}$, $\bar{\sigma}_{1,4}$]

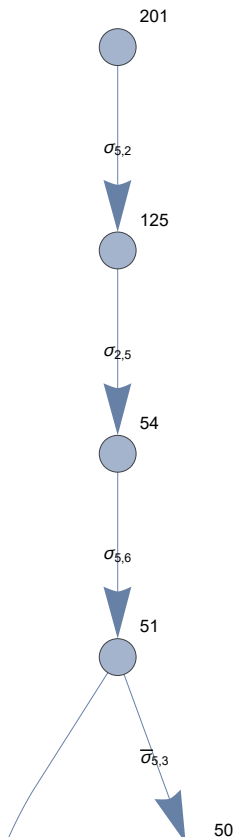


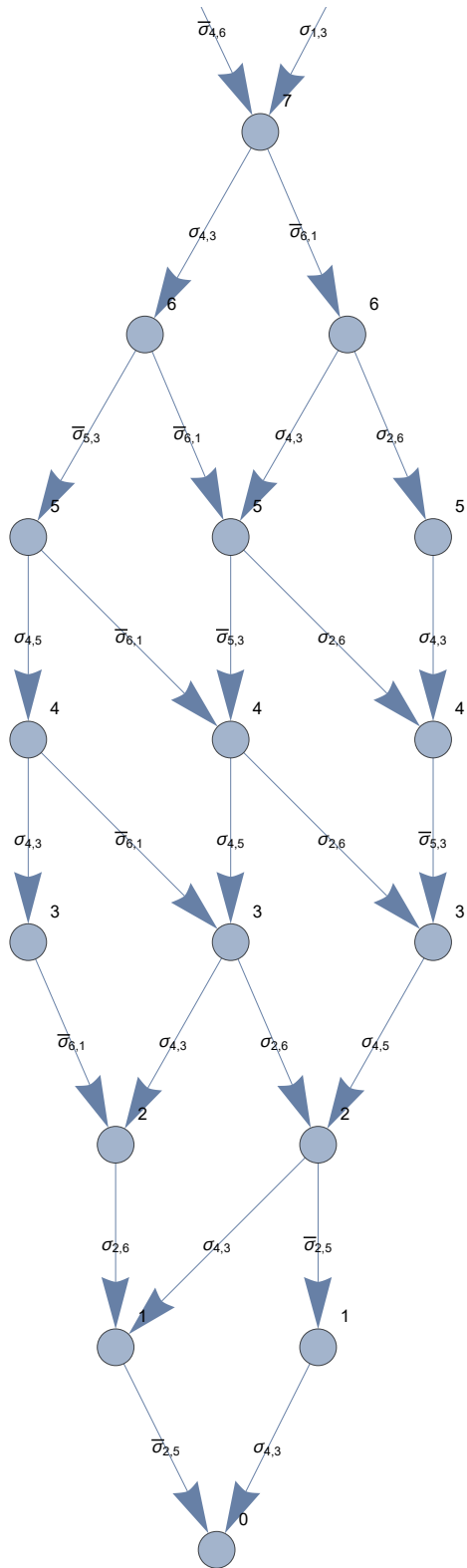


VPB[3, $\bar{\sigma}_{1,3}$, $\sigma_{1,3}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{3,2}$, $\sigma_{1,2}$, $\bar{\sigma}_{2,3}$, $\sigma_{1,3}$, $\bar{\sigma}_{3,1}$, $\sigma_{3,1}$, $\sigma_{3,2}$, $\bar{\sigma}_{3,1}$, $\sigma_{3,1}$, $\bar{\sigma}_{1,2}$, $\sigma_{2,1}$, $\sigma_{2,3}$]

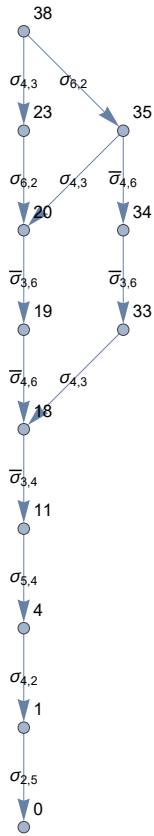


VPB [6, $\sigma_{5,2}$, $\sigma_{2,5}$, $\sigma_{5,6}$, $\sigma_{5,4}$, $\bar{\sigma}_{4,3}$, $\sigma_{1,3}$, $\bar{\sigma}_{4,6}$, $\sigma_{4,2}$, $\bar{\sigma}_{4,6}$, $\sigma_{4,3}$, $\bar{\sigma}_{6,1}$, $\bar{\sigma}_{5,3}$, $\sigma_{2,6}$, $\sigma_{4,5}$, $\sigma_{4,3}$, $\bar{\sigma}_{2,5}$]

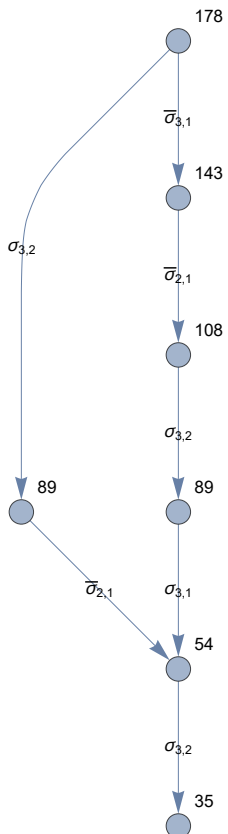


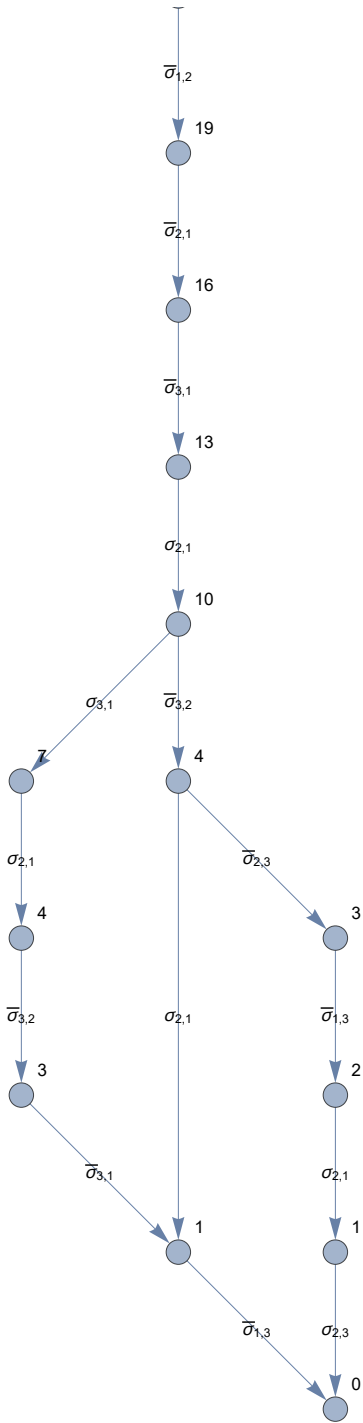


VPB [6, $\sigma_{6,2}$, $\bar{\sigma}_{4,6}$, $\bar{\sigma}_{3,6}$, $\sigma_{4,3}$, $\bar{\sigma}_{3,4}$, $\sigma_{5,4}$, $\sigma_{4,2}$, $\sigma_{2,5}$]

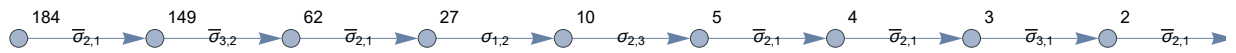


VPB [3, $\sigma_{3,2}$, $\bar{\sigma}_{2,1}$, $\sigma_{3,2}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{3,1}$, $\sigma_{2,1}$, $\bar{\sigma}_{3,1}$, $\sigma_{3,1}$, $\bar{\sigma}_{3,2}$, $\sigma_{2,1}$, $\bar{\sigma}_{1,3}$]

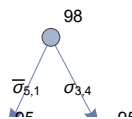


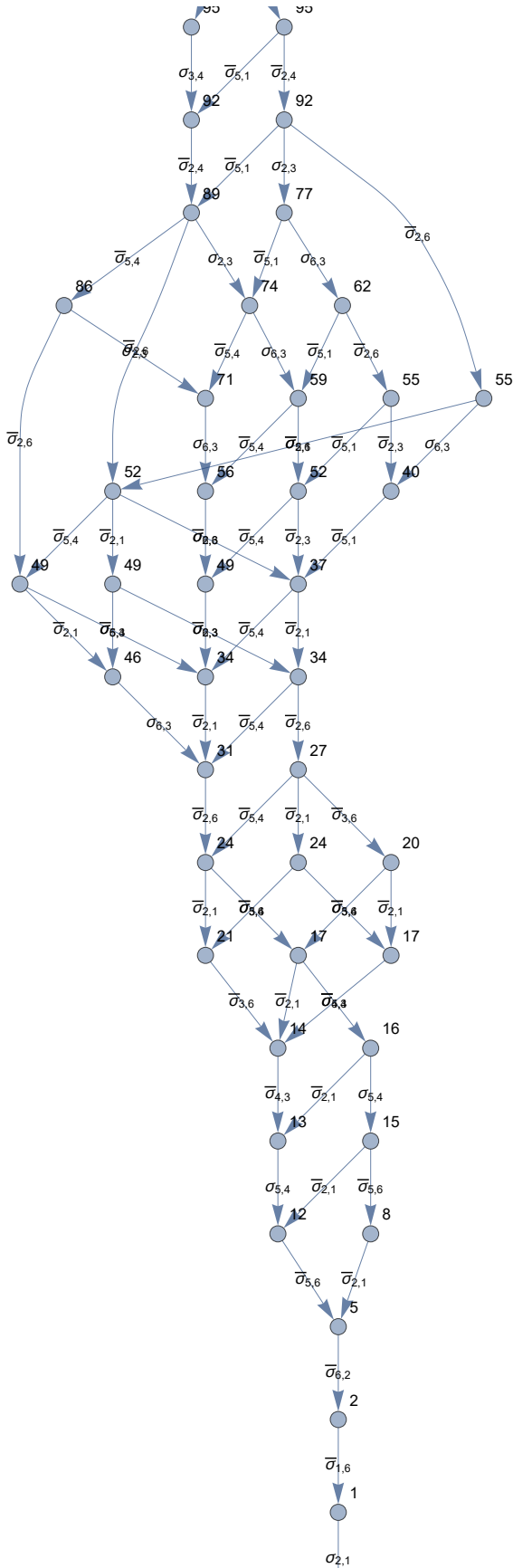


VPB [3, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{3,2}$, $\bar{\sigma}_{2,1}$, $\sigma_{1,2}$, $\sigma_{2,3}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{3,1}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{3,2}$]



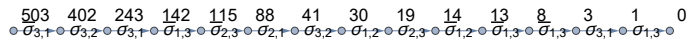
VPB [6, $\bar{\sigma}_{5,1}$, $\sigma_{3,4}$, $\bar{\sigma}_{2,4}$, $\bar{\sigma}_{2,6}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{5,4}$, $\sigma_{6,3}$, $\bar{\sigma}_{2,6}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{3,6}$, $\bar{\sigma}_{4,3}$, $\sigma_{5,4}$, $\bar{\sigma}_{5,6}$, $\bar{\sigma}_{6,2}$, $\bar{\sigma}_{1,6}$, $\sigma_{2,1}$]







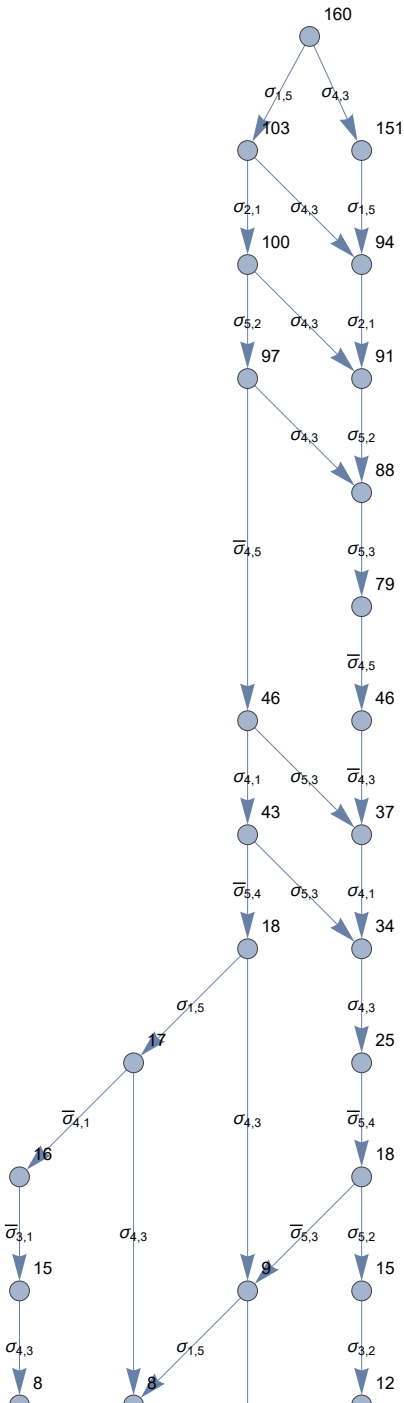
VPB [3, $\bar{\sigma}_{3,1}$, $\sigma_{3,2}$, $\sigma_{3,1}$, $\bar{\sigma}_{1,3}$, $\bar{\sigma}_{2,3}$, $\sigma_{2,1}$, $\sigma_{3,2}$, $\sigma_{1,2}$, $\sigma_{2,3}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{2,3}$, $\sigma_{2,3}$, $\bar{\sigma}_{1,3}$, $\bar{\sigma}_{1,3}$, $\sigma_{3,1}$, $\sigma_{1,3}$]

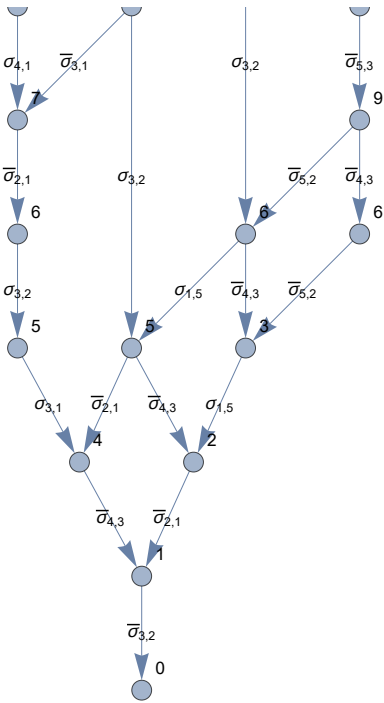


VPB [4, $\bar{\sigma}_{1,4}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{2,1}$, $\sigma_{1,2}$, $\sigma_{1,4}$, $\sigma_{3,4}$, $\sigma_{2,3}$]

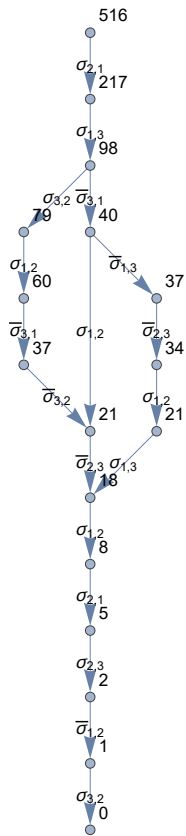


VPB [5, $\sigma_{1,5}$, $\sigma_{2,1}$, $\sigma_{5,2}$, $\bar{\sigma}_{4,5}$, $\sigma_{4,1}$, $\bar{\sigma}_{5,4}$, $\sigma_{1,5}$, $\bar{\sigma}_{1,4}$, $\sigma_{1,4}$, $\sigma_{4,3}$, $\sigma_{3,2}$, $\bar{\sigma}_{2,1}$, $\bar{\sigma}_{4,3}$, $\bar{\sigma}_{3,4}$, $\sigma_{3,4}$, $\bar{\sigma}_{3,2}$]





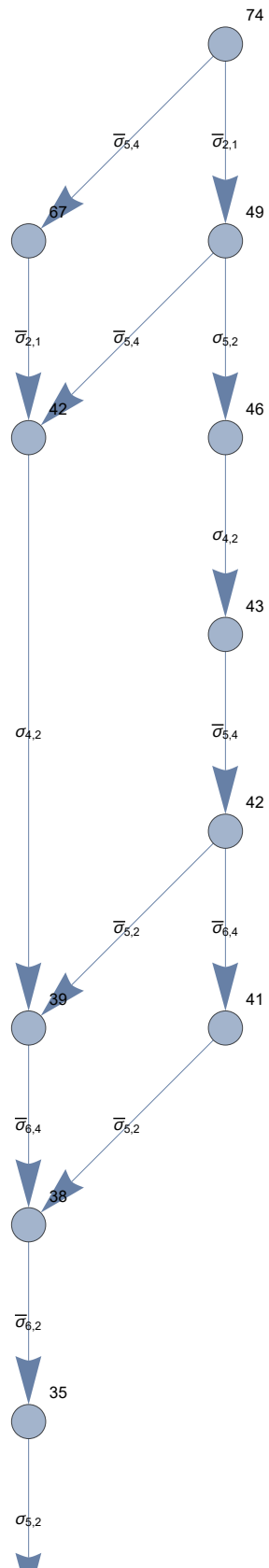
VPB[3, sigma_{2,1}, sigma_{1,3}, sigma-bar_{3,1}, sigma_{1,2}, sigma_{1,3}, sigma_{1,2}, sigma_{2,1}, sigma-bar_{1,3}, sigma-bar_{1,2}, sigma_{3,2}]

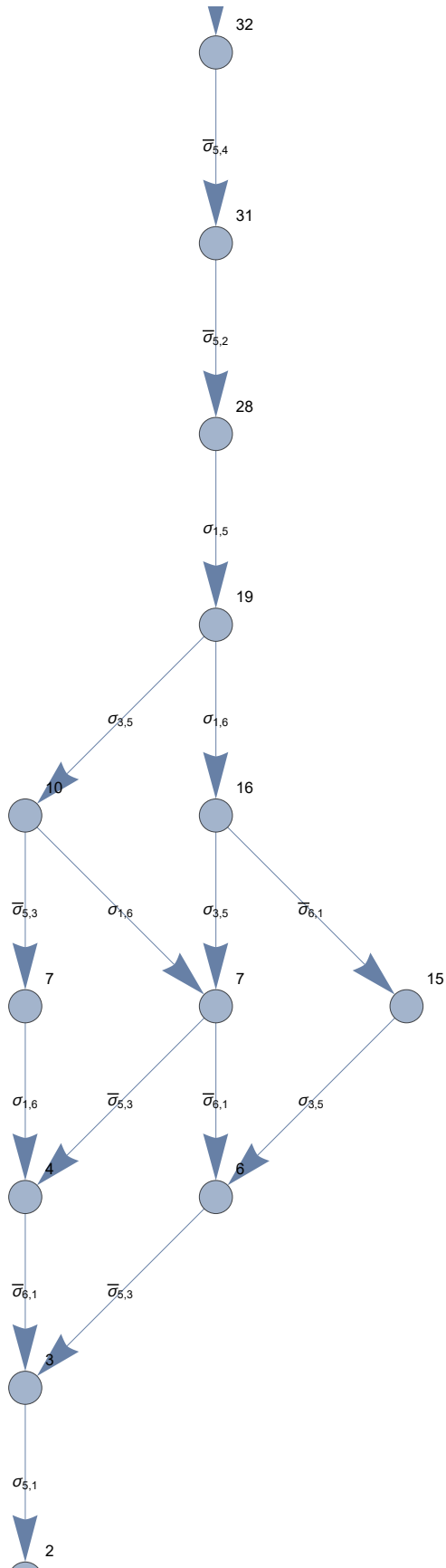


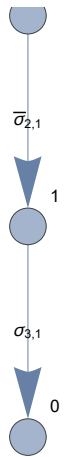
VPB[3, sigma-bar_{1,2}, sigma-bar_{3,1}, sigma-bar_{1,2}, sigma-bar_{1,3}, sigma_{2,1}, sigma-bar_{2,1}, sigma_{1,2}, sigma_{3,2}]



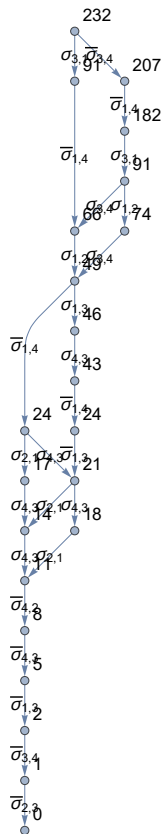
VPB [6, $\bar{\sigma}_{5,4}$, $\bar{\sigma}_{2,1}$, $\sigma_{4,2}$, $\bar{\sigma}_{6,4}$, $\bar{\sigma}_{6,2}$, $\sigma_{5,2}$, $\bar{\sigma}_{5,4}$, $\bar{\sigma}_{5,2}$, $\sigma_{1,5}$, $\sigma_{3,5}$, $\bar{\sigma}_{5,3}$, $\sigma_{1,6}$, $\bar{\sigma}_{6,1}$, $\sigma_{5,1}$, $\bar{\sigma}_{2,1}$, $\sigma_{3,1}$]







VPB [4, $\sigma_{3,1}$, $\bar{\sigma}_{1,4}$, $\sigma_{1,2}$, $\bar{\sigma}_{1,4}$, $\sigma_{4,3}$, $\sigma_{2,1}$, $\sigma_{4,3}$, $\bar{\sigma}_{4,2}$, $\bar{\sigma}_{4,3}$, $\bar{\sigma}_{1,3}$, $\bar{\sigma}_{3,4}$, $\bar{\sigma}_{2,3}$]



VPB [4, $\sigma_{2,4}$, $\bar{\sigma}_{3,2}$, $\bar{\sigma}_{4,3}$, $\bar{\sigma}_{2,4}$, $\sigma_{3,1}$, $\bar{\sigma}_{1,3}$, $\sigma_{1,2}$, $\bar{\sigma}_{4,3}$, $\bar{\sigma}_{4,2}$, $\bar{\sigma}_{3,2}$, $\sigma_{2,3}$, $\bar{\sigma}_{1,2}$, $\bar{\sigma}_{3,1}$, $\sigma_{1,4}$, $\bar{\sigma}_{4,2}$, $\sigma_{1,2}$]

Out[*]= \$Aborted

In[]:= **ExtractionGraph**[VPB[3, $\bar{\sigma}_{1,3}$, $\bar{\sigma}_{2,3}$, $\sigma_{1,2}$]]

