Pensieve header: Counting virtual pure braids: Cartesian product diagrams immediately serialized.

An ERO is an Equivalence Relation Object, as in EquivalenceRelations.nb.

(Alt) In[•]:=
```
Print@"Warning: risky $m1 and $m2 in EROAdjoin!"
```

```
Warning: risky $m1 and $m2 in EROAdjoin!
```

(Alt) In[•]:=
```
SetAttributes[{EROMake, EROPeek, EROAdjoin}, HoldFirst];
```

(Alt) In[•]:=
```
EROMake[er_, n_Integer] := er = Table[0, $EROLength = n];
```

(Alt) In[•]:=
```
EROPeek[er_, n_Integer] := (
   If[n > $EROLength,
    Echo@n;
    Echo@CountVPB$Locals;
    Echo@ReleaseHold@CountVPB$Locals;
    Abort[]
   ];
   If[er[[n]] == 0, n, er[[n]] = EROPeek[er, er[[n]]]]
  );
```

(Alt) In[•]:=
```
EROAdjoin[er_, n1_Integer ⟼ n2_Integer] := (
  $m1 = EROPeek[er, n1]; $m2 = EROPeek[er, n2];
  Switch[Order[$m1, $m2], 0, $m1, 1, er[[$m2]] = $m1, -1, er[[$m1]] = $m2])
```

(Alt) In[•]:=
```
VPB[n_, gs_List] := 1 + Sum[(2 n (n - 1))^s, {s, 0, Length[gs] - 1}] + FromDigits[gs //.
     {σ_i_,j_ :> (n - 1) (i - 1) + If[j < i, j - 1, j - 2], σ̄_i_,j_ :> n (n - 1) + σ_i,j}, 2 n (n - 1)];
```

(Alt) In[•]:=
```
VPB[n_, c_Integer] := Module[{c1, cc, r = 0, s, i, j, d},
  c1 = cc = c - 1;
  While[(c1 = cc - (2 n (n - 1))^r) ≥ 0, cc = c1; ++r];
  Table[
   {r, i, j} = 1 + IntegerDigits[d, MixedRadix[{2, n, n - 1}], 3];
   If[j ≥ i, ++j];
   If[r == 1, σ_i,j, σ̄_i,j],
   {d, IntegerDigits[cc, 2 n (n - 1), r]}
  ]]
```

(Alt) In[•]:=
```
VPB[3, #] & /@ Range[50]
```

(Alt) Out[•]=
$$\Big\{\{\},\ \{\sigma_{1,2}\},\ \{\sigma_{1,3}\},\ \{\sigma_{2,1}\},\ \{\sigma_{2,3}\},\ \{\sigma_{3,1}\},\ \{\sigma_{3,2}\},\ \{\overline{\sigma}_{1,2}\},\ \{\overline{\sigma}_{1,3}\},\ \{\overline{\sigma}_{2,1}\},\ \{\overline{\sigma}_{2,3}\},\ \{\overline{\sigma}_{3,1}\},$$
$$\{\overline{\sigma}_{3,2}\},\ \{\sigma_{1,2},\sigma_{1,2}\},\ \{\sigma_{1,2},\sigma_{1,3}\},\ \{\sigma_{1,2},\sigma_{2,1}\},\ \{\sigma_{1,2},\sigma_{2,3}\},\ \{\sigma_{1,2},\sigma_{3,1}\},\ \{\sigma_{1,2},\sigma_{3,2}\},$$
$$\{\sigma_{1,2},\overline{\sigma}_{1,2}\},\ \{\sigma_{1,2},\overline{\sigma}_{1,3}\},\ \{\sigma_{1,2},\overline{\sigma}_{2,1}\},\ \{\sigma_{1,2},\overline{\sigma}_{2,3}\},\ \{\sigma_{1,2},\overline{\sigma}_{3,1}\},\ \{\sigma_{1,2},\overline{\sigma}_{3,2}\},\ \{\sigma_{1,3},\sigma_{1,2}\},$$
$$\{\sigma_{1,3},\sigma_{1,3}\},\ \{\sigma_{1,3},\sigma_{2,1}\},\ \{\sigma_{1,3},\sigma_{2,3}\},\ \{\sigma_{1,3},\sigma_{3,1}\},\ \{\sigma_{1,3},\sigma_{3,2}\},\ \{\sigma_{1,3},\overline{\sigma}_{1,2}\},$$
$$\{\sigma_{1,3},\overline{\sigma}_{1,3}\},\ \{\sigma_{1,3},\overline{\sigma}_{2,1}\},\ \{\sigma_{1,3},\overline{\sigma}_{2,3}\},\ \{\sigma_{1,3},\overline{\sigma}_{3,1}\},\ \{\sigma_{1,3},\overline{\sigma}_{3,2}\},\ \{\sigma_{2,1},\sigma_{1,2}\},$$
$$\{\sigma_{2,1},\sigma_{1,3}\},\ \{\sigma_{2,1},\sigma_{2,1}\},\ \{\sigma_{2,1},\sigma_{2,3}\},\ \{\sigma_{2,1},\sigma_{3,1}\},\ \{\sigma_{2,1},\sigma_{3,2}\},\ \{\sigma_{2,1},\overline{\sigma}_{1,2}\},$$
$$\{\sigma_{2,1},\overline{\sigma}_{1,3}\},\ \{\sigma_{2,1},\overline{\sigma}_{2,1}\},\ \{\sigma_{2,1},\overline{\sigma}_{2,3}\},\ \{\sigma_{2,1},\overline{\sigma}_{3,1}\},\ \{\sigma_{2,1},\overline{\sigma}_{3,2}\},\ \{\sigma_{2,3},\sigma_{1,2}\}\Big\}$$

(Alt) In[•]:=
```
Range[50] === (VPB[3, #] & /@ (VPB[3, #] & /@ Range[50]))
```

(Alt) Out[•]=
```
True
```

(Alt) In[ ]:=
```
CountVPB[n_, m_] := CountVPB[n, {m, m}];
CountVPB[n_, {m1_, m2_}] :=
 Module[{σ, gens, dc, s, t, VPB, T, ij, ijk, ijkl, i, j, k, l, perm, p, q},
  CountVPB$Locals =
   Hold[{n, {m1, m2}, σ, gens, dc, s, t, VPB, T, ij, ijk, ijkl, i, j, k, l, perm, p, q}];
  {σ_i_,j_ := (n - 1) (i - 1) + If[j < i, j - 1, j - 2], σ̄_i_,j_ := n (n - 1) + σ_i,j};
  gens = Range[2 n (n - 1)] - 1;
  dc[m_] := dc[m] = Sum[(2 n (n - 1))^s, {s, 0, m}];
  Print[dc /@ {m1, m2}, " diagrams..."];
  ERAMake[$er, dc[m2]];
  VPB[_, gs_List] := 1 + dc[Length[gs] - 1] + FromDigits[gs, 2 n (n - 1)];
  T[b1_, b2_] := ERAdjoin[$er, b1 ⊷ b2];
  Do[{i, j} = ij; {
    T[VPB[n, Join[p, {σ_i,j, σ̄_i,j}, q]], VPB[n, Join[p, q]]],
    T[VPB[n, Join[p, {σ̄_i,j, σ_i,j}, q]], VPB[n, Join[p, q]]]
   },
   {s, 0, m2 - 2}, {t, 0, s}, {p, Tuples[gens, t]}, {q, Tuples[gens, s - t]},
   {ij, Join @@ (Permutations /@ Subsets[Range[n], {2}])}
  ];
  Do[{i, j, k} = ijk; {
    T[VPB[n, Join[p, {σ_i,j, σ_i,k, σ_j,k}, q]], VPB[n, Join[p, {σ_j,k, σ_i,k, σ_i,j}, q]]],
    T[VPB[n, Join[p, {σ̄_j,i, σ_i,k, σ_j,k}, q]], VPB[n, Join[p, {σ_j,k, σ_i,k, σ̄_j,i}, q]]],
    T[VPB[n, Join[p, {σ_i,j, σ_i,k, σ̄_k,j}, q]], VPB[n, Join[p, {σ̄_k,j, σ_i,k, σ_i,j}, q]]],
    T[VPB[n, Join[p, {σ_i,j, σ̄_k,i, σ̄_k,j}, q]], VPB[n, Join[p, {σ̄_k,j, σ̄_k,i, σ_i,j}, q]]],
    T[VPB[n, Join[p, {σ̄_j,i, σ̄_k,i, σ_j,k}, q]], VPB[n, Join[p, {σ_j,k, σ̄_k,i, σ̄_j,i}, q]]],
    T[VPB[n, Join[p, {σ̄_j,i, σ̄_k,i, σ̄_k,j}, q]], VPB[n, Join[p, {σ̄_k,j, σ̄_k,i, σ̄_j,i}, q]]]
   },
   {s, 0, m2 - 3}, {t, 0, s}, {p, Tuples[gens, t]}, {q, Tuples[gens, s - t]},
   {ijk, Join @@ (Permutations /@ Subsets[Range[n], {3}])}
  ];
  Do[{i, j, k, l} = ijkl; {
    T[VPB[n, Join[p, {σ_i,j, σ_k,l}, q]], VPB[n, Join[p, {σ_k,l, σ_i,j}, q]]],
    T[VPB[n, Join[p, {σ̄_i,j, σ_k,l}, q]], VPB[n, Join[p, {σ_k,l, σ̄_i,j}, q]]],
    T[VPB[n, Join[p, {σ_i,j, σ̄_k,l}, q]], VPB[n, Join[p, {σ̄_k,l, σ_i,j}, q]]],
    T[VPB[n, Join[p, {σ̄_i,j, σ̄_k,l}, q]], VPB[n, Join[p, {σ̄_k,l, σ̄_i,j}, q]]]
   },
   {s, 0, m2 - 2}, {t, 0, s}, {p, Tuples[gens, t]}, {q, Tuples[gens, s - t]},
   {ijkl, Join @@ (Permutations /@ Subsets[Range[n], {4}])}
  ];
  Count[Take[$er, dc[m1]], 0]
 ]
```

(Alt) In[ ]:= `VPB[4, {σ_4,1, σ̄_2,3}]`

(Alt) Out[ ]= 258

(Alt) In[•]:= **CountVPB[2, 1]**

{5, 5} diagrams...

(Alt) Out[•]= 5

(Alt) In[•]:= **CountVPB[2, 2]**

{21, 21} diagrams...

(Alt) Out[•]= 17

(Alt) In[•]:= **Timing@CountVPB[2, 3]**

{85, 85} diagrams...

(Alt) Out[•]= {0., 53}

(Alt) In[•]:= **Timing@CountVPB[2, {3, 4}]**

{85, 341} diagrams...

(Alt) Out[•]= {0.015625, 53}

(Alt) In[•]:= **Timing@CountVPB[2, 4]**

{341, 341} diagrams...

(Alt) Out[•]= {0.015625, 161}

(Alt) In[•]:= **Timing@CountVPB[2, 5]**

{1365, 1365} diagrams...

(Alt) Out[•]= {0.0625, 485}

(Alt) In[•]:= **Timing@CountVPB[2, 6]**

{5461, 5461} diagrams...

(Alt) Out[•]= {0.25, 1457}

(Alt) In[•]:= **Timing@CountVPB[3, 1]**

{13, 13} diagrams...

(Alt) Out[•]= {0., 13}

(Alt) In[•]:= **Timing@CountVPB[3, 2]**

{157, 157} diagrams...

(Alt) Out[•]= {0., 145}

(Alt) In[•]:= **Timing@CountVPB[3, 3]**

{1885, 1885} diagrams...

(Alt) Out[•]= {0.015625, 1561}

(Alt) In[•]:= **Timing@CountVPB[3, {3, 4}]**

{1885, 22621} diagrams...

(Alt) Out[•]= {0.25, 1561}

(Alt) In[•]:= **Timing@CountVPB[3, 4]**

{22621, 22621} diagrams...

(Alt) Out[•]= {0.203125, 16741}

(Alt) In[•]:= **Timing@CountVPB[3, {4, 5}]**

{22621, 271453} diagrams...

(Alt) Out[•]= {3.125, 16741}

(Alt) In[•]:= **Timing@CountVPB[3, {4, 6}]**

{22621, 3257437} diagrams...

(Alt) Out[•]= {46.5938, 16717}

(Alt) In[•]:= **Timing@CountVPB[3, {4, 7}]**

{22621, 39089245} diagrams...

(Alt) Out[•]= {746.359, 16717}

(Alt) In[•]:= **Timing@CountVPB[3, 5]**

{271453, 271453} diagrams...

(Alt) Out[•]= {3.17188, 179401}

(Alt) In[•]:= **Timing@CountVPB[3, {5, 6}]**

{271453, 3257437} diagrams...

(Alt) Out[•]= {49.6094, 179377}

(Alt) In[•]:= **Timing@CountVPB[3, {5, 7}]**

{271453, 39089245} diagrams...

(Alt) Out[•]= {679.078, 178873}

(Alt) In[•]:= **Timing@CountVPB[4, 1]**

{25, 25} diagrams...

(Alt) Out[•]= {0., 25}

(Alt) In[•]:= **Timing@CountVPB[4, 2]**

{601, 601} diagrams...

(Alt) Out[•]= {0., 529}

(Alt) In[•]:= **Timing@CountVPB[4, 3]**

{14 425, 14 425} diagrams...

(Alt) Out[•]= {0.34375, 10 873}

(Alt) In[•]:= **Timing@CountVPB[4, 4]**

{346 201, 346 201} diagrams...

(Alt) Out[•]= {5.65625, 222 385}

(Alt) In[•]:= **Timing@CountVPB[4, {4, 5}]**

{346 201, 8 308 825} diagrams...

(Alt) Out[•]= {251.719, 222 385}

(Alt) In[•]:= **Timing@CountVPB[4, {4, 6}]**

{346 201, 199 411 801} diagrams...

(Alt) Out[•]= {8379.89, 222 289}

(Alt) In[•]:= **Timing@CountVPB[5, 1]**

{41, 41} diagrams...

(Alt) Out[•]= {0., 41}

(Alt) In[•]:= **Timing@CountVPB[5, 2]**

{1641, 1641} diagrams...

(Alt) Out[•]= {0.078125, 1361}

(Alt) In[•]:= **Timing@CountVPB[5, 3]**

{65 641, 65 641} diagrams...

(Alt) Out[•]= {3.625, 43 121}

(Alt) In[•]:= **Timing@CountVPB[5, 4]**

{2 625 641, 2 625 641} diagrams...

(Alt) Out[•]= {156.203, 1 351 721}

(Alt) In[•]:= **Timing@CountVPB[5, {4, 6}]**

{2 625 641, 4 201 025 641} diagrams...

» 68 786 585 600

» Hold[{5, {4, 6}, σ$2350, gens$2350, dc$2350, s$2350, t$2350, VPB$2350, T$2350, ij$2350, ijk$2350, ijkl$2350, i$2350, j$2350, k$2350, l$2350, perm$2350, p$2350, q$2350}]

» {5, {4, 6}, σ$2350, {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39}, dc$2350, 4, 1, VPB$2350, T$2350, ij$2350, ijk$2350, {4, 1, 5, 3}, 4, 1, 5, 3, perm$2350, {20}, {20, 13, 5}}

(Alt) Out[•]= $Aborted

(Alt) In[•]:= 2^{32}

(Alt) Out[•]= 4 294 967 296

(Alt) In[•]:= `VPB[5, 68 786 585 600]`

(Alt) Out[•]= $\left\{\sigma_{4,5},\ \overline{\sigma}_{3,4},\ \overline{\sigma}_{3,1},\ \overline{\sigma}_{3,2},\ \sigma_{4,3},\ \overline{\sigma}_{5,3},\ \overline{\sigma}_{5,3}\right\}$

(Alt) In[•]:= `Max[$er]`

(Alt) Out[•]= 68 786 585 603

(Alt) In[•]:= `Timing@CountVPB[6, 1]`

```
{61, 61} diagrams...
```

(Alt) Out[•]= {0., 61}

(Alt) In[•]:= `Timing@CountVPB[6, 2]`

```
{3661, 3661} diagrams...
```

(Alt) Out[•]= {0.046875, 2881}

(Alt) In[•]:= `Timing@CountVPB[6, 3]`

```
{219 661, 219 661} diagrams...
```

(Alt) Out[•]= {5.26563, 127 021}