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META-MONOIDS, META-BICROSSED PRODUCTS, AND THE ALEXANDER POLYNOMIAL

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ABSTRACT

We introduce a new invariant of tangles along with an algebraic framework. We claim that the invariant contains the classical Alexander polynomial of knots and its multivariable extension to links. We argue that of the computationally efficient members of the family of Alexander invariants, it is the most meaningful. These are lecture notes for talks given by the first author, written and completed by the second. The talks, with handouts and videos, are available at http://www.math.toronto.edu/drorbn/Talks/Regina-1206/. See also further comments at http://www.math.toronto.edu/drorbn/Talks/Caen-1206/#June8.

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1. Warm-up: The Baby Invariant, Z^G

Let T be an oriented tangle diagram. Let G be a monoid,[a] and suppose we are given two pairs $R^{\pm} = (g_o^{\pm}, g_u^{\pm})$ of elements of G. At each positive (respectively, negative)[b] crossing of T, assign g_o^+ (respectively, g_o^-) to the upper strand and g_u^+ (respectively, g_u^-) to the lower strand, as in Fig. 1. Then, for every strand, multiply all elements assigned to it in the order that they appear and store the end result. If T has n strands, we get a collection of n elements of G. Call this collection $Z^G(T)$.

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[a] A monoid is like a group, but without inverses: It is a set with an associative binary operation and a unit. Every group is also a monoid.

[b] Signs are determined by the "right-hand rule": If the right-hand thumb points along the direction of the upper strand of a positive crossing, then the fingers curl in the direction of the lower strand.

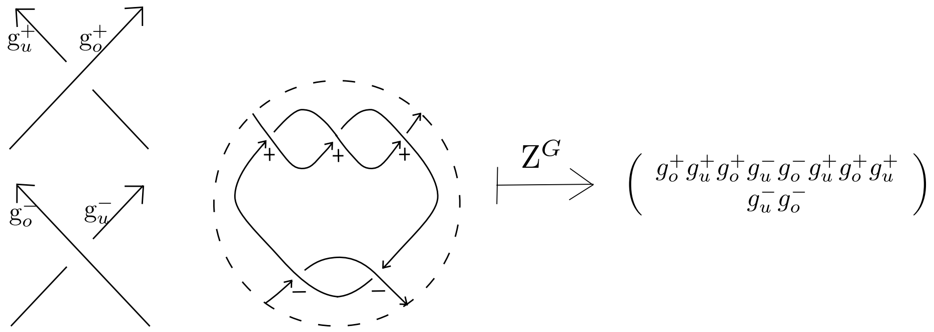

Fig. 1. Computing Z^G of a tangle: (a) Assigning values to crossings. (b) Collecting along strands.

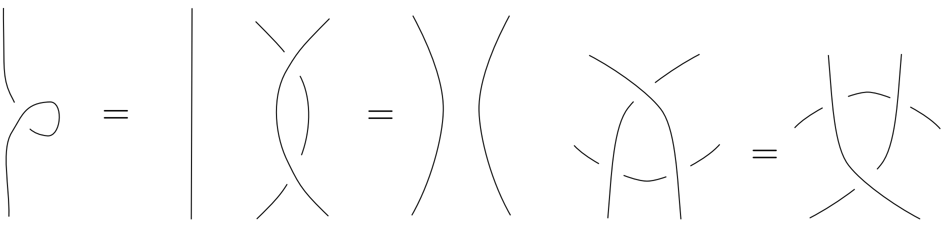

Fig. 2. The three Reidemeister moves: *I*, *II*, *III*.

Unfortunately, the gods are not so kind and Z^G is not worth much more than the effort that went in it. Indeed, invariance under the Reidemeister *II* move (see Fig. 2) demands $g_o^- = (g_o^+)^{-1}$ and $g_u^- = (g_u^+)^{-1}$, while Reidemeister *III* adds that g_o^+ and g_u^+, as well as g_o^- and g_u^-, commute. As a result, every component of $Z^G(T)$ collapses to the form $g_o^a g_u^b$ for some integers a and b, so all the information to bring home is the signed number of times a given strand crosses over or under other strands. It will turn out, nevertheless, that a generalized version of this procedure yields an amply non-trivial invariant with novel properties.

2. A Better Invariant: Z^β

The invariant that we wish to introduce can be thought of as taking values in a meta-monoid. This is a generalization of what we call a "monoid computer".

2.1. *Preliminary*: *A monoid computer*

If X is a finite set and G is a monoid we let G^X denote the set of all possible assignments of elements of G to the set X; these are "G-valued datasets, with registers labeled by the elements of X".

A monoid computer can manipulate registers in some prescribed ways. For example, if X does not contain x, y and z, define $m_z^{xy} : G^{X\cup\{x,y\}} \to G^{X\cup\{z\}}$ using

AQ: Please cite Fig. 3 in text.

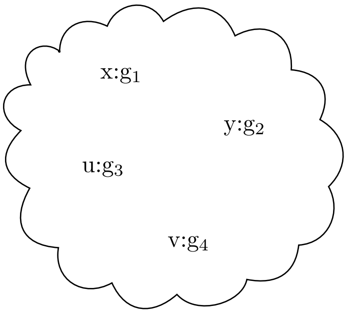

Fig. 3. A typical element of $G^{\{x,y,u,v\}}$.

the monoid multiplication, $\{x : g_1, y : g_2\} \mapsto \{z : g_1 g_2\}$. There are obvious operations for renaming or deleting a register, and inserting the identity in a new register, respectively denoted ρ^x_y, d^x and e_y, and respectively implemented on $G^{X\cup\{x\}}$ by fixing the content of X and mapping $\{x : g\}$ to $\{y : g\}$, $\{\}$ and $\{x : g, y : e\}$. In addition, there is a binary operation for merging data sets, $\bigcup : G^X \times G^Y \to G^{X\cup Y}$, which takes two data sets P and Q and forms their disjoint union $P \cup Q$. We can compose the aforementioned maps if labels match correctly, and we do so from left to right with the aid of the notation $/\!\!/$. For example, we write $P /\!\!/ \rho^x_y /\!\!/ \rho^y_z$ to rename the register x of P first to y, then to z.

2.2. *Meta-monoids*

The operations on a monoid computer obey a certain set of basic set-theoretic axioms as well as axioms inherited from the monoid G. A meta-monoid is an abstract computer that satisfies some but not all of those axioms. We postpone the precise definition to Sec. 3. It may be best to begin with examples and a prototypical one is as follows. Let $G_X := M_{X\times X}(\mathbf{Z})$ denote (not in reference to any monoid G) the set of $|X| \times |X|$ matrices of integers with rows and columns labeled by X. The operation of "multiplication", on say, 3×3 matrices, $m^{xy}_z : G_{\{x,y,w\}} \to G_{\{z,w\}}$, is defined by simultaneously adding rows and columns labeled by x and y:

$$\begin{array}{cc} & \begin{array}{ccc} x & y & w \end{array} \\ \begin{array}{c} x \\ y \\ w \end{array} & \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \end{array} \mapsto \begin{array}{cc} & \begin{array}{cc} z & w \end{array} \\ \begin{array}{c} z \\ w \end{array} & \begin{pmatrix} a+b+d+e & c+f \\ g+h & i \end{pmatrix} \end{array}.$$

While still satisfying the associativity condition $m^{xy}_u /\!\!/ m^{uv}_w = m^{yv}_u /\!\!/ m^{xu}_w$, this example differs from a monoid computer by the failure of a critical axiom: if $P \in G_{\{x,y\}}$,

$$d_y P \cup d_x P \neq P.$$

Indeed, if $P \in G_{\{x,y\}}$ is the matrix $\begin{array}{c|cc} & x & y \\ \hline x & a & b \\ y & c & d \end{array}$, then

$$d_y P \cup d_x P = \begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \end{array} \neq P.$$

2.3. *Meta-bicrossed products*

Suppose a group G is given as the product $G = TH$ of two of its subgroups, where $T \cap H = \{e\}$. Then also $G = HT$[c] and every element of G has unique[d] representations of the form th and $h't'$ where $h, h' \in H$ and $t, t' \in T$. Accordingly there is a "swap" map $sw : T \times H \to H \times T$, $(t, h) \mapsto (h', t')$ such that if $g = th$ then $g = h't'$ also. The swap map satisfies some relations; in monoid-computer language, the important ones are as in Fig. 4. Conversely, provided that the swap map satisfies the relations in Fig. 4, the data (H, T, sw) determines a monoid G, with product given by $\{(h_1, t_1), (h_2, t_2)\} \mapsto (h_1 h_2', t_1' t_2)$ where $sw(t_1, h_2) = (h_2', t_1')$. G is called the bicrossed product of H and T, which we could denote by $(H \times T)_{sw}$. In a semidirect product, one of H or T is normal (say T) and the swap map is $sw : (t, h) \mapsto (h, h^{-1}th)$.

The corresponding notion of a meta-bicrossed product is a collection of sets $\beta(\eta, \tau)$ indexed by all *pairs* of finite sets η and τ (η for "heads", τ for "tails"), and equipped with multiplication maps tm_z^{xy} (x, y and z tail labels), hm_z^{xy} (x, y and z head labels), and a swap map sw_{xy}^{th} (where t and h indicate that x is a tail label and y is a head label — note that sw_{yx}^{ht} is in general a different map) satisfying (a) and (b).

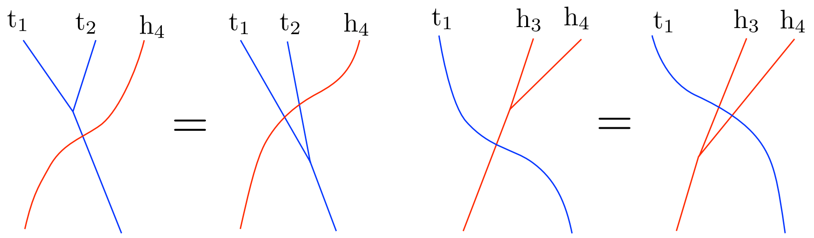

Fig. 4. Swap operation axioms. tm and hm stand for multiplication in T and H, respectively: (a) $tm_1^{12} /\!/ sw_{14} = sw_{24} /\!/ sw_{14} /\!/ tm_1^{12}$. (b) $hm_3^{34} /\!/ sw_{13} = sw_{13} /\!/ sw_{14} /\!/ hm_3^{34}$.

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[c] Indeed, if $g^{-1} = th$, then $g = h^{-1}t^{-1}$, so $g^{-1} \in TH$ implies $g \in HT$, and as $TH = G$, also $HT = G$.

[d] Separation of variables: suppose $g = h_1 t_1 = h_2 t_2$. Then we have $h_2^{-1} h_1 = t_2 t_1^{-1}$, which implies that $h_1 = h_2$ and $t_1 = t_2$ since $h_2^{-1} h_1 \in H$, $t_2 t_1^{-1} \in T$, and $H \cap T = \{e\}$.

Given the above we can make a "monoid multiplication" map out of the head and tail multiplication maps via $gm_z^{xy} := sw_{xy}^{th} \mathbin{/\!\!/} tm_z^{xy} \mathbin{/\!\!/} hm_z^{xy}$. Thus, a meta-bicrossed product defines a meta-monoid with $\Gamma_X = \beta(X,X)$. An example of a meta-bicrossed product is given by the rectangular matrices, $\mu(\eta,\tau) := M_{\tau\times\eta}(\mathbf{Z})$, with tm_z^{xy} and hm_z^{xy} corresponding to adding two rows and adding two columns, and swap being the trivial operation. Here, Γ_X is the same as the first example of Sec. 2.2. An example with a non-trivial swap map will shortly follow.

$$\begin{tikzcd}
 & \begin{array}{cc} & h_1 \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{pmatrix} a+b \\ c+d \\ e+f \end{pmatrix} \end{array} \\
\begin{array}{cc} & \begin{matrix} h_1 & h_2 \end{matrix} \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array} \arrow[ur, "hm_1^{1,2}"] \arrow[r, "tm_1^{1,2}"] \arrow[dr, "sw_{1,2}^{th}"'] & \begin{array}{cc} & \begin{matrix} h_1 & h_2 \end{matrix} \\ \begin{matrix} t_1 \\ t_3 \end{matrix} & \begin{pmatrix} a+c & b+d \\ e & f \end{pmatrix} \end{array} \\
 & \begin{array}{cc} & \begin{matrix} h_1 & h_2 \end{matrix} \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \end{array}
\end{tikzcd}$$

2.4. β *Calculus*

The β calculus has an arcane origin [1][e] which we will not discuss. We expect that it can be presented in a much simpler and fitting context than that in which it was discovered. Accordingly we will simply pull it out of a hat. Though note that many of our formulas bear close resemblance to formulas in [2, 4, 5].

Let $\beta(\eta,\tau)$ be (again, in reference to *sets* η and τ) the collection of arrays with rows labeled by $t_i \in \tau$ and columns labeled by $h_j \in \eta$, along with a distinguished element ω. Such arrays are conveniently presented in the following format:

$$\begin{array}{c|ccc}
\omega & h_1 & h_2 & \dots \\
\hline
t_1 & \alpha_{11} & \alpha_{12} & \cdot \\
t_2 & \alpha_{21} & \alpha_{22} & \cdot \\
\vdots & \cdot & \cdot & \cdot
\end{array}$$

The α_{ij} and ω are rational functions of variables T_i, which are in bijection with the row labels t_i.

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[e] In which, among other things, the "heads and tails" vocabulary is motivated.

$\beta(\eta,\tau)$ is equipped with a peculiar set of operations. Despite being repulsive at sight, they are completely elementary. They are defined as follows:

$$tm^{xy}_z : \begin{array}{c|c} \omega & \dots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \dots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}$$

Here, α and β are rows and γ is a matrix. The sum $\alpha+\beta$ is accompanied by the corresponding change of variables $T_x, T_y \mapsto T_z$.

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & h_x & h_y & \dots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \dots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array}$$

Here, α and β are columns, γ is a matrix, and $\langle\alpha\rangle = \sum_i \alpha_i$.

$$sw^{th}_{xy} : \begin{array}{c|cc} \omega & h_y & \dots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \dots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array}$$

Here, α is a single entry, β is a row, γ is a column, and δ is a matrix comprised of the rest. $\epsilon = 1+\alpha$. Note also that $\gamma\beta$ is the matrix product of the column γ with the row β and hence has the same dimensions as the matrix δ.

We also need the disjoint union, defined by

$$\begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array}$$

We make β into a meta-monoid via the "monoid-multiplication" map $gm^{xy}_z := sw^{th}_{xy} \mathbin{/\!\!/} tm^{xy}_z \mathbin{/\!\!/} hm^{xy}_z$. We will later set out to make proper definitions, write down the remaining operations, and establish the following theorem.

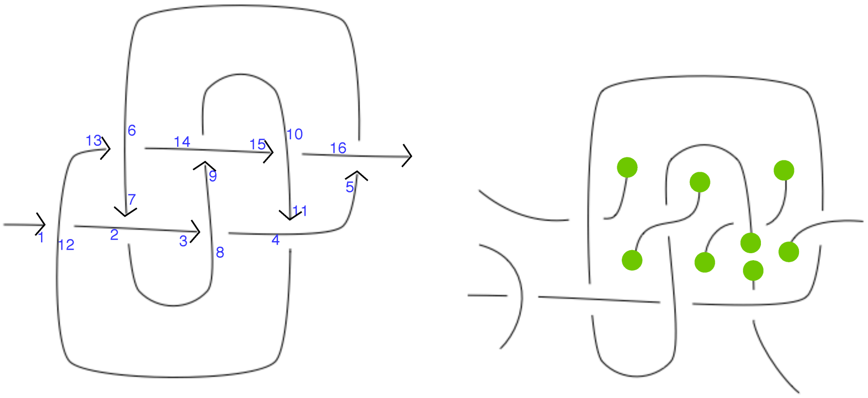
Theorem 2.1. *β is a meta-bicrossed product.*

Finally, there are two elements which will serve as a pair of "R-matrices", analogous to the pair of pairs $(g^\pm_o, g^\pm_u)$ of Z^G:

$$R^+_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x-1 \\ t_y & 0 & 0 \end{array} \qquad R^-_{xy} = \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x^{-1}-1 \\ t_y & 0 & 0 \end{array}$$

2.5. Z^β

Let T be again an oriented tangle diagram. At each crossing, assign a number to the upper strand and to the lower strand. Using the $R^\pm_{xy}$ of above, from the disjoint union $\bigcup_{\{i,j\}} R^\pm_{ij}$ where $\{i,j\}$ runs over all pairs assigned to crossings, with i labeling



(a) (b)

Fig. 5. (Color online) The knot 8_{17}: (a) With crossings labeled. (b) After attaching crossings 1 through 10. The arcs with green dots cannot make it out to the boundary disk.

the upper strand and j labeling the lower strand, and where $\pm$ is determined by the sign of the given crossing. Now for each strand multiply all the labels in the order in which they appear. That is, if the first label on the strand is k, repeatedly apply gm_k^{kl} where l runs over all labels subsequently encountered on the strand (in order). If T has n strands, the result is an $n \times n$ array with an extra corner element. Call this array $Z^\beta(T)$. Those were a lot of words, so take for example the knot 8_{17} illustrated in Fig. 5. In this case, form the disjoint union[f]

$$R^-_{12,1}R^-_{2,7}R^-_{8,3}R^-_{4,11}R^+_{16,5}R^+_{6,13}R^+_{14,3}R^+_{10,15},$$

which is given by the following array[g]:

1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$T_2^{-1}-1$	0	0	0	0
t_4	0	0	0	0	0	$T_4^{-1}-1$	0	0
t_6	0	0	0	0	0	0	T_6-1	0
t_8	0	$T_8^{-1}-1$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$T_{10}-1$
t_{12}	$T_{12}^{-1}-1$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$T_{14}-1$	0	0	0
t_{16}	0	0	$T_{16}-1$	0	0	0	0	0

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[f] From now on, we omit the $\cup$ in disjoint unions: $\beta_1\beta_2 := \beta_1 \cup \beta_2$.

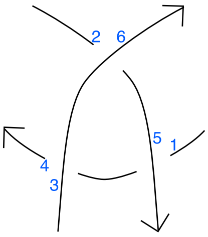
[g] We suppress rows/columns of zeros.

Then apply the multiplications gm_1^{1k}, with k running from 2 to 16, to get the following 1×1 array with corner element:

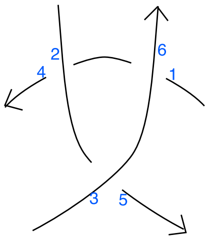
$$\begin{array}{c|c} -T_1^{-3}+4T_1^{-2}-8T_1^{-1}+11-8T_1+4T_1^2-T_1^3 & h_1 \\ \hline t_1 & 0 \end{array}$$

Theorem 2.2. *Z^β is an invariant of oriented tangle diagrams.*

Proof. Straightforward check. We do the computation for the Reidemeister *III* move to illustrate. The disjoint unions for each side of the equality are given by:



$$R^-_{1,5}R^-_{6,2}R^+_{3,4} = \begin{array}{c|ccc} 1 & h_1 & h_2 & h_4 \\ \hline t_3 & 0 & 0 & T_3-1 \\ t_5 & T_5^{-1}-1 & 0 & 0 \\ t_6 & 0 & T_6^{-1}-1 & 0 \end{array}$$



$$R^+_{6,1}R^-_{2,4}R^-_{3,5} = \begin{array}{c|ccc} 1 & h_1 & h_4 & h_5 \\ \hline t_3 & 0 & T_2^{-1}-1 & 0 \\ t_5 & 0 & 0 & T_3^{-1}-1 \\ t_6 & T_6-1 & 0 & 0 \end{array}$$

Then one checks that indeed

$$R^-_{1,5}R^-_{6,2}R^+_{3,4} \mathbin{/\!\!/} gm_1^{1,4} \mathbin{/\!\!/} gm_2^{2,5} \mathbin{/\!\!/} gm_3^{3,6} = R^+_{6,1}R^-_{2,4}R^-_{3,5} \mathbin{/\!\!/} gm_1^{1,4} \mathbin{/\!\!/} gm_2^{2,5} \mathbin{/\!\!/} gm_3^{3,6}$$
$$= \begin{array}{c|cc} 1 & h_1 & h_2 \\ \hline t_1 & T_2^{-1}-1 & 0 \\ t_2 & T_2^{-1}(T_3-1) & T_3^{-1}-1 \end{array}. \qquad \square$$

One philosophically appealing major property of Z^β is that the operations used to compute it have a literal interpretation of gluing crossings together. In particular, at every stage of the computation we get an invariant of the tangle[h] made of all the crossings but only those for which the corresponding gm was carried out have been glued. Additionally, unlike other existing extensions of the Alexander polynomial to tangles, Z^β takes values in spaces of polynomial size, at every step of the calculation.

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[h] The careful reader may wish to peek ahead at Sec. 3.1 for a better grasp of this statement.

2.6. *Knots and links*

Conjecture 2.1. *Restricted to long knots* (*which are the same as round knots*), *the corner element of* Z^β *is the Alexander polynomial. Restricted to string links* (*which map surjectively to links*), Z^β *contains the multivariable Alexander polynomial.*

While we are shy of a formal proof, the computer evidence behind Conjecture 2.1 is overwhelming. See Sec. 4.3.

3. More on Meta-Monoids

3.1. *The meta-monoid of colored v-tangles*

When one tries to follow the interpretation of the computation of Z^β as progressively attaching crossings together to form a tangle, one will in general encounter a step where the tangle becomes non-planar (a strand will have to go through another in an "artificial" crossing to reach the boundary disk). See Fig. 5. Such tangles are called virtual or v-tangles and constitute a rich subject of study on their own; see [3]. We will be content with acknowledging their existence and giving them a name.

If X is a finite set, oriented X-colored pure[i] virtual tangles form a meta-monoid. The operation m^{xy}_z attaches the head of strand x to the tail of strand y (possibly through a few virtual crossings) and names the resulting strand z.[j]

3.2. *Some familiar invariants*

We have already suggested that Z^G and Z^β take values in meta-monoids. Some more traditional invariants can also be cast in meta-monoid context. Note that Z^G is in fact very traditional, being nothing more than linking numbers. We invite the reader familiar with the fundamental group of the complement of a tangle to consider the following set-up:

Let $G_{\{x_1,\ldots,x_n\}} = \{(\Gamma, m_1, l_1, \ldots, m_n, l_n); \Gamma \text{ is a group}; m_i, l_i \in \Gamma\}$. The multiplication map that corresponds to what happens to the meridians and longitudes when one plugs a strand into another is

$$m^{ij}_i(\Gamma, m_1, l_1, \ldots, m_n, l_n) = (\Gamma/(m_j = l_i^{-1} m_i l_i), m_1, l_1 l_2, \ldots, \widehat{m_j}, \widehat{l_j}, \ldots, m_n, l_n).$$

Also the fundamental group of the complement of two disjoint tangles is the free product of the respective fundamental groups, so we define also

$$\begin{aligned}(\Gamma^1, m^1_1, l^1_1, \ldots, m^1_n, l^1_n) \cup (\Gamma^2, m^2_1, l^2_1, \ldots, m^2_k, l^2_k)\\ = (\Gamma^1 \star \Gamma^2, m^1_1, l^1_1, \ldots, m^1_n, l^1_n, m^2_1, l^2_1, \ldots, m^2_k, l^2_k).\end{aligned}$$

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[i] Pure means that the tangles have no closed component.

[j] Remark: This is *not* a meta-generalization of the group structure on braids.

3.3. *Definitions*

We now proceed to laying down the details of the definitions of meta-monoids and meta-bicrossed products.

A meta-monoid is a collection of sets Γ indexed by all finite sets, equipped with operations $m^{xy}_z : \Gamma_{\{x,y\}\cup X} \to \Gamma_{\{z\}\cup X}$, $e_x : \Gamma_X \to \Gamma_{\{x\}\cup X}$, $d_x : \Gamma_{\{x\}\cup X} \to \Gamma_X$, and $\bigcup : \Gamma_X \times \Gamma_Y \to \Gamma_{X\cup Y}$ satisfying the following:

"Monoid theory" axioms

- $e_x /\!\!/ m^{xy}_z = \rho^y_z$ (left identity),
- $e_y /\!\!/ m^{xy}_z = \rho^x_z$ (right identity),
- $m^{xy}_u /\!\!/ m^{uz}_v = m^{yz}_u /\!\!/ m^{xu}_v$ (associativity).

"Set manipulation" axioms

- $\rho^y_x /\!\!/ \rho^x_y = \mathrm{id}$,
- $\rho^x_y /\!\!/ \rho^y_z = \rho^x_z$,
- $\rho^x_y /\!\!/ d_y = d_x$,
- $m^{xy}_z /\!\!/ d_z = d_x /\!\!/ d_y$,
- $e_x /\!\!/ d_x = \mathrm{id}$,
- $m^{xy}_z /\!\!/ \rho^z_u = m^{xy}_u$,
- $\rho^x_u /\!\!/ m^{uy}_z = m^{xy}_z$,
- $e_x /\!\!/ \rho^x_y = e_y$,
- operations involving disjoint sets of labels commute (e.g. $e_x /\!\!/ e_y = e_y /\!\!/ e_x$).

A meta-bicrossed product is a collection of sets Γ indexed by all pairs of finite sets, equipped with maps hm, tm, and sw, such that:

- $hm^{xy}_z : \Gamma(\eta \cup \{x,y\}, \tau_0) \to \Gamma(\eta \cup \{z\}, \tau_0)$ and $tm^{xy}_z : \Gamma(\eta_0, \tau \cup \{x,y\}) \to \Gamma(\eta_0, \tau \cup \{z\})$ define a meta-monoid structure for each fixed choice of τ_0 and η_0, respectively.
- sw_{xy} satisfies the following relations (recall Fig. 4):
  - — $tm^{xy}_x /\!\!/ sw_{xz} = sw_{xz} /\!\!/ sw_{yz} /\!\!/ tm^{xy}_x$,
  - — $hm^{yz}_y /\!\!/ sw_{xy} = sw_{xy} /\!\!/ sw_{xz} /\!\!/ hm^{yz}_y$,
  - — $sw_{xy} /\!\!/ t\rho^x_u = t\rho^x_u /\!\!/ sw_{uy}$,
  - — $sw_{xy} /\!\!/ h\rho^y_u = h\rho^y_u /\!\!/ sw_{xu}$,
  - — $te_x /\!\!/ sw_{xy} = te_x$,
  - — $he_y /\!\!/ sw_{xy} = he_y$.

Note that in a meta-bicrossed product, $m^{xy}_z = sw_{xy} /\!\!/ hm^{h_x h_y}_{h_z} /\!\!/ tm^{t_x t_y}_{t_z}$ always defines a meta-monoid with $\Gamma_X = \Gamma(X,X)$.

4. Some Verifications: Computer Program

Using *Mathematica*, it is possible to write a very concise implementation of β-calculus, and use to carry out the algebraic manipulations that prove Theorem 2.1 and verify Conjecture 2.1 on a convincing number of knots and links. We do that in several parts below, with all code included.

AQ: We have changed "Theorem 1" and "Conjecture 1" to "Theorem 2.1" and "Conjecture 2.1". Please check.

4.1. *The program*

We start by loading the *Mathematica* package `KnotTheory‘`. This is not strictly necessary, and it is only used for comparison with standard evaluations of the Alexander polynomial:

```
<< KnotTheory`
Loading KnotTheory` version of February 5, 2013, 3:48:46.4762.
Read more at http://katlas.org/wiki/KnotTheory.
```

We then move on to our main program.

The first part of the program is mostly cosmetic. Its main part is the routine `βForm` used for pretty-printing β-calculus outputs:

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], t_{s_} ⧴ s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_{s_} ⧴ s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_{#1} t_{#2}]] &, hs, ts];
   PrependTo[M, t_{#} & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_{#} & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B ⧴ βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

In the main part of the program, a β matrix is represented as a polynomial in two variables: $\mu = \sum \alpha_{ij} t_i h_j$. This makes some calculations very simple! Selecting the content of column i is achieved by taking a derivative with respect to h_i; setting all the t's equal to 1 computes its column sum. The disjoint union of two matrices is simply the sum of their polynomials:

```
⟨μ_⟩ := μ /. t_ → 1;
tm_{x_ y_→z_}[β_] := βCollect[β /. {t_{x|y} → t_{z}, T_{x|y} → T_{z}}];
hm_{x_ y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_{x}], β = D[Λ, h_{y}], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_{z} + γ] // βCollect];
sw_{x_ y_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_{y} t_{x}]; β = D[Λ, t_{x}] /. h_{y} → 0;
   γ = D[Λ, h_{y}] /. t_{x} → 0;   δ = Λ /. h_{y} | t_{x} → 0;
```

```
    ϵ  = 1 + α;
    B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
           + γ / ϵ h_y                + δ - γ * β / ϵ
     ] // βCollect];
gm_{x_ y_ →z_}[β_] := β // sw_{xy} // hm_{xy→z} // tm_{xy→z};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
(R⁺)_{x_ y_}  := B[1, (T_x - 1) t_x h_y];
(R⁻)_{x_ y_}  := B[1, ((T_x)^-1 - 1) t_x h_y];
```

4.2. *Proof of Theorem* 2.1

To establish Theorem 2.1 we just need to check that the operations of β-calculus satisfy the axioms of a meta-bicrossed product listed in Sec. 3.3. Our only concern is with the non-obvious axioms, the associativity of tm and of hm, and the two swap axioms of Fig. 4. Even this we do the lazy way — we have a computer implementation of the β-calculus operations. Why not use it to check the relations?

As a first check, we check the meta-associativity of tm — we input a generic 4-tail and 2-head β matrix, let O_1 and O_2 be the outputs of evaluating $tm_1^{12} \,/\!\!/\, tm_1^{13}$ and $tm_2^{23} \,/\!\!/\, tm_1^{12}$ on β, and finally we print the logical value of $O_1 = O_2$. Nicely, it comes out to be `True`:

```
{β = B[ω, Sum[α_{2 i+j-6} t_i h_j, {i, 1, 4},
      {j, 5, 6}]],
  O_1 = β // tm_{12→1} // tm_{13→1},
  O_2 = β // tm_{23→2} // tm_{12→1},
  O_1 == O_2} // ColumnForm
```

```
( ω   h5  h6 )
( t1  α1  α2 )
( t2  α3  α4 )
( t3  α5  α6 )
( t4  α7  α8 )
( ω        h5              h6       )
( t1  α1 + α3 + α5  α2 + α4 + α6 )
( t4       α7              α8       )
( ω        h5              h6       )
( t1  α1 + α3 + α5  α2 + α4 + α6 )
( t4       α7              α8       )
True
```

We then do the same for hm, except we now use a β matrix with 2 tails and 4 heads, and we suppress the printing of O_2. Nicely, the logical value of $O_1 = O_2$ is again `True`. (So we did not lose much by not printing O_2). Note that to keep our output from overflowing the width of the page, we have to denote α_i by $\hat{i}$:

```
{β = B[ω, Sum[α_{4 i+j-6} t_i h_j, {i, 1, 2}, {j, 3, 6}]],
   O_1 = β // hm_{34→3} // hm_{35→3},
   O_2 = β // hm_{45→4} // hm_{34→3};
   O_1 == O_2} /. α_{i_} :→ î // ColumnForm
```

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 & h_6 \\ t_1 & \hat{1} & \hat{2} & \hat{3} & \hat{4} \\ t_2 & \hat{5} & \hat{6} & \hat{7} & \hat{8} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h_3 & h_6 \\ t_1 & \hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{3}+\hat{1}\,\hat{3}+\hat{2}\,\hat{3}+\hat{1}\,\hat{2}\,\hat{3}+\hat{2}\,\hat{5}+\hat{3}\,\hat{5}+\hat{2}\,\hat{3}\,\hat{5}+\hat{3}\,\hat{6}+\hat{1}\,\hat{3}\,\hat{6}+\hat{3}\,\hat{5}\,\hat{6} & \hat{4} \\ t_2 & \hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{5}\,\hat{6}+\hat{7}+\hat{1}\,\hat{7}+\hat{2}\,\hat{7}+\hat{1}\,\hat{2}\,\hat{7}+\hat{5}\,\hat{7}+\hat{2}\,\hat{5}\,\hat{7}+\hat{6}\,\hat{7}+\hat{1}\,\hat{6}\,\hat{7}+\hat{5}\,\hat{6}\,\hat{7} & \hat{8} \end{pmatrix}$$

```
True
```

Next comes the two swap axioms:

```
{β = B[ω, Sum[α_{2 i+j-5} t_i h_j, {i, 1, 3},
    {j, 4, 5}]],
 O_1 = β // tm_{12→1} // sw_{14},
 O_2 = β // sw_{24} // sw_{14} // tm_{12→1};
 O_1 == O_2}
```

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_1 & \alpha_2 \\ t_2 & \alpha_3 & \alpha_4 \\ t_3 & \alpha_5 & \alpha_6 \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha_1+\alpha_3) & h_4 & h_5 \\ t_1 & \frac{(\alpha_1+\alpha_3)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} & \frac{(\alpha_2+\alpha_4)\,(1+\alpha_1+\alpha_3+\alpha_5)}{1+\alpha_1+\alpha_3} \\ t_3 & \frac{\alpha_5}{1+\alpha_1+\alpha_3} & \frac{-\alpha_2\,\alpha_5-\alpha_4\,\alpha_5+\alpha_6+\alpha_1\,\alpha_6+\alpha_3\,\alpha_6}{1+\alpha_1+\alpha_3} \end{pmatrix}, \text{True}\right\}$$

Note that for the second swap axiom, some algebraic simplification must take place, using the routine β`Collect`:

```
{β = B[ω, Sum[α_{3 i+j-5} t_i h_j, {i, 1, 2}, {j, 3, 5}]],
   O_1 = β // hm_{34→3} // sw_{13} // βCollect,
   O_2 = β // sw_{13} // sw_{14} // hm_{34→3} // βCollect;
   O_1 == O_2
  } /. α_{i_} :→ î // ColumnForm
```

$$\begin{pmatrix} \omega & h_3 & h_4 & h_5 \\ t_1 & \hat{1} & \hat{2} & \hat{3} \\ t_2 & \hat{4} & \hat{5} & \hat{6} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right) & h_3 & h_5 \\ t_1 & \frac{\left(1+\hat{1}+\hat{4}\right)\left(\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{\hat{3}\left(1+\hat{1}+\hat{4}\right)\left(1+\hat{2}+\hat{5}\right)}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \\ t_2 & \frac{\hat{4}+\hat{5}+\hat{1}\,\hat{5}+\hat{4}\,\hat{5}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} & \frac{-\hat{3}\,\hat{4}-\hat{3}\,\hat{5}-\hat{1}\,\hat{3}\,\hat{5}-\hat{3}\,\hat{4}\,\hat{5}+\hat{6}+\hat{1}\,\hat{6}+\hat{2}\,\hat{6}+\hat{1}\,\hat{2}\,\hat{6}+\hat{2}\,\hat{4}\,\hat{6}}{1+\hat{1}+\hat{2}+\hat{1}\,\hat{2}+\hat{2}\,\hat{4}} \end{pmatrix}$$

```
True
```

Just for completeness, we verify the third Reidemeister move once again:

$$\{(R^-)_{51}\,(R^-)_{62}\,(R^+)_{34}\;//\;gm_{14\to 1}\;//\;gm_{25\to 2}\;//\;gm_{36\to 3},\\ (R^+)_{61}\,(R^-)_{24}\,(R^-)_{35}\;//\;gm_{14\to 1}\;//\;gm_{25\to 2}\;//\;gm_{36\to 3}\}$$

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix},\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+T_2}{T_2} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} \end{pmatrix}\right\}$$

4.3. *Testing Conjecture* 2.1

Our next task is to carry out some computations for knots and links in support of Conjecture 2.1. As our first demonstration, we compute $Z^\beta(8_{17})$ in several steps. The first step is to generate the invariant of the tangle consisting of the disjoint union of 8 crossings, labeled as the crossings of 8_{17} are labeled but not yet connected to each other:

$$\beta = (R^-)_{12,1}\,(R^-)_{27}\,(R^-)_{83}\,(R^-)_{4,11}\,(R^+)_{16,5}\,(R^+)_{6,13}\,(R^+)_{14,9}\,(R^+)_{10,15}$$

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+T_2}{T_2} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+T_4}{T_4} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_6 & 0 \\ t_8 & 0 & -\frac{-1+T_8}{T_8} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{10} \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+T_{14} & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+T_{16} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Next, we *partially* concatenate the strands of these 8 crossings to each other, making only 9 of the required 15 connections. The result is 3-component tangle that approximates 8_{17}, and a chance to see what an intermediate step of the computation looks like:

$$\mathrm{Do}[\beta = \beta\;//\;gm_{1k\to 1},\ \{k, 2, 10\}];\ \beta$$

$$\begin{pmatrix} \frac{T_1^2+T_{16}-T_1T_{16}}{T_1^2} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+T_1)\,T_{14}\,(T_1^3+T_{16}^2)}{T_1^2\,T_{12}\,(T_1^2+T_{16}-T_1T_{16})} & -\frac{(-1+T_1)\,(1-T_1+T_1^2)\,T_{14}\,T_{16}}{T_1\,(T_1^2+T_{16}-T_1T_{16})} & \frac{(-1+T_1)\,(1-T_1+T_1^2)\,T_{14}}{T_1^2+T_{16}-T_1T_{16}} & -1+T_1 \\ t_{12} & -\frac{-1+T_{12}}{T_{12}} & 0 & 0 & 0 \\ t_{14} & \frac{(-1+T_{14})\,(-T_1+T_1^2+T_{16})}{T_{12}\,(T_1^2+T_{16}-T_1T_{16})} & \frac{(-1+T_1)\,(1-T_1+T_1^2)\,(-1+T_{14})\,T_{16}}{T_1\,(T_1^2+T_{16}-T_1T_{16})} & -\frac{(-1+T_1)\,(1-T_1+T_1^2)\,(-1+T_{14})}{T_1^2+T_{16}-T_1T_{16}} & 0 \\ t_{16} & \frac{T_1\,(-1+T_{16})}{T_{12}\,(T_1^2+T_{16}-T_1T_{16})} & \frac{(-1+T_1)\,T_1\,(-1+T_{16})}{T_1^2+T_{16}-T_1T_{16}} & -\frac{(-1+T_1)^2\,(-1+T_{16})}{T_1^2+T_{16}-T_1T_{16}} & 0 \end{pmatrix}$$

We then complete the sewing together of 8_{17}, obtaining $Z^\beta(8_{17})$. Note that the "matrix part" of the invariant is completely suppressed by our printing routine, because it is 0:

$$\texttt{Do}[\beta = \beta\ //\ \texttt{gm}_{1k\to 1},\ \{k,\ 11,\ 16\}];\ \beta$$

$$\left(\ -\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^3}\ \right)$$

For completeness, we compare with the pre-computed value of the Alexander polynomial, as known to `KnotTheory‘`. As can be fairly expected, it differs from the computed value of $Z^\beta(8_{17})$ by a unit:

```
Alexander[Knot[8, 17]][X]
KnotTheory::loading: Loading precomputed data in PD4Knots`.
```

$$11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8\,X+4\,X^2-X^3$$

We next make it systematic by writing a short program that compute Z^β of an arbitrary input link:

```
βZ[L_] := Module[{s, β, c, k},
  s = Skeleton[L];
  β = Times @@ PD[L] /. X[i_, j_, k_, l_] :→ If[
      PositiveQ[X[i, j, k, l]],
      (R⁺)_{l,i}, (R⁻)_{j,i}];
  Do[β = β // gm_{s[[c,1]], s[[c,k]]→s[[c,1]]},
   {c, Length[s]}, {k, 2, Length[s[[c]]]}];
  β]
```

We verify that for all knots with up to 8 crossings, the ratio of Z^β and the Alexander polynomial is always a unit. At home we have verified the same thing for all knots with up to 11 crossings:

$$\texttt{Factor}\left[\frac{\beta\texttt{Z[\#][[1]]}}{\texttt{Alexander[\#][}T_1\texttt{]}}\right]\ \texttt{\& /@ AllKnots[\{3, 8\}]}$$

$$\left\{\frac{1}{T_1},\ T_1,\ \frac{1}{T_1^2},\ \frac{1}{T_1^2},\ 1,\ 1,\ 1,\ \frac{1}{T_1^3},\ \frac{1}{T_1^3},\ T_1^4,\ T_1^4,\ \frac{1}{T_1^3},\right.$$
$$\frac{1}{T_1},\ T_1^2,\ \frac{1}{T_1},\ \frac{1}{T_1},\ T_1,\ T_1,\ T_1^3,\ \frac{1}{T_1},\ T_1,\ T_1,\ T_1,$$
$$\left.T_1,\ \frac{1}{T_1},\ T_1,\ T_1,\ \frac{1}{T_1},\ \frac{1}{T_1^3},\ \frac{1}{T_1},\ T_1,\ 1,\ T_1^4,\ 1,\ \frac{1}{T_1}\right\}$$

Next is the program for extracting the multivariable Alexander polynomial from the information in Z^β:

```
βMVA[L_Link] := Module[{ηs, ω, μ, M},
  {ω, μ} = List @@ βZ[L];
  ηs = Rest[h_# & /@ (First /@ Skeleton[L])];
  M = Outer[
    Coefficient[μ - (μ /. t_ → 1 /. h_a_ :→ t_a h_a), #1 * #2] &,
    ηs, ηs /. h_a_ :→ t_a];
  Factor[ω Det[M] / (1 - T_Skeleton[L][[1,1]])]]
```

It works for the Borromean rings!

```
βMVA[Link["L6a4"]]
KnotTheory::loading : Loading precomputed data in PD4Links`.
```

$$-\frac{(-1+T_1)\ (-1+T_5)\ (-1+T_9)}{T_1\ T_5}$$

And also for all links with up to 7 crossings. At home we have verified the same for all links with up to 11 crossings:

```
Factor[(MultivariableAlexander[#][T] /. T[i_] :→ T_Skeleton[#][[i,1]]) / βMVA[#]] & /@
 AllLinks[{2, 7}]
KnotTheory::loading : Loading precomputed data in MultivariableAlexander4Links`.
```

$$\Big\{T_1^2\ T_3,\ T_1^{3/2}\ T_5^{3/2},\ \sqrt{T_1}\ T_5^{3/2},\ T_1^{3/2}\ \sqrt{T_5},\ T_1^2\ T_7^2,\ T_1^2\ T_7^2,$$
$$-\frac{\sqrt{T_1}\ \sqrt{T_5}}{\sqrt{T_9}},\ -T_1^{3/2}\ T_5^{3/2}\ T_9^{3/2},\ -\frac{\sqrt{T_1}\ \sqrt{T_5}}{T_9^{3/2}},\ \sqrt{T_1}\ \sqrt{T_5},\ T_1^{3/2}\ T_5^{7/2},$$
$$\frac{\sqrt{T_1}}{T_5^{3/2}},\ \frac{\sqrt{T_1}}{T_5^{3/2}},\ T_1\ T_7^2,\ \frac{1}{T_7},\ -\frac{T_1^{3/2}\ \sqrt{T_5}}{\sqrt{T_9}},\ T_1^{3/2}\ T_5^{7/2},\ \sqrt{T_1}\ T_5^{5/2}\Big\}$$

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