Pensieve header: The free-Lie meta-monoid-action structure for http://www.math.toronto.edu/~drorbn/papers/KBH/.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\KBH"];
```

The Program

We load FreeLie.m and modify the formating of CWSeries:

LoadFreeLie

```
<< FreeLie.m
$SeriesShowDegree = 3; $SeriesCompareDegree = 5;

Format[s_CWSeries, StandardForm] := MapAt[
   Style[#, GrayLevel[0.6]] &,
   TopBracketForm[s[{$SeriesShowDegree}]],
   1];
CWSeries[ser_Symbol][{dd_Integer}] := MapAt[
   Style[#, GrayLevel[0.6]] &,
   TopBracketForm[CWS @@ Table[ser[d], {d, dd}]],
   1];
```

BCHDemo1

```
t1 = BCH[⟨u⟩, ⟨v⟩]
```

BCHDemo1

```
LS[u + v, uv/2, 1/12 uuv + 1/12 uvv]
```

BCHDemo2

```
t1@{6}
```

BCHDemo2

```
LS[u + v, uv/2, 1/12 uuv + 1/12 uvv, 1/24 uuvv,
 -1/720 uuuuv + 1/180 uuuvv + 1/180 uuvvv + 1/120 uvuvv + 1/360 uuvuv - 1/720 uvvvv,
 -uuuuvv/1440 + 1/360 uuuvvv + 1/240 uuvuvv + 1/720 uuuvuv - uuvvvv/1440]
```

Some set theoretic definitions:

SetTheory

```
Domain[f_List] := First /@ f;
f_ \ key_ := DeleteCases[f, key → _];
f_ \ keys_List := Fold[#1 \ #2 &, f, keys];
f1_List ≡ f2_List := Domain[f1] === Domain[f2] && (And @@ (
      ((# /. f1) ≡ (# /. f2)) & /@ Domain[f1]));
LieMorphism[mor_][f_List] := MapAt[LieMorphism[mor], f, {All, 2}];
M[λ1_, ω1_] ≡ M[λ2_, ω2_] := (λ1 ≡ λ2) && (ω1 ≡ ω2);
```

LieDefs

```
RCu_[γ_LieSeries, ub_][s_] := StableApply[LieMorphism[⟨u⟩ → Ad[γ][⟨ub⟩]], s];
RCu_[γ_LieSeries][s_] := s // RCu[γ, ⟨υ⟩] // LieMorphism[⟨υ⟩ → ⟨u⟩];
Ju_[γ_] :=
  Module[{s}, ∫_0^1 (γ // RCu[s γ] // divu // LieMorphism[u → Ad[-s γ][u]]) ds];
```

JBCH

```
Jv[t1]@{4}
```

JBCH

$$\mathrm{CWS}\left[\overline{\mathrm{v}},\ \overline{\mathrm{uv}},\ \frac{\overline{\mathrm{uuv}}}{2}-\frac{\overline{\mathrm{uvv}}}{2},\ \frac{\overline{\mathrm{uuuv}}}{6}+\frac{3\,\overline{\mathrm{uuvv}}}{4}-\frac{3\,\overline{\mathrm{uvuv}}}{2}+\frac{\overline{\mathrm{uvvv}}}{6}\right]$$

MMADefs

```
M /: M[λ1_, ω1_] ** M[λ2_, ω2_] := M[λ1 ⋃ λ2, ω1 + ω2];
tm[u_, v_, w_][λ_List] := λ // LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
tm[u_, v_, w_][M[λ_, ω_]] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩] /@ M[λ, ω];
hm[x_, y_, z_][λ_List] := Union[λ \ {x, y}, {z → BCH[x /. λ, y /. λ]}];
hm[x_, y_, z_][M[λ_, ω_]] := M[λ // hm[x, y, z], ω];
tha[u_, x_][λ_List] := MapAt[RCu[x /. λ], λ, {All, 2}];
tha[u_, x_][M[λ_, ω_]] := M[λ // tha[u, x], (ω + Ju[x /. λ]) // RCu[x /. λ]];
```

rho

```
hϵ[x_] := M[{x → MakeLieSeries[0]}, MakeCWSeries[0]]
ρ⁺[u_, x_] := M[{x → MakeLieSeries[⟨u⟩]}, MakeCWSeries[0]];
ρ⁻[u_, x_] := M[{x → MakeLieSeries[-⟨u⟩]}, MakeCWSeries[0]];
```

udefs

```
R⁺[a_, b_] := ρ⁺[a, b] ** hϵ[a]; R⁻[a_, b_] := ρ⁻[a, b] ** hϵ[a];
dm[a_, b_, c_][μ_] := μ // tha[⟨a⟩, b] // tm[⟨a⟩, ⟨b⟩, ⟨c⟩] // hm[a, b, c];
```

Testing Properties and Relations

Testing tm

Testing_tm

```
Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩};
   1 → (t1 = M[{
         x → MakeLieSeries[u + v + w], y → MakeLieSeries[b[u, v] + b[v, w]]
       }, MakeCWSeries[CW["uvw"]]]),
   2 → (t1 // tm[u, v, u]),
   3 → (t2 = t1 // tm[u, v, u] // tm[u, w, u]),
   4 → (t1 // tm[v, w, v]),
   5 → (t3 = t1 // tm[v, w, v] // tm[u, v, u]),
   6 → (t2 ≡ t3)};
```

Testing_tm

$1 \to M\left[\left\{x \to LS\left[\overline{u}+\overline{v}+\overline{w},\ 0,\ 0\right],\ y \to LS\left[0,\ \overline{u\,v}+\overline{v\,w},\ 0\right]\right\},\ CWS\left[0,\ 0,\ \overset{\frown}{uvw}\right]\right]$

Testing_tm

$2 \to M\left[\left\{x \to LS\left[2\,\overline{u}+\overline{w},\ 0,\ 0\right],\ y \to LS\left[0,\ \overline{u\,w},\ 0\right]\right\},\ CWS\left[0,\ 0,\ \overset{\frown}{uuw}\right]\right]$

Testing_tm

$3 \to M\left[\left\{x \to LS\left[3\,\overline{u},\ 0,\ 0\right],\ y \to LS[0,\ 0,\ 0]\right\},\ CWS\left[0,\ 0,\ \overset{\frown}{uuu}\right]\right]$

Testing_tm

$4 \to M\left[\left\{x \to LS\left[\overline{u}+2\,\overline{v},\ 0,\ 0\right],\ y \to LS\left[0,\ \overline{u\,v},\ 0\right]\right\},\ CWS\left[0,\ 0,\ \overset{\frown}{uvv}\right]\right]$

Testing_tm

$5 \to M\left[\left\{x \to LS\left[3\,\overline{u},\ 0,\ 0\right],\ y \to LS[0,\ 0,\ 0]\right\},\ CWS\left[0,\ 0,\ \overset{\frown}{uuu}\right]\right]$

Testing_tm

$6 \to \text{True}$

Testing hm

Testing_hm

```
Print /@ {
   1 → (t1 = ρ⁺[u, x] ρ⁺[v, y] ρ⁺[w, z]),
   2 → (t1 // hm[x, y, x]),
   3 → (t2 = t1 // hm[x, y, x] // hm[x, z, x]),
   4 → (t1 // hm[y, z, y]),
   5 → (t3 = t1 // hm[y, z, y] // hm[x, y, x]),
   6 → (t2 ≡ t3)};
```

Testing_hm

$1 \to M\left[\left\{x \to LS\left[\overline{u},\ 0,\ 0\right],\ y \to LS\left[\overline{v},\ 0,\ 0\right],\ z \to LS\left[\overline{w},\ 0,\ 0\right]\right\},\ CWS[0,\ 0,\ 0]\right]$

Testing_hm

$2 \to M\left[\left\{x \to LS\left[\overline{u}+\overline{v},\ \frac{\overline{u\,v}}{2},\ \frac{1}{12}\,\overline{u\,\overline{u\,v}}+\frac{1}{12}\,\overline{\overline{u\,v}\,v}\right],\ z \to LS\left[\overline{w},\ 0,\ 0\right]\right\},\ CWS[0,\ 0,\ 0]\right]$

Testing_hm

$3 \to M\left[\left\{x \to LS\left[\overline{u}+\overline{v}+\overline{w},\ \frac{\overline{u\,v}}{2}+\frac{\overline{u\,w}}{2}+\frac{\overline{v\,w}}{2},\ \frac{1}{12}\,\overline{u\,\overline{u\,v}}+\frac{1}{12}\,\overline{u\,\overline{u\,w}}+\frac{1}{3}\,\overline{u\,\overline{v\,w}}+\frac{1}{12}\,\overline{v\,\overline{v\,w}}+\frac{1}{12}\,\overline{\overline{u\,v}\,v}+\frac{1}{6}\,\overline{\overline{u\,w}\,v}+\frac{1}{12}\,\overline{\overline{u\,w}\,w}+\frac{1}{12}\,\overline{\overline{v\,w}\,w}\right]\right\},\ CWS[0,\ 0,\ 0]\right]$

Testing_hm

$4 \to M\left[\left\{x \to LS\left[\overline{u},\ 0,\ 0\right],\ y \to LS\left[\overline{v}+\overline{w},\ \frac{\overline{v\,w}}{2},\ \frac{1}{12}\,\overline{v\,\overline{v\,w}}+\frac{1}{12}\,\overline{\overline{v\,w}\,w}\right]\right\},\ CWS[0,\ 0,\ 0]\right]$

Testing_hm

$5 \to M\left[\left\{x \to LS\left[\overline{u}+\overline{v}+\overline{w},\ \frac{\overline{u\,v}}{2}+\frac{\overline{u\,w}}{2}+\frac{\overline{v\,w}}{2},\ \frac{1}{12}\,\overline{u\,\overline{u\,v}}+\frac{1}{12}\,\overline{u\,\overline{u\,w}}+\frac{1}{3}\,\overline{u\,\overline{v\,w}}+\frac{1}{12}\,\overline{v\,\overline{v\,w}}+\frac{1}{12}\,\overline{\overline{u\,v}\,v}+\frac{1}{6}\,\overline{\overline{u\,w}\,v}+\frac{1}{12}\,\overline{\overline{u\,w}\,w}+\frac{1}{12}\,\overline{\overline{v\,w}\,w}\right]\right\},\ CWS[0,\ 0,\ 0]\right]$

Testing_hm

$6 \to \text{True}$

Testing t-action

taction

```
Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩, t = ⟨"t"⟩};
   1 → (t1 = M[{
        x → MakeLieSeries[u + b[u, t]], y → MakeLieSeries[u + b[u, t]]
      }, MakeCWSeries[CW["uu"] + CW["tuv"]]]),
   2 → (t2 = t1 // tm[u, v, w] // tha[w, x]),
   3 → (t3 = t1 // tha[u, x] // tha[v, x] // tm[u, v, w]),
   4 → (t2 ≡ t3)};
```

taction

$$1 \to M\left[\left\{x \to LS\left[\overline{u}, -\overline{tu}, 0\right], y \to LS\left[\overline{u}, -\overline{tu}, 0\right]\right\}, CWS\left[0, \widehat{uu}, \widehat{tuv}\right]\right]$$

taction

$$2 \to M\left[\left\{x \to LS\left[\overline{w}, -\overline{tw}, -\overline{\overline{tw}w}\right], y \to LS\left[\overline{w}, -\overline{tw}, -\overline{\overline{tw}w}\right]\right\}, CWS\left[\widehat{w}, -\widehat{tw} + \widehat{ww}, \frac{3\,\widehat{tww}}{2}\right]\right]$$

taction

$$3 \to M\left[\left\{x \to LS\left[\overline{w}, -\overline{tw}, -\overline{\overline{tw}w}\right], y \to LS\left[\overline{w}, -\overline{tw}, -\overline{\overline{tw}w}\right]\right\}, CWS\left[\widehat{w}, -\widehat{tw} + \widehat{ww}, \frac{3\,\widehat{tww}}{2}\right]\right]$$

taction

4 → True

Testing h-action

haction

```
Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩};
   1 → (t1 = M[{
        x → MakeLieSeries[u + b[u, v]], y → MakeLieSeries[v + b[u, v]]
      }, MakeCWSeries[CW["uu"] + CW["uvv"]]]),
   2 → (t2 = t1 // hm[x, y, z] // tha[u, z]),
   3 → (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]),
   4 → (t2 ≡ t3)};
```

haction

$$1 \to M\left[\left\{x \to LS\left[\overline{u}, \overline{uv}, 0\right], y \to LS\left[\overline{v}, \overline{uv}, 0\right]\right\}, CWS\left[0, \widehat{uu}, \widehat{uvv}\right]\right]$$

haction

$$2 \to M\left[\left\{z \to LS\left[\overline{u} + \overline{v}, \frac{3\,\overline{uv}}{2}, -\frac{17}{12}\overline{u\overline{uv}} - \frac{17}{12}\overline{\overline{uv}v}\right]\right\}, CWS\left[\widehat{u}, \widehat{uu} - 2\,\widehat{uv}, \frac{\widehat{uuv}}{2} + \frac{\widehat{uvv}}{2}\right]\right]$$

haction

$$3 \to M\left[\left\{z \to LS\left[\overline{u} + \overline{v}, \frac{3\,\overline{uv}}{2}, -\frac{17}{12}\overline{u\overline{uv}} - \frac{17}{12}\overline{\overline{uv}v}\right]\right\}, CWS\left[\widehat{u}, \widehat{uu} - 2\,\widehat{uv}, \frac{\widehat{uuv}}{2} + \frac{\widehat{uvv}}{2}\right]\right]$$

haction

4 → True

Testing the Conjugation Relation

TestingConjugationRelation

```
Print /@ {
   1 → (t1 = ρ⁺[u, x] ρ⁺[v, y] ρ⁺[w, z]),
   2 → (t2 = t1 // tm[v, w, v] // hm[x, y, x] // tha[u, z]),
   3 → (t3 = ρ⁺[v, x] ρ⁺[w, z] ρ⁺[u, y]),
   4 → (t4 = t3 // tm[v, w, v] // hm[x, y, x]),
   5 → (t2 ≡ t4)};
```

TestingConjugationRelation

$1 \to M\left[\left\{x \to LS\left[\overline{u}, 0, 0\right], y \to LS\left[\overline{v}, 0, 0\right], z \to LS\left[\overline{w}, 0, 0\right]\right\}, CWS[0, 0, 0]\right]$

TestingConjugationRelation

$2 \to M\left[\left\{x \to LS\left[\overline{u} + \overline{v}, -\frac{\overline{uv}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{\overline{uv}v}\right], z \to LS\left[\overline{v}, 0, 0\right]\right\}, CWS[0, 0, 0]\right]$

TestingConjugationRelation

$3 \to M\left[\left\{x \to LS\left[\overline{v}, 0, 0\right], y \to LS\left[\overline{u}, 0, 0\right], z \to LS\left[\overline{w}, 0, 0\right]\right\}, CWS[0, 0, 0]\right]$

TestingConjugationRelation

$4 \to M\left[\left\{x \to LS\left[\overline{u} + \overline{v}, -\frac{\overline{uv}}{2}, \frac{1}{12}\overline{u\overline{uv}} + \frac{1}{12}\overline{\overline{uv}v}\right], z \to LS\left[\overline{v}, 0, 0\right]\right\}, CWS[0, 0, 0]\right]$

TestingConjugationRelation

$5 \to$ True

Demo 1 - The Knot 8_{17}

817-1

```
μ1 = R⁻[12, 1] R⁻[2, 7] R⁻[8, 3] R⁻[4, 11] R⁺[16, 5] R⁺[6, 13] R⁺[14, 9] R⁺[10, 15]
```

817-1

$M\left[\left\{1 \to LS[-\overline{c}, 0, 0], 2 \to LS[0, 0, 0], 3 \to LS\left[-\overline{8}, 0, 0\right], 4 \to LS[0, 0, 0], 5 \to LS[\overline{g}, 0, 0], 6 \to LS[0, 0, 0], 7 \to LS\left[-\overline{2}, 0, 0\right], 8 \to LS[0, 0, 0], 9 \to LS[\overline{e}, 0, 0], 10 \to LS[0, 0, 0], 11 \to LS\left[-\overline{4}, 0, 0\right], 12 \to LS[0, 0, 0], 13 \to LS\left[\overline{6}, 0, 0\right], 14 \to LS[0, 0, 0], 15 \to LS[\overline{a}, 0, 0], 16 \to LS[0, 0, 0]\right\}, CWS[0, 0, 0]\right]$

817-2

```
Do[μ1 = μ1 // dm[1, k, 1], {k, 2, 16}];
Last[μ1]@{6}
```

817-2

$CWS\left[0, -\widehat{11}, 0, -\frac{31\,\widehat{1111}}{12}, 0, -\frac{1351\,\widehat{111111}}{360}\right]$

817-3

```
Series[Log[-1/X^3 + 4/X^2 - 8/X + 11 - 8 X + 4 X^2 - X^3 /. X → ⅇ^x], {x, 0, 6}]
```

817-3

$-x^2 - \frac{31 x^4}{12} - \frac{1351 x^6}{360} + O[x]^7$

Demo 2 - The Borromean Tangle

Borromean1

```
μ2 = R⁻[r, 6] R⁺[2, 4] R⁻[g, 9] R⁺[5, 7] R⁻[b, 3] R⁺[8, 1];
(Do[μ2 = μ2 // dm[r, k, r], {k, 1, 3}]; Do[μ2 = μ2 // dm[g, k, g], {k, 4, 6}];
 Do[μ2 = μ2 // dm[b, k, b], {k, 7, 9}]; μ2)
```

Borromean1

$M\left[\left\{b \to LS\left[0, \overline{gr}, \frac{1}{2}\overline{g\overline{gr}} + \overline{\overline{br}g} + \frac{1}{2}\overline{\overline{gr}r}\right], g \to LS\left[0, -\overline{br}, \frac{1}{2}\overline{b\overline{br}} - \overline{b\overline{gr}} - \overline{\overline{br}g} + \frac{1}{2}\overline{\overline{br}r}\right], r \to LS\left[0, \overline{bg}, \frac{1}{2}\overline{b\overline{bg}} + \overline{b\overline{gr}} + \frac{1}{2}\overline{\overline{bg}g}\right]\right\}, CWS\left[0, 0, 2\,\widehat{bgr}\right]\right]$

Borromean2

```
(r /. First[μ2])@{5}
```

Borromean2

$$\mathrm{LS}\Big[0,\ \overline{bg},\ \frac{1}{2}\overline{b\overline{bg}}+\overline{b\overline{gr}}+\frac{1}{2}\overline{\overline{bg}g},$$
$$\frac{1}{6}\overline{b\overline{b\overline{bg}}}+\frac{1}{2}\overline{b\overline{b\overline{gr}}}+\frac{1}{2}\overline{b\overline{g\overline{gr}}}+\frac{1}{4}\overline{b\overline{\overline{bg}g}}+\frac{1}{2}\overline{b\overline{\overline{gr}r}}+\frac{1}{6}\overline{\overline{\overline{bg}g}g},$$
$$\frac{1}{24}\overline{b\overline{b\overline{b\overline{bg}}}}+\frac{1}{6}\overline{b\overline{b\overline{b\overline{gr}}}}+\frac{1}{4}\overline{b\overline{b\overline{g\overline{gr}}}}+\frac{1}{12}\overline{b\overline{b\overline{\overline{bg}g}}}+\frac{1}{4}\overline{b\overline{b\overline{\overline{gr}r}}}+$$
$$\frac{1}{6}\overline{b\overline{g\overline{g\overline{gr}}}}+\frac{1}{4}\overline{b\overline{g\overline{\overline{gr}r}}}-\overline{b\overline{\overline{b\overline{gr}}g}}+\frac{1}{12}\overline{b\overline{\overline{\overline{bg}g}g}}-2\,\overline{b\overline{\overline{\overline{br}g}g}}+\frac{1}{6}\overline{b\overline{\overline{\overline{gr}r}r}}+$$
$$\frac{1}{2}\overline{\overline{bg}\,\overline{b\overline{gr}}}-\overline{\overline{bg}\,\overline{\overline{br}g}}-\frac{1}{12}\overline{\overline{b\overline{bg}}\,\overline{bg}}-\frac{1}{2}\overline{\overline{b\overline{gr}}\,\overline{gr}}+\frac{1}{24}\overline{\overline{\overline{\overline{bg}g}g}g}\Big]$$

Borromean3

```
Last[μ2]@{5}
```

Borromean3

$$\mathrm{CWS}\Big[0,\ 0,\ 2\,\widehat{bgr},\ \widehat{bbgr}-\widehat{bgbr}+\widehat{bggr}-\widehat{bgrg}+\widehat{bgrr}-\widehat{brgr},$$
$$\frac{\widehat{bbbgr}}{3}-\frac{\widehat{bbgbr}}{2}+\frac{\widehat{bbggr}}{2}+\frac{\widehat{bbgrg}}{2}+\frac{\widehat{bbgrr}}{2}+\frac{\widehat{bbrbg}}{2}-\frac{3\,\widehat{bbrgr}}{2}+\frac{\widehat{bgbrr}}{2}-\frac{3\,\widehat{bggbr}}{2}+$$
$$\frac{\widehat{bgggr}}{3}-\frac{\widehat{bggrg}}{2}+\frac{\widehat{bggrr}}{2}+\frac{\widehat{bgrgg}}{2}-\frac{3\,\widehat{bgrrg}}{2}+\frac{\widehat{bgrrr}}{3}+\frac{\widehat{brggr}}{2}-\frac{\widehat{brgrr}}{2}+\frac{\widehat{brrgr}}{2}\Big]$$

Further Runs for the KBH paper

ExampleZeta

```
T0 = R⁻[3, a] R⁺[b, 2] R⁺[1, 4];
T0 // dm[2, 1, 1] // dm[4, b, b] // dm[1, a, a] // dm[3, a, a]
```

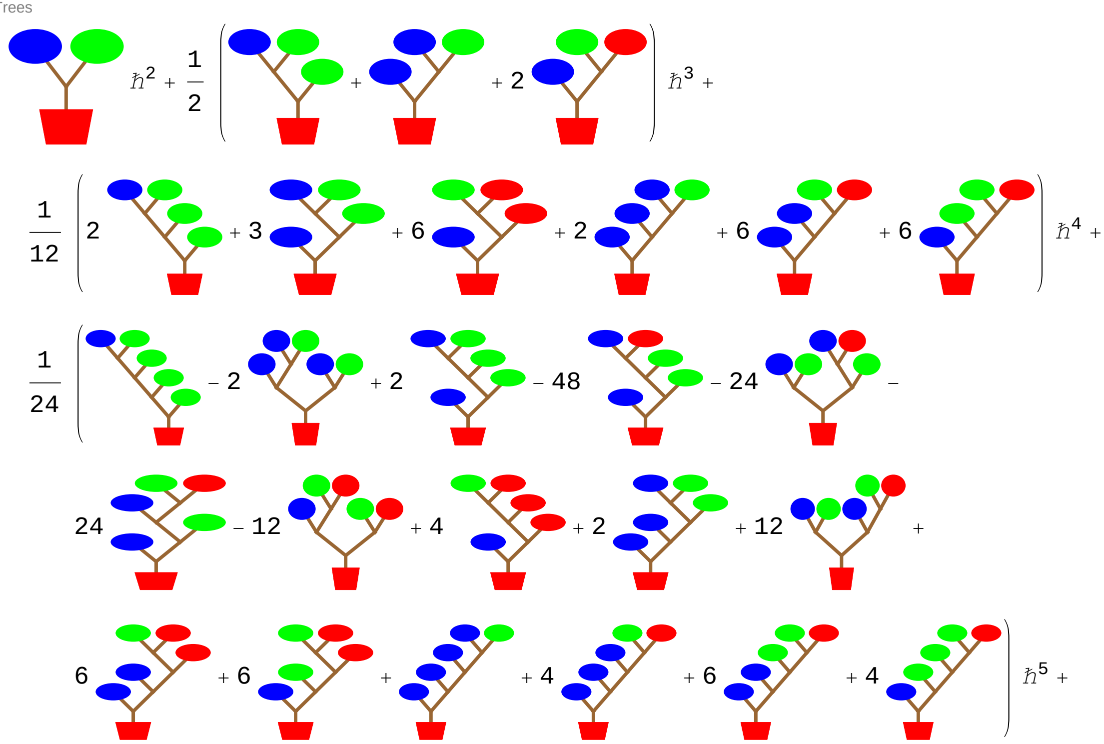
ExampleZeta

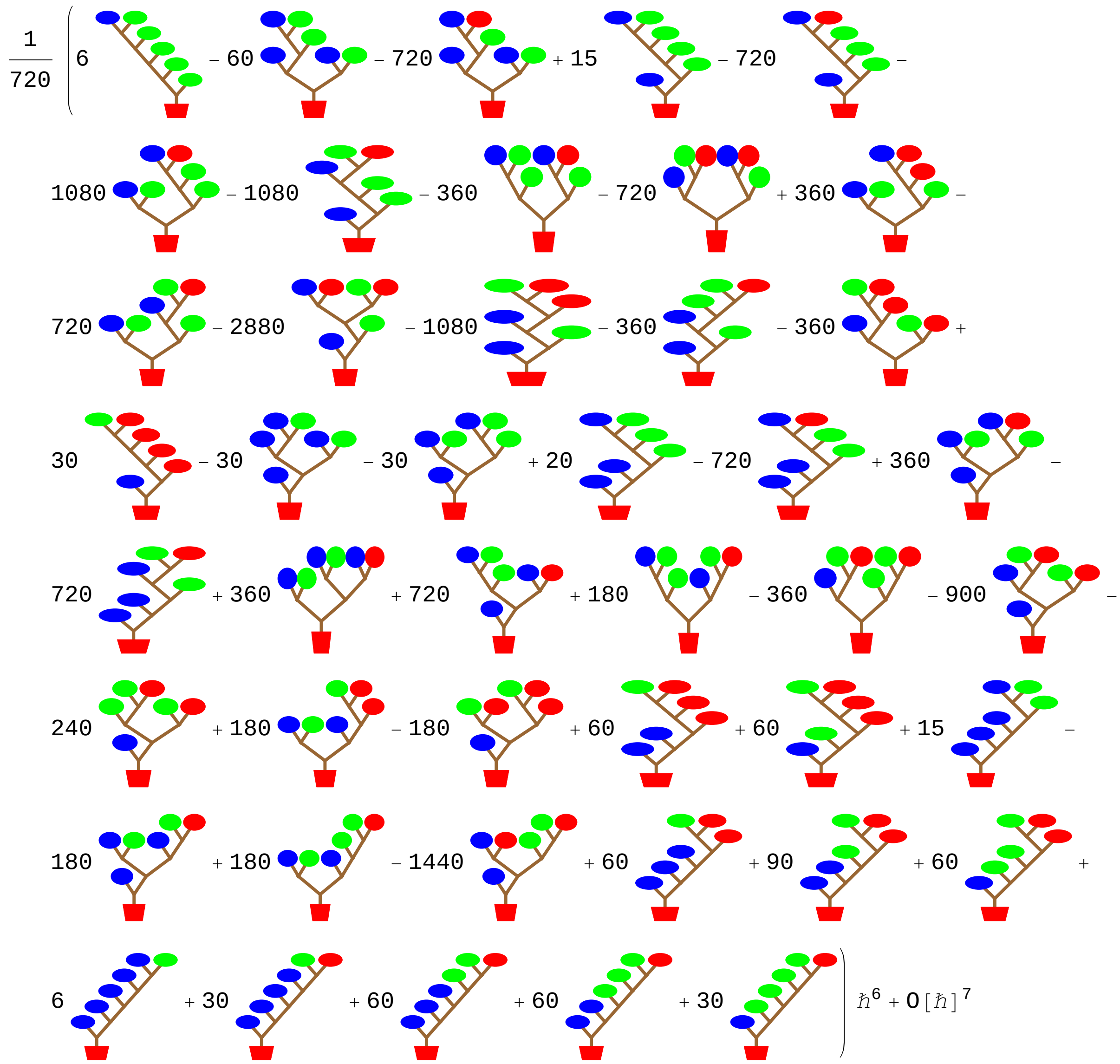
$$\mathrm{M}\Big[\Big\{a\to\mathrm{LS}\Big[-\overline{a}+\overline{b},\ \frac{3\,\overline{ab}}{2},\ \frac{13}{12}\overline{a\overline{ab}}-\frac{13}{12}\overline{\overline{ab}b}\Big],\ b\to\mathrm{LS}\Big[\overline{a},\ 0,\ -\overline{a\overline{ab}}\Big]\Big\},$$
$$\mathrm{CWS}\Big[-\widehat{a},\ -\widehat{ab},\ -\frac{\widehat{aab}}{2}-\frac{\widehat{abb}}{2}\Big]\Big]$$

BorromeanTrees

```
trees = Table[(r /. First[μ2]) @k, {k, 6}];
t1 = Series[(List @@ trees //. w_LW :→ B @@ Reverse[LyndonFactorization[w]] /.
        B[s_] :→ s /. t_B :→ Tree[t]).ħ^Range[Length[trees]],
    {ħ, 0, Length[trees]}
  ] /. {"r" → r, "g" → g, "b" → b};
t1 /. t_Tree :→ TreeForm[t,
    VertexRenderingFunction → (Switch[#2,
        Tree, {
         Red,
         Polygon[
          {{-0.4, 0.4} - #1, {0.4, 0.4} - #1, {0.3, -0.4} - #1, {-0.3, -0.4} - #1}]
        },
        B, {},
        _, {
         ReleaseHold[#2 /. {r → Red, g → Green, b → Blue}],
         Disk[-#1, 0.4]
        }
      ] &),
    EdgeRenderingFunction → ({
        Brown, Thickness[0.03],
        Line[-#]
      } &),
    PlotRangePadding → 0, ImageSize → 60, AspectRatio → 1
  ]
```

BorromeanTrees

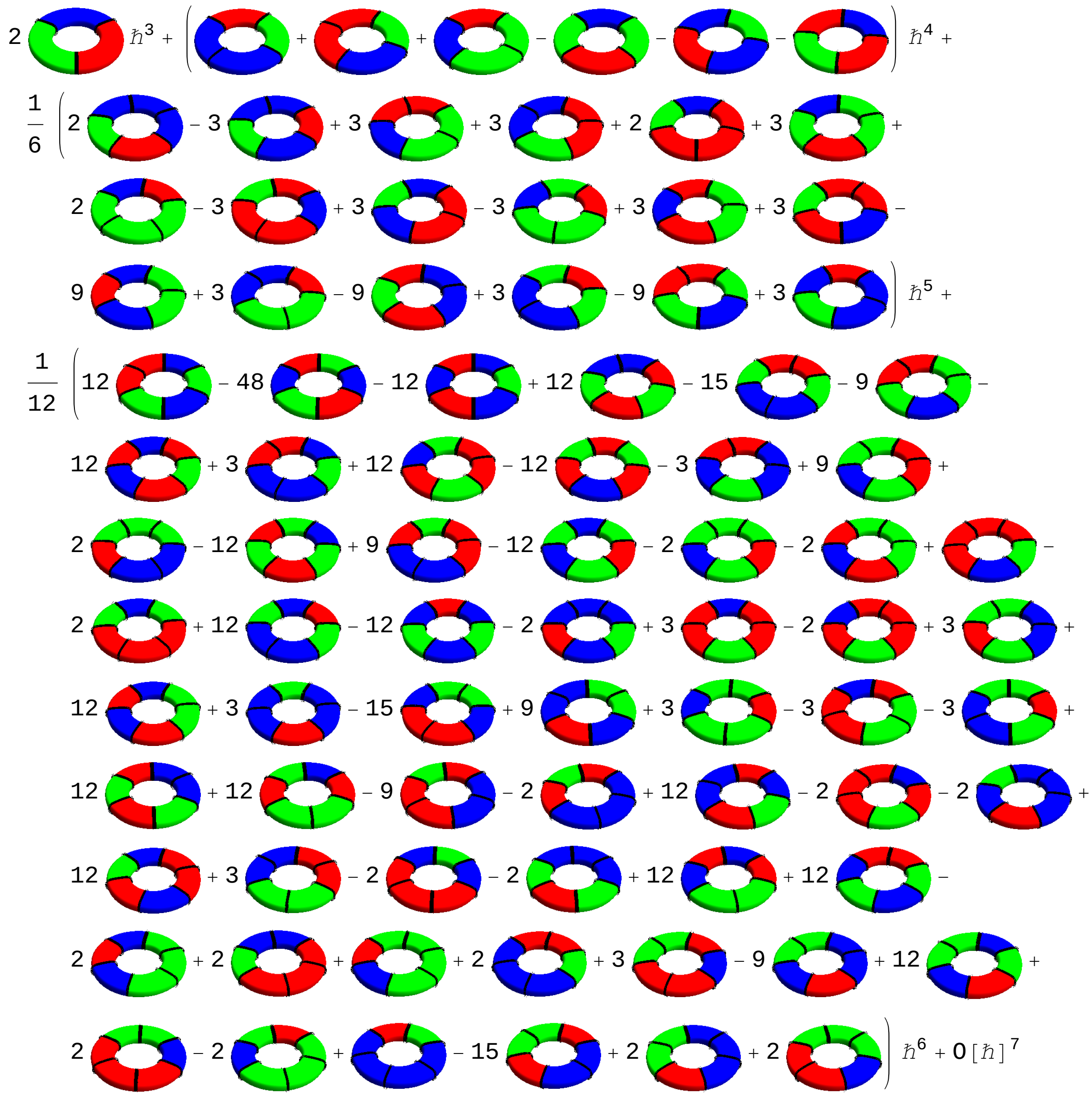




BorromeanWheels

```
n = 6;
wheels = Table[Last[μ2]@k, {k, n}];
SetOptions[Rasterize, {RasterSize → 256, ImageSize → 256}];
Collect[
  Expand[(Plus @@ wheels)] /.
   CW[s_String] :→ ℏ^StringLength[s] Show[ImageCrop[PieChart3D[
        Table[1, {StringLength[s]}],
        ChartStyle :→ (Characters[s] /. {"r" → Red, "g" → Green, "b" → Blue}),
        SectorOrigin → {{RandomReal[{0, 2 π}], "Counterclockwise"}, 1},
        ChartBaseStyle → EdgeForm[{Thickness[0.03], Black}],
        ChartElementFunction → "ProfileSector3D",
        ImagePadding → 0, ImageMargins → 0, PlotRangePadding → 0
       ]], ImageSize → 52],
  ℏ, Factor] + O[ℏ]^(n+1)
```

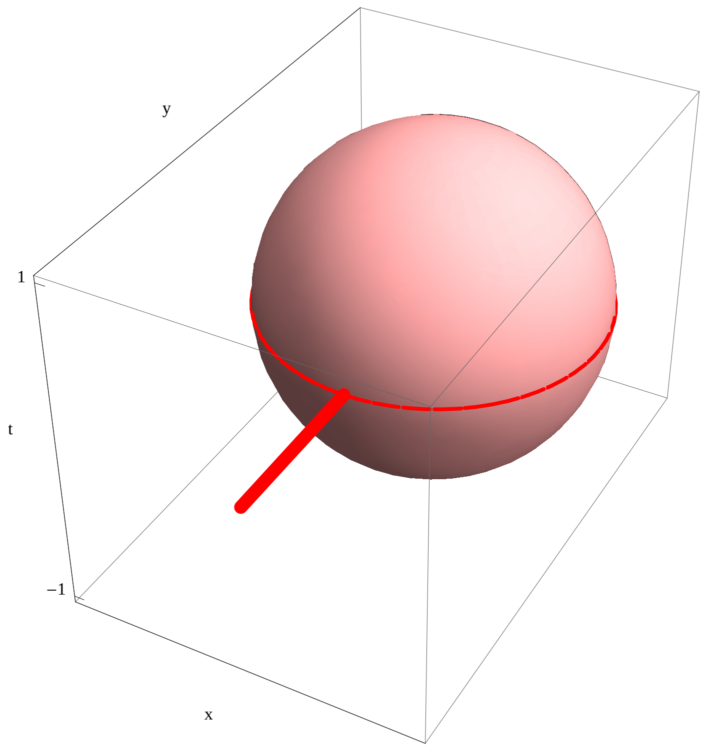
BorromeanWheels



TheTrivialBalloon

```
Graphics3D[{
  Red, Thickness[0.02], Line[{{0, -2, 0}, {0, -1, 0}}],
  Thick, Line[Table[
    {Cos[θ], Sin[θ], 0},
    {θ, 0, 2 π, 2 π / 72}
   ]],
  RGBColor[1, 0.65, 0.65], Sphere[{0, 0, 0}, 1]
 },
 Axes → True, AxesLabel → {"x", "y", "t"}, (* LabelStyle → Directive[Large],*)
 Ticks → {{}, {}, {-1, 1}},
 Lighting → "Neutral"
]
```

TheTrivialBalloon



eab

$$e[a\_, b\_] := \text{MatrixExp}\left[\begin{pmatrix} b\, c_\beta & -a\, c_\beta \\ -b\, c_\alpha & a\, c_\alpha \end{pmatrix}\right];$$

$$e[1, 0].e[0, 1] == e\left[\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \frac{e^{c_\alpha} - 1}{c_\alpha}, \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta}\right] \text{ // Simplify}$$

eab

True