Pensieve header: Exponentiation in ybax algebras.

Startup

In[∘]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 2;
<< Engine.m
<< Objects.m
<< KT.m
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Out[∘]= {2021, 8, 13, 15, 3, 21.4316886}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

Exponentials

Task. Define $\mathrm{Exp}_m[\mathrm{U\_}(\mathbb{U}_{\{\_\_\_\}\to\{\mathrm{i\_}\}}]$ to compute $e^{\mathbb{O}(U)}$ to order $\epsilon^{\mathrm{Length@}\{U\}-1}$ using the $m_{i,i\to i}$ multiplication, where U is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.
Example: $\mathrm{Exp}_{\mathrm{dm},1}[\mathbb{U}_{\{\}\to\{2\}}[b_2\,a_2 + y_2\,x_2,\,0]]$ is the exponential of the arrow on strand 2, computed to degree 1.

In[ ]:=
```
Exp_m_[𝕌_{is_→{i_}}[U__]] :=
 Module[{λ, k, n, F, f, j, lhs, rhs, sol, MI (*multi-index*), mis, mi, yax},
  MI /: Coefficient[ℰ_, MI[p_, n_, q_]] :=
   Coefficient[Coefficient[Coefficient[ℰ, y_i, p], a_i, n], x_i, q];
  yax /: yax^MI[p_,n_,q_] := y_i^p a_i^n x_i^q;
  F = 𝔼_{}→{i}[f[λ] a_i];
  lhs =
   (∂_μ First[F (F /. {λ → μ, i → j}) // m_{i,j→i}]) /. μ → 0 /. {f[0] → 0, f'[0] → ∂_{a_i} {U}[[1]]};
  rhs = (∂_μ First[F /. λ → λ + μ]) /. μ → 0;
  F = F /. First@DSolve[lhs == rhs ∧ f[0] == 0, f, λ];
  mis = {MI[0, 0, 0], MI[1, 0, 0], MI[0, 0, 1], MI[1, 0, 1]};
  F[[1]] += Sum[f_mi[λ] yax^mi, {mi, mis}];
  lhs = (∂_μ U2l@Last[F (F /. {λ → μ, i → j}) // m_{i,j→i}]) /. μ → 0 /. f_[0] → 0 /.
    Table[f_mi'[0] → Coefficient[{U}[[1]], mi], {mi, mis}];
  rhs = (∂_μ Last[F /. λ → λ + μ]) /. μ → 0;
  F = F /. First@DSolve[Table[Coefficient[lhs - rhs, mi] == 0 ∧ f_mi[0] == 0, {mi, mis}],
      Table[f_mi, {mi, mis}], λ];
  Do[
   mis =
    Flatten@Table[MI[p, n, q], {n, 0, 2 k + 2, 2}, {p, 0, 2 k + 2 - 2 n}, {q, 0, 2 k + 2 - 2 n - p}];
   AppendTo[F, Sum[f_mi[λ] yax^mi, {mi, mis}]];
   lhs = (∂_μ U2l@Last[F (F /. {λ → μ, i → j}) // m_{i,j→i}]) /. μ → 0 /. f_[0] → 0 /.
     Table[f_mi'[0] → Coefficient[{U}[[k + 1]], mi], {mi, mis}];
   rhs = (∂_μ Last[F /. λ → λ + μ]) /. μ → 0;
   F = F /. First@DSolve[Table[Coefficient[lhs - rhs, mi] == 0 ∧ f_mi[0] == 0, {mi, mis}],
       Table[f_mi, {mi, mis}], λ],
   {k, Length[{U}] - 1}
  ];
  CF@l2U[F /. λ → 1]
 ]
```

In[ ]:=
```
Exp_cm[𝕌_{}→{i}[ℏ a_i b_i + ℏ x_i y_i, c_1 (x_i + y_i), 0]]
```

Out[ ]=
$$\mathbb{E}_{\{\}\to\{i\}}\Big[\hbar\, a_i b_i - \frac{(-1+B_i)\, x_i y_i}{b_i},\ c_1 x_i + c_1 y_i - \frac{\hbar\,(-1+B_i)\, x_i y_i}{b_i} + \frac{\left(1 - 4 B_i + 3 B_i^2 + 2\hbar\, b_i B_i^2\right) x_i^2 y_i^2}{4 b_i^3},$$
$$\frac{1}{2} b_i c_1^2 + \frac{1}{2}\hbar\, c_1 x_i + \frac{1}{2}\hbar\, c_1 y_i - \frac{\hbar^2 (-1+B_i)\, x_i y_i}{2 b_i} - \frac{a_i^2\left(-2 + 2 B_i + 2\hbar\, b_i B_i + \hbar^2 b_i^2 B_i\right) x_i y_i}{2 b_i^3} -$$
$$\frac{\left(-3 + 2\hbar\, b_i + 4 B_i - B_i^2\right) c_1 x_i y_i^2}{4\hbar\, b_i^3} + \frac{\left(1 + 2\hbar\, b_i - 8 B_i - 8\hbar\, b_i B_i + 7 B_i^2 + 12\hbar\, b_i B_i^2 + 6\hbar^2 b_i^2 B_i^2\right) x_i^2 y_i^2}{4 b_i^4} -$$
$$\frac{\left(-2 + 15 B_i - 30 B_i^2 - 12\hbar\, b_i B_i^2 + 17 B_i^3 + 18\hbar\, b_i B_i^3 + 6\hbar^2 b_i^2 B_i^3\right) x_i^3 y_i^3}{12 b_i^5}\Big]$$

Step by step

In[∘]:= U = $\mathbb{U}_{\{\}\to\{i\}}[0, c_1 (x_i + y_i), c_2 x_i^2 y_i]$

Out[∘]= $\mathbb{U}_{\{\}\to\{i\}}[0, c_1 (x_i + y_i), c_2 x_i^2 y_i]$

In[∘]:= m = cm; U = $\mathbb{U}_{\{\}\to\{i\}}[\hbar a_i b_i + \hbar x_i y_i, c_1 (x_i + y_i), c_2 x_i^2 y_i]$

Out[∘]= $\mathbb{U}_{\{\}\to\{i\}}[\hbar a_i b_i + \hbar x_i y_i, c_1 (x_i + y_i), c_2 x_i^2 y_i]$

In[∘]:= Module[{λ, k, F, f, i, j, lhs, rhs, sol, MI (*multi-index*), mis, mi, yax},]

In[∘]:= F = $\mathbb{E}_{\{\}\to\{i\}}[f[\lambda]\, a_i]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}[f[\lambda]\, a_i]$

In[∘]:= lhs = $(\partial_\mu$ First[F (F /. {λ → μ, i → j}) // $m_{i,j\to i}$]) /. μ → 0 /. {f[0] → 0, f'[0] → ∂_{a_i} U⟦1⟧}

Out[∘]= $\hbar a_i b_i$

In[∘]:= rhs = (∂_μ First[F /. λ → λ + μ]) /. μ → 0

Out[∘]= $a_i f'[\lambda]$

In[∘]:= F = F /. First@DSolve[lhs == rhs ∧ f[0] == 0, f, λ]

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}[\lambda \hbar a_i b_i]$

In[∘]:= mis = {MI[0, 0, 0], MI[1, 0, 0], MI[0, 0, 1], MI[1, 0, 1]}

Out[∘]= {MI[0, 0, 0], MI[1, 0, 0], MI[0, 0, 1], MI[1, 0, 1]}

In[∘]:= MI /: Coefficient[ℰ_, MI[p_, m_, q_]] :=
Coefficient[Coefficient[Coefficient[ℰ, y_i, p], a_i, m], x_i, q];
yax /: $\text{yax}^{\text{MI}[p\_,m\_,q\_]}$:= $y_i^p a_i^m x_i^q$

In[∘]:= F⟦-1⟧ += Sum[$f_{mi}[\lambda]\, \text{yax}^{mi}$, {mi, mis}]

Out[∘]= $\lambda \hbar a_i b_i + f_{\text{MI}[0,0,0]}[\lambda] + x_i f_{\text{MI}[0,0,1]}[\lambda] + y_i f_{\text{MI}[1,0,0]}[\lambda] + x_i y_i f_{\text{MI}[1,0,1]}[\lambda]$

In[∘]:= lhs = (∂_μ U2l@Last[F (F /. {λ → μ, i → j}) // $m_{i,j\to i}$]) /. μ → 0 /. f_[0] → 0 /.
Table[f_{mi}'[0] → Coefficient[U⟦1⟧, mi], {mi, mis}]

Out[∘]= $\hbar a_i b_i + e^{-\lambda \hbar b_i} \hbar x_i y_i$

In[∘]:= rhs = (∂_μ Last[F /. λ → λ + μ]) /. μ → 0

Out[∘]= $\hbar a_i b_i + f_{\text{MI}[0,0,0]}'[\lambda] + x_i f_{\text{MI}[0,0,1]}'[\lambda] + y_i f_{\text{MI}[1,0,0]}'[\lambda] + x_i y_i f_{\text{MI}[1,0,1]}'[\lambda]$

In[∘]:= F = F /. First@DSolve[
Table[Coefficient[lhs - rhs, mi] == 0 ∧ f_{mi}[0] == 0, {mi, mis}], Table[f_{mi}, {mi, mis}], λ]

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[\lambda \hbar a_i b_i + \frac{e^{-\lambda \hbar b_i}\left(-1 + e^{\lambda \hbar b_i}\right) x_i y_i}{b_i}\right]$

In[•]:=
```
Do[
  mis =
   Flatten@Table[MI[p, m, q], {m, 0, 2 k + 2, 2}, {p, 0, 2 k + 2 - 2 m}, {q, 0, 2 k + 2 - 2 m - p}];
  AppendTo[F, Sum[f_mi[λ] yax^mi, {mi, mis}]];
  lhs = (∂_μ U2l@Last[F (F /. {λ → μ, i → j}) // m_{i,j→i}]) /. μ → 0 /. f_[0] → 0 /.
    Table[f_mi'[0] → Coefficient[U⟦k + 1⟧, mi], {mi, mis}];
  rhs = (∂_μ Last[F /. λ → λ + μ]) /. μ → 0;
  F = F /. First@DSolve[Table[Coefficient[lhs - rhs, mi] == 0 ∧ f_mi[0] == 0, {mi, mis}],
      Table[f_mi, {mi, mis}], λ],
  {k, Length[U] - 1}
 ];
F
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\Big[\lambda\hbar a_i b_i + \frac{e^{-\lambda\hbar b_i}\left(-1+e^{\lambda\hbar b_i}\right)x_i y_i}{b_i},$$

$$\lambda c_1 x_i + \lambda c_1 y_i + \frac{e^{-\lambda\hbar b_i}\left(-1+e^{\lambda\hbar b_i}\right)\lambda\hbar x_i y_i}{b_i} + \frac{e^{-2\lambda\hbar b_i}\left(3-4e^{\lambda\hbar b_i}+e^{2\lambda\hbar b_i}+2\lambda\hbar b_i\right)x_i^2 y_i^2}{4 b_i^3},$$

$$\frac{1}{2}\lambda^2 b_i c_1^2 + \frac{1}{2}\lambda^2\hbar c_1 x_i + \frac{1}{2}\lambda^2\hbar c_1 y_i + \frac{e^{-\lambda\hbar b_i}\left(-1+e^{\lambda\hbar b_i}\right)\lambda^2\hbar^2 x_i y_i}{2 b_i} +$$

$$\frac{e^{-\lambda\hbar b_i} a_i^2\left(-2+2e^{\lambda\hbar b_i}-2\lambda\hbar b_i-\lambda^2\hbar^2 b_i^2\right)x_i y_i}{2 b_i^3} + \lambda c_2 x_i^2 y_i -$$

$$\frac{e^{-2\lambda\hbar b_i}\left(-1+4e^{\lambda\hbar b_i}-3e^{2\lambda\hbar b_i}+2e^{2\lambda\hbar b_i}\lambda\hbar b_i\right)c_1 x_i y_i^2}{4\hbar b_i^3} +$$

$$\frac{e^{-2\lambda\hbar b_i}\left(7-8e^{\lambda\hbar b_i}+e^{2\lambda\hbar b_i}+12\lambda\hbar b_i-8e^{\lambda\hbar b_i}\lambda\hbar b_i+2e^{2\lambda\hbar b_i}\lambda\hbar b_i+6\lambda^2\hbar^2 b_i^2\right)x_i^2 y_i^2}{4 b_i^4} +$$

$$\frac{e^{-3\lambda\hbar b_i}\left(-17+30e^{\lambda\hbar b_i}-15e^{2\lambda\hbar b_i}+2e^{3\lambda\hbar b_i}-18\lambda\hbar b_i+12e^{\lambda\hbar b_i}\lambda\hbar b_i-6\lambda^2\hbar^2 b_i^2\right)x_i^3 y_i^3}{12 b_i^5}\Big]$$

In[•]:=
```
CF@l2U[F /. λ → 1]
```

Out[•]=
$$\mathbb{E}_{\{\}\to\{i\}}\Big[\hbar a_i b_i - \frac{(-1+B_i)x_i y_i}{b_i},\ c_1 x_i + c_1 y_i - \frac{\hbar(-1+B_i)x_i y_i}{b_i} + \frac{\left(1-4B_i+3B_i^2+2\hbar b_i B_i^2\right)x_i^2 y_i^2}{4 b_i^3},$$

$$\frac{1}{2}b_i c_1^2 + \frac{1}{2}\hbar c_1 x_i + \frac{1}{2}\hbar c_1 y_i - \frac{\hbar^2(-1+B_i)x_i y_i}{2 b_i} - \frac{a_i^2\left(-2+2B_i+2\hbar b_i B_i+\hbar^2 b_i^2 B_i\right)x_i y_i}{2 b_i^3} + c_2 x_i^2 y_i -$$

$$\frac{\left(-3+2\hbar b_i+4B_i-B_i^2\right)c_1 x_i y_i^2}{4\hbar b_i^3} + \frac{\left(1+2\hbar b_i-8B_i-8\hbar b_i B_i+7B_i^2+12\hbar b_i B_i^2+6\hbar^2 b_i^2 B_i^2\right)x_i^2 y_i^2}{4 b_i^4} -$$

$$\frac{\left(-2+15B_i-30B_i^2-12\hbar b_i B_i^2+17B_i^3+18\hbar b_i B_i^3+6\hbar^2 b_i^2 B_i^3\right)x_i^3 y_i^3}{12 b_i^5}\Big]$$

In[∘]:= **CF@l2U@F**

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\Big[\lambda\hbar\, a_i\, b_i - \frac{(-1+B_i^{\lambda})\, x_i\, y_i}{b_i},$

$\lambda\, c_1\, x_i + \lambda\, c_1\, y_i - \frac{\lambda\hbar\,(-1+B_i^{\lambda})\, x_i\, y_i}{b_i} + \frac{(1-4B_i^{\lambda}+3B_i^{2\lambda}+2\lambda\hbar\, b_i\, B_i^{2\lambda})\, x_i^2\, y_i^2}{4 b_i^3},$

$\frac{1}{2}\lambda^2 b_i\, c_1^2 + \frac{1}{2}\lambda^2\hbar\, c_1\, x_i + \frac{1}{2}\lambda^2\hbar\, c_1\, y_i - \frac{\lambda^2\hbar^2\,(-1+B_i^{\lambda})\, x_i\, y_i}{2 b_i} -$

$\frac{a_i^2\,(-2+2B_i^{\lambda}+2\lambda\hbar\, b_i\, B_i^{\lambda}+\lambda^2\hbar^2 b_i^2 B_i^{\lambda})\, x_i\, y_i}{2 b_i^3} + \lambda\, c_2\, x_i^2\, y_i - \frac{(-3+2\lambda\hbar\, b_i+4B_i^{\lambda}-B_i^{2\lambda})\, c_1\, x_i\, y_i^2}{4\hbar\, b_i^3} +$

$\frac{(1+2\lambda\hbar\, b_i-8B_i^{\lambda}-8\lambda\hbar\, b_i\, B_i^{\lambda}+7B_i^{2\lambda}+12\lambda\hbar\, b_i\, B_i^{2\lambda}+6\lambda^2\hbar^2 b_i^2 B_i^{2\lambda})\, x_i^2\, y_i^2}{4 b_i^4} -$

$\frac{(-2+15B_i^{\lambda}-30B_i^{2\lambda}-12\lambda\hbar\, b_i\, B_i^{2\lambda}+17B_i^{3\lambda}+18\lambda\hbar\, b_i\, B_i^{3\lambda}+6\lambda^2\hbar^2 b_i^2 B_i^{3\lambda})\, x_i^3\, y_i^3}{12 b_i^5}\Big]$

In[∘]:= **U2l[B_i]**

Out[∘]= $e^{-\hbar\, b_i}$