Pensieve header: Profile with encapsulation of Zip3-Inner. Time to K31@$k=3: 5816.8.

Startup

In[•]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
$k = 1;
<< Objects.m
<< KT.m
```

Out[•]= {2021, 1, 3, 8, 46, 22.7844035}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

Engine

Canonical Forms:

In[•]:=
```
CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@Together[ℰ]; (*Coefficient Canonical Form *)
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | a | x | η | β | τ | ξ)_, ∞] ∪ {y, a, x, η, β, τ, ξ}},
    Total[(CCF[#[[2]]] × (Times @@ vs^#[[1]])) & /@ CoefficientRules[ℰ, vs]]
   ];
CF[ℰ_List] := CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[𝔼_sp___[𝒮___]] := CF /@ 𝔼_sp[𝒮];
```

Variables and their duals:

In[•]:=
```
{t*, b*, y*, a*, x*, z*, τ*, β*, η*, α*, ξ*, ξ*} = {τ, β, η, α, ξ, ξ, t, b, y, a, x, z};
(vs_List)* := (v ↦ v*) /@ vs;
(u_i_)* := (u*)_i;
```

Weights:

In[•]:=
```
Clear[Wt];
Evaluate[Wt /@ {y, b, t, a, x, η, β, τ, α, ξ}] = {1, 0, 0, 2, 1, 1, 2, 2, 0, 1};
Wt[u_i_] := Wt[u];
```

The maximal weight $n, i.e. the *n* of gl(*n*). Initially and for a long while this will not be tested beyond $n == 2.

In[•]:=
```
$n = 2;
```

Upper to lower and lower to Upper:

```
U2l[ℰ_] := ℰ /. {B_i^p_. :> e^(-p ħ b_i), B^p_. :> e^(-p ħ b), T_i^p_. :> e^(p ħ t_i), T^p_. :> e^(p ħ t), 𝒜_i^p_. :> e^(p α_i), 𝒜^p_. :> e^(p α)};
l2U[ℰ_] := ℰ //. {e^(c_. b_i_ + d_.) :> B_i^(-c/ħ) e^d, e^(c_. b + d_.) :> B^(-c/ħ) e^d, e^(c_. t_i_ + d_.) :> T_i^(c/ħ) e^d, e^(c_. t + d_.) :> T^(c/ħ) e^d,
    e^(c_. α_i_ + d_.) :> 𝒜_i^c e^d, e^(c_. α + d_.) :> 𝒜^c e^d, e^𝒳_ :> e^Expand@𝒳};
l2U[r_Rule] := Module[{U = r[[1]] /. {b → B, t → T, α → 𝒜}}, U → l2U[U2l[U] /. r]];
AlsoUpper[rs_List] := rs ⋃ (l2U /@ rs);
```

Derivatives in the presence of exponentiated variables:

```
D_b[f_] := ∂_b f - ħ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ħ B_i ∂_B_i f;
D_t[f_] := ∂_t f + ħ T ∂_T f; D_t_i_[f_] := ∂_t_i f + ħ T_i ∂_T_i f;
D_α[f_] := ∂_α f + 𝒜 ∂_𝒜 f; D_α_i_[f_] := ∂_α_i f + 𝒜_i ∂_𝒜_i f;
D_ν_[f_] := ∂_ν f;
```

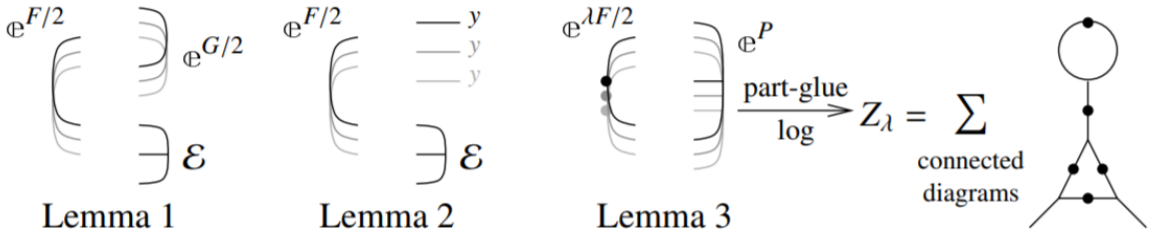
𝔼 operations:

```
ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
ℰ_𝔼[k_Integer] := ℰ[[k + 1]]; 𝔼_[ℰs___][k_Integer] := {ℰs}[[k + 1]];
𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
𝔼_(d1_→r1_)[ℰ1s___] ≡ 𝔼_(d2_→r2_)[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
𝔼_(d1_→r1_)[ℰ1s___] 𝔼_(d2_→r2_)[ℰ2s___] ^:= 𝔼_((d1⋃d2)→(r1⋃r2)) @@ (𝔼[ℰ1s] × 𝔼[ℰ2s]);
```

```
𝔼_(d1_→r1_)[ℰ1s___] // 𝔼_(d2_→r2_)[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs},
  lvs = Flatten@Table[{y_$@i, b_$@i, t_$@i, a_$@i, x_$@i}, {i, is}];
  𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ (Zip_lvs⋃lvs*[{lvs*.lvs, Times[
       𝔼[ℰ1s] /. Table[(v : b | B | t | T | a | x | y)_i → v_$@i, {i, is}],
       𝔼[ℰ2s] /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_$@i, {i, is}]
      ]}])
 ]
```

```
Λ2𝔼_(d_→r_)[Λ_] := Module[{k}, 𝔼_(d→r) @@ l2U@Table[SeriesCoefficient[Λ, {ϵ, 0, k}], {k, 0, $k}]];
```

Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.




```
Zip_vs_[{ℱ_, ℰ_}] := {ℱ, ℰ} // Zip1_vs // Zip2_Select[vs, (0<Wt[#]<$n)&] // EZip3_Select[vs, (0<Wt[#]<$n)&] //
    Zip2_Select[vs, (Wt[#]==0∨Wt[#]==$n)&] // Zip3_Select[vs, (Wt[#]==0∨Wt[#]==$n)&] // Last;
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

```
In[•]:= Zip1{} = Identity;
Zip1vs_@{ℱ_, 𝔼[Q_, P___]} := PPZip1@Module[{ℐ, F, G, u, v},
   ℐ = IdentityMatrix@Length@vs;
   F = Table[If[Wt[u] + Wt[v] == $n, ∂u*,v* ℱ, 0], {u, vs}, {v, vs}];
   G = Table[If[Wt[u] + Wt[v] == $n, ∂u,v Q, 0], {u, vs}, {v, vs}];
   {CF[vs*.(F.Inverse[ℐ - G.F]).vs* / 2], 𝔼[CF[Q - Log[Det[ℐ - G.F]] / 2 - vs.G.vs / 2], P]}
  ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F:\, \mathcal{E}\, \mathbb{P}^{\sum_{i\in B} y_i z_i}\right\rangle_B = \mathbb{P}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F:\ \mathcal{E}|_{z_B \to z_B + F y_B}\right\rangle_B$.

```
In[•]:= Zip2{} = Identity;
Zip2vs_@{ℱ_, 𝔼[Q_, P___]} := PPZip2@Module[{F, Y, u, v},
   F = Table[If[Wt[u] + Wt[v] == $n, ∂u*,v* ℱ, 0], {u, vs}, {v, vs}];
   Y = Table[∂v Q, {v, vs}] /. AlsoUpper@Table[v → 0, {v, vs}];
   CF /@ ({ℱ, 𝔼[Q - Y.vs + Y.F.Y / 2, P]} /. AlsoUpper@Thread[vs → vs + F.Y])
  ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\, \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

```
In[•]:= Zip3vs_@{ℱ_, ℰ_𝔼} := PPZip3@Module[{F, u, v, Z, $k, kk, jj, $m = 0, m, n},
  $k = Length[ℰ] - 1;
  Do[Z[0, kk] = ℰ[[kk + 1]], {kk, 0, $k}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == $n, ∂u*,v* ℱ, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ $k, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] === 0, 0, F[u, v]/(2 (m + 1))
        (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
     ℱ - Sum[F[u, v] u* v* / 2, {u, vs}, {v, vs}],
     𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, $k}]
    } /. AlsoUpper@Table[v → 0, {v, vs}])
 ]
```

Encapsulation.

```
In[•]:= EZip3vs_@{ℱ_, ℰ_𝔼} := PPEZip3@Module[
  {nℰ, nℱ, rc, ps, rr = {(*release rules*)}},
  rc = 0; nℰ = Total[
     CoefficientRules[#, vs] /. (ps_ → c_) :> (AppendTo[rr, cℰ[++rc] → c]; cℰ[rc] × (Times @@ vs^ps))
    ] & /@ ℰ;
  rc = 0; nℱ = Total[CoefficientRules[ℱ, vs*] /.
    (ps_ → c_) :> (AppendTo[rr, cℱ[++rc] → c]; cℱ[rc] × (Times @@ (vs*)^ps))];
  CF[Expand[{nℱ, nℰ} // Zip3vs] /. rr]
 ]
```

Profiling

```
In[•]:= BeginProfile[];
```

```
In[•]:= Timing@Block[{$k = 1}, Z[Knot[3, 1]]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[•]= {17.4844, 𝔼{}→{0}[1/2 × (-4 t ℏ - Log[(1/T^3 - 2/T^2 + 2/T)^2] - Log[(1 + T/(1 - 2 T + 2 T^2) - T^2/(1 - 2 T + 2 T^2))^2]),
  a (-2 ℏ + 2 T^2 ℏ)/(1 - T + T^2) + (-2 ℏ + 3 T ℏ - 2 T^2 ℏ + T^3 ℏ)/(1 - 2 T + 3 T^2 - 2 T^3 + T^4) + x y (-2 ℏ^2 - 2 T ℏ^2)/(1 - T + T^2)]}
```

In[•]:= **PrintProfile[]**

Out[•]=
```
ProfileRoot is root. Profiled time: 17.438
  (    1)  0.093/ 17.440 above Z
  (    1)     0/    0 above RVK
CF: called 13065 times, time in 6.18/11.876
  (   84)  0.139/ 0.391 under Z
  (   76)  0.079/ 0.221 under Boot
  (  135)  0.451/ 0.983 under EZip3
  (   90)  0.094/ 0.299 under Zip1
  (  270)  1.715/ 5.406 under Zip2
  ( 12410)  3.702/ 4.576 under Zip3
  ( 8889)  5.696/ 5.696 above CCF
CCF: called 8889 times, time in 5.696/5.696
  ( 8889)  5.696/ 5.696 under CF
Zip3: called 90 times, time in 2.486/7.062
  (   22)  0.719/ 2.436 under Z
  (   23)  1.000/ 2.611 under Boot
  (   45)  0.767/ 2.015 under EZip3
  ( 12410)  3.702/ 4.576 above CF
Zip1: called 45 times, time in 1.105/1.404
  (   22)  0.328/ 0.516 under Z
  (   23)  0.777/ 0.888 under Boot
  (   90)  0.094/ 0.299 above CF
EZip3: called 45 times, time in 0.969/3.967
  (   22)  0.814/ 2.437 under Z
  (   23)  0.155/ 1.530 under Boot
  (  135)  0.451/ 0.983 above CF
  (   45)  0.767/ 2.015 above Zip3
Zip2: called 90 times, time in 0.847/6.253
  (   44)  0.313/ 5.220 under Z
  (   46)  0.534/ 1.033 under Boot
  (  270)  1.715/ 5.406 above CF
Z: called 1 times, time in 0.093/17.438
  (    1)  0.093/ 17.440 under ProfileRoot
  (    5)     0/ 6.345 above Boot
  (   84)  0.139/ 0.391 above CF
  (   22)  0.814/ 2.437 above EZip3
  (   22)  0.328/ 0.516 above Zip1
  (   44)  0.313/ 5.220 above Zip2
  (   22)  0.719/ 2.436 above Zip3
Boot: called 23 times, time in 0.062/18.626
  (    5)     0/ 6.345 under Z
  (   18)  0.062/ 12.280 under Boot
  (   18)  0.062/ 12.280 above Boot
  (   76)  0.079/ 0.221 above CF
  (   23)  0.155/ 1.530 above EZip3
  (   23)  0.777/ 0.888 above Zip1
  (   46)  0.534/ 1.033 above Zip2
  (   23)  1.000/ 2.611 above Zip3
RVK: called 1 times, time in 0./0.
  (    1)     0/    0 under ProfileRoot
```

In[•]:=
```
Timing@Block[{$k = 1}, Z[Knot[8, 17]]]
```

Out[•]=

$$\left\{66.0469,\ \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-2\,t\,\hbar-\text{Log}\left[\left(-1-\frac{1}{T^4}+\frac{4}{T^3}-\frac{6}{T^2}+\frac{5}{T}\right)^2\right]-\right.\right.\right.$$

$$\text{Log}\left[\left(1+\frac{T}{1-4\,T+6\,T^2-5\,T^3+T^4}-\frac{2\,T^2}{1-4\,T+6\,T^2-5\,T^3+T^4}+\frac{T^3}{1-4\,T+6\,T^2-5\,T^3+T^4}\right)^2\right]-$$

$$\text{Log}\left[\left(1-\frac{T}{1-3\,T+4\,T^2-4\,T^3+T^4}+\frac{4\,T^2}{1-3\,T+4\,T^2-4\,T^3+T^4}-\frac{7\,T^3}{1-3\,T+4\,T^2-4\,T^3+T^4}+\right.\right.$$

$$\left.\left.\left.\left.\frac{7\,T^4}{1-3\,T+4\,T^2-4\,T^3+T^4}-\frac{4\,T^5}{1-3\,T+4\,T^2-4\,T^3+T^4}+\frac{T^6}{1-3\,T+4\,T^2-4\,T^3+T^4}\right)^2\right]\right),$$

$$\frac{-3\,\hbar+8\,T\,\hbar-8\,T^2\,\hbar+8\,T^4\,\hbar-8\,T^5\,\hbar+3\,T^6\,\hbar}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}+\frac{a\left(-6\,\hbar+16\,T\,\hbar-16\,T^2\,\hbar+16\,T^4\,\hbar-16\,T^5\,\hbar+6\,T^6\,\hbar\right)}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}+$$

$$\left.\left.\frac{x\,y\left(-6\,\hbar^2+10\,T\,\hbar^2-6\,T^2\,\hbar^2-6\,T^3\,\hbar^2+10\,T^4\,\hbar^2-6\,T^5\,\hbar^2\right)}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}\right]\right\}$$

In[•]:=
```
PrintProfile[]
```

Out[•]=
```
ProfileRoot is root. Profiled time: 83.485
  (    2)   0.309/ 83.485 above Z
  (    2)       0/     0 above RVK
CCF: called 26186 times, time in 34.842/34.842
  ( 26186)  34.842/ 34.842 under CF
CF: called 27296 times, time in 30.591/65.433
  (  298)   1.007/  2.304 under Z
  (   88)   0.109/  0.251 under Boot
  (  318)   2.980/  6.437 under EZip3
  (  212)   0.313/  0.785 under Zip1
  (  636)  16.357/ 40.972 under Zip2
  ( 25744)   9.825/ 14.684 under Zip3
  ( 26186)  34.842/ 34.842 above CCF
EZip3: called 106 times, time in 8.204/18.654
  (   79)   8.018/ 16.969 under Z
  (   27)   0.186/  1.685 under Boot
  (  318)   2.980/  6.437 above CF
  (  106)   1.549/  4.013 above Zip3
Zip3: called 212 times, time in 5.367/20.051
  (   79)   2.725/ 13.224 under Z
  (   27)   1.093/  2.814 under Boot
  (  106)   1.549/  4.013 under EZip3
  ( 25744)   9.825/ 14.684 above CF
Zip2: called 212 times, time in 2.058/43.03
  (  158)   1.428/ 41.870 under Z
  (   54)   0.630/  1.160 under Boot
  (  636)  16.357/ 40.972 above CF
Zip1: called 106 times, time in 2.036/2.821
  (   79)   1.211/  1.855 under Z
  (   27)   0.825/  0.966 under Boot
  (  212)   0.313/  0.785 above CF
Z: called 2 times, time in 0.309/83.485
  (    2)   0.309/ 83.485 under ProfileRoot
  (    7)       0/  6.954 above Boot
  (  298)   1.007/  2.304 above CF
  (   79)   8.018/ 16.969 above EZip3
  (   79)   1.211/  1.855 above Zip1
  (  158)   1.428/ 41.870 above Zip2
  (   79)   2.725/ 13.224 above Zip3
Boot: called 27 times, time in 0.078/19.579
  (    7)       0/  6.954 under Z
  (   20)   0.078/ 12.625 under Boot
  (   20)   0.078/ 12.625 above Boot
  (   88)   0.109/  0.251 above CF
  (   27)   0.186/  1.685 above EZip3
  (   27)   0.825/  0.966 above Zip1
  (   54)   0.630/  1.160 above Zip2
  (   27)   1.093/  2.814 above Zip3
RVK: called 2 times, time in 0./0.
  (    2)       0/     0 under ProfileRoot
```

In[•]:=
```
Timing@Block[{$k = 2}, Z[Knot[3, 1]]]
```

Out[•]=

$$\left\{143.844,\ \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-4\,t\,\hbar-\text{Log}\left[\left(\frac{1}{T^3}-\frac{2}{T^2}+\frac{2}{T}\right)^2\right]-\text{Log}\left[\left(1+\frac{T}{1-2T+2T^2}-\frac{T^2}{1-2T+2T^2}\right)^2\right]\right),\right.\right.$$

$$\frac{a\left(-2\hbar+2T^2\hbar\right)}{1-T+T^2}+\frac{-2\hbar+3T\hbar-2T^2\hbar+T^3\hbar}{1-2T+3T^2-2T^3+T^4}+\frac{xy\left(-2\hbar^2-2T\hbar^2\right)}{1-T+T^2},\ \frac{a^2\left(2T\hbar^2-8T^2\hbar^2+2T^3\hbar^2\right)}{1-2T+3T^2-2T^3+T^4}+$$

$$\frac{a\left(2T\hbar^2-14T^2\hbar^2+12T^3\hbar^2-6T^4\hbar^2+2T^5\hbar^2\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{T\hbar^2-11T^2\hbar^2+16T^3\hbar^2-12T^4\hbar^2+8T^5\hbar^2-3T^6\hbar^2+T^7\hbar^2}{2-8T+20T^2-32T^3+38T^4-32T^5+20T^6-8T^7+2T^8}+$$

$$\left.\left.\frac{axy\left(8T\hbar^3-8T^2\hbar^3-4T^3\hbar^3\right)}{1-2T+3T^2-2T^3+T^4}+\frac{xy\left(-2\hbar^3-2T^2\hbar^3-6T^3\hbar^3+2T^5\hbar^3\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{x^2y^2\left(\hbar^4+5T\hbar^4+T^2\hbar^4\right)}{1-2T+3T^2-2T^3+T^4}\right]\right\}$$

In[•]:=
```
PrintProfile[]
```

Out[•]=
```
ProfileRoot is root. Profiled time: 227.329
  (    3)   0.466/ 227.330 above Z
  (    3)      0/     0 above RVK
CCF: called 47610 times, time in 110.268/110.268
  ( 47610)  110.270/ 110.270 under CF
CF: called 41270 times, time in 82.741/193.009
  (  424)   1.430/  3.385 under Z
  (  202)   0.202/  0.485 under Boot
  (  498)  17.104/ 36.795 under EZip3
  (  302)   0.424/  1.117 under Zip1
  (  996)  31.594/ 92.989 under Zip2
  ( 38848)  31.987/ 58.238 under Zip3
  ( 47610)  110.270/ 110.270 above CCF
EZip3: called 151 times, time in 18.36/62.125
  (  101)  18.001/ 58.814 under Z
  (   50)   0.359/  3.311 under Boot
  (  498)  17.104/ 36.795 above CF
  (  151)   2.742/  6.970 above Zip3
Zip3: called 302 times, time in 9.438/67.676
  (  101)   4.723/ 55.610 under Z
  (   50)   1.973/  5.096 under Boot
  (  151)   2.742/  6.970 under EZip3
  ( 38848)  31.987/ 58.238 above CF
Zip2: called 302 times, time in 2.929/95.918
  (  202)   1.913/ 93.511 under Z
  (  100)   1.016/  2.407 under Boot
  (  996)  31.594/ 92.989 above CF
Zip1: called 151 times, time in 2.923/4.04
  (  101)   1.567/  2.370 under Z
  (   50)   1.356/  1.670 under Boot
  (  302)   0.424/  1.117 above CF
Z: called 3 times, time in 0.466/227.329
  (    3)   0.466/ 227.330 under ProfileRoot
  (   12)   0.016/ 13.173 above Boot
  (  424)   1.430/  3.385 above CF
  (  101)  18.001/ 58.814 above EZip3
  (  101)   1.567/  2.370 above Zip1
  (  202)   1.913/ 93.511 above Zip2
  (  101)   4.723/ 55.610 above Zip3
Boot: called 47 times, time in 0.204/35.61
  (   12)   0.016/ 13.173 under Z
  (   35)   0.188/ 22.437 under Boot
  (   35)   0.188/ 22.437 above Boot
  (  202)   0.202/  0.485 above CF
  (   50)   0.359/  3.311 above EZip3
  (   50)   1.356/  1.670 above Zip1
  (  100)   1.016/  2.407 above Zip2
  (   50)   1.973/  5.096 above Zip3
RVK: called 3 times, time in 0./0.
  (    3)      0/     0 under ProfileRoot
```

In[•]:=
```
Timing@Block[{$k = 2}, Z[Knot[8, 17]]]
```

Out[•]=

$$\left\{1747.75,\ \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-2\,t\,\hbar-\text{Log}\left[\left(-1-\frac{1}{T^4}+\frac{4}{T^3}-\frac{6}{T^2}+\frac{5}{T}\right)^2\right]-\right.\right.\right.$$

$$\text{Log}\left[\left(1+\frac{T}{1-4\,T+6\,T^2-5\,T^3+T^4}-\frac{2\,T^2}{1-4\,T+6\,T^2-5\,T^3+T^4}+\frac{T^3}{1-4\,T+6\,T^2-5\,T^3+T^4}\right)^2\right]-$$

$$\text{Log}\left[\left(1-\frac{T}{1-3\,T+4\,T^2-4\,T^3+T^4}+\frac{4\,T^2}{1-3\,T+4\,T^2-4\,T^3+T^4}-\frac{7\,T^3}{1-3\,T+4\,T^2-4\,T^3+T^4}+\right.\right.$$

$$\left.\left.\left.\left.\frac{7\,T^4}{1-3\,T+4\,T^2-4\,T^3+T^4}-\frac{4\,T^5}{1-3\,T+4\,T^2-4\,T^3+T^4}+\frac{T^6}{1-3\,T+4\,T^2-4\,T^3+T^4}\right)^2\right]\right)\right],$$

$$\frac{-3\,\hbar+8\,T\,\hbar-8\,T^2\,\hbar+8\,T^4\,\hbar-8\,T^5\,\hbar+3\,T^6\,\hbar}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}+\frac{a\left(-6\,\hbar+16\,T\,\hbar-16\,T^2\,\hbar+16\,T^4\,\hbar-16\,T^5\,\hbar+6\,T^6\,\hbar\right)}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}+$$

$$\frac{x\,y\left(-6\,\hbar^2+10\,T\,\hbar^2-6\,T^2\,\hbar^2-6\,T^3\,\hbar^2+10\,T^4\,\hbar^2-6\,T^5\,\hbar^2\right)}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6},$$

$$\left(a\left(8\,T\,\hbar^2-64\,T^2\,\hbar^2+262\,T^3\,\hbar^2-608\,T^4\,\hbar^2+952\,T^5\,\hbar^2-1096\,T^6\,\hbar^2+952\,T^7\,\hbar^2-608\,T^8\,\hbar^2+262\,T^9\,\hbar^2-64\,T^{10}\,\hbar^2+8\,T^{11}\,\hbar^2\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(a^2\left(8\,T\,\hbar^2-64\,T^2\,\hbar^2+262\,T^3\,\hbar^2-608\,T^4\,\hbar^2+952\,T^5\,\hbar^2-1096\,T^6\,\hbar^2+952\,T^7\,\hbar^2-608\,T^8\,\hbar^2+262\,T^9\,\hbar^2-64\,T^{10}\,\hbar^2+8\,T^{11}\,\hbar^2\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(4\,T\,\hbar^2-50\,T^2\,\hbar^2+307\,T^3\,\hbar^2-1160\,T^4\,\hbar^2+3062\,T^5\,\hbar^2-6127\,T^6\,\hbar^2+9760\,T^7\,\hbar^2-12\,754\,T^8\,\hbar^2+13\,916\,T^9\,\hbar^2-12\,754\,T^{10}\,\hbar^2+9760\,T^{11}\,\hbar^2-6127\,T^{12}\,\hbar^2+3062\,T^{13}\,\hbar^2-1160\,T^{14}\,\hbar^2+307\,T^{15}\,\hbar^2-50\,T^{16}\,\hbar^2+4\,T^{17}\,\hbar^2\right)\Big/$$

$$\left(2-24\,T+144\,T^2-578\,T^3+1728\,T^4-4056\,T^5+7708\,T^6-12\,072\,T^7+15\,744\,T^8-17\,194\,T^9+15\,744\,T^{10}-12\,072\,T^{11}+7708\,T^{12}-4056\,T^{13}+1728\,T^{14}-578\,T^{15}+144\,T^{16}-24\,T^{17}+2\,T^{18}\right)+$$

$$\left(a\,x\,y\left(28\,T\,\hbar^3-168\,T^2\,\hbar^3+544\,T^3\,\hbar^3-1000\,T^4\,\hbar^3+1248\,T^5\,\hbar^3-1096\,T^6\,\hbar^3+656\,T^7\,\hbar^3-216\,T^8\,\hbar^3-20\,T^9\,\hbar^3+40\,T^{10}\,\hbar^3-12\,T^{11}\,\hbar^3\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(x\,y\left(-18\,\hbar^3+78\,T\,\hbar^3-146\,T^2\,\hbar^3+110\,T^3\,\hbar^3+78\,T^4\,\hbar^3-274\,T^5\,\hbar^3+274\,T^6\,\hbar^3-78\,T^7\,\hbar^3-110\,T^8\,\hbar^3+146\,T^9\,\hbar^3-78\,T^{10}\,\hbar^3+18\,T^{11}\,\hbar^3\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left.\left.\left(x^2\,y^2\left(3\,\hbar^4-37\,T^2\,\hbar^4+153\,T^3\,\hbar^4-261\,T^4\,\hbar^4+325\,T^5\,\hbar^4-261\,T^6\,\hbar^4+153\,T^7\,\hbar^4-37\,T^8\,\hbar^4+3\,T^{10}\,\hbar^4\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)\right]\right\}$$

In[•]:=
```
PrintProfile[]
```

Out[•]=
```
ProfileRoot is root. Profiled time: 1975.08
  (     4)    1.484/ 1975.080 above Z
  (     4)        0/       0 above RVK
CCF: called 104354 times, time in 940.437/940.437
  ( 104354)  940.437/ 940.437 under CF
CF: called 56959 times, time in 838.722/1779.16
  (   745)    7.192/  16.274 under Z
  (   220)    0.248/   0.547 under Boot
  (   742)  244.828/ 463.079 under EZip3
  (   424)    0.565/   1.568 under Zip1
  (  1484)  307.379/ 756.116 under Zip2
  ( 53344)  278.510/ 541.575 under Zip3
  ( 104354)  940.437/ 940.437 above CCF
EZip3: called 212 times, time in 165.566/639.85
  (   158)  165.177/ 636.352 under Z
  (    54)    0.389/   3.498 under Boot
  (   742)  244.828/ 463.079 above CF
  (   212)    4.510/  11.205 above Zip3
Zip3: called 424 times, time in 20.098/561.673
  (   158)   13.568/ 545.122 under Z
  (    54)    2.020/   5.346 under Boot
  (   212)    4.510/  11.205 under EZip3
  ( 53344)  278.510/ 541.575 above CF
Zip2: called 424 times, time in 4.72/760.836
  (   316)    3.643/ 758.226 under Z
  (   108)    1.077/   2.610 under Boot
  (  1484)  307.379/ 756.116 above CF
Zip1: called 212 times, time in 3.801/5.369
  (   158)    2.413/   3.667 under Z
  (    54)    1.388/   1.702 under Boot
  (   424)    0.565/   1.568 above CF
Z: called 4 times, time in 1.484/1975.08
  (     4)    1.484/ 1975.080 under ProfileRoot
  (    14)    0.016/  13.954 above Boot
  (   745)    7.192/  16.274 above CF
  (   158)  165.177/ 636.352 above EZip3
  (   158)    2.413/   3.667 above Zip1
  (   316)    3.643/ 758.226 above Zip2
  (   158)   13.568/ 545.122 above Zip3
Boot: called 51 times, time in 0.251/36.86
  (    14)    0.016/  13.954 under Z
  (    37)    0.235/  22.906 under Boot
  (    37)    0.235/  22.906 above Boot
  (   220)    0.248/   0.547 above CF
  (    54)    0.389/   3.498 above EZip3
  (    54)    1.388/   1.702 above Zip1
  (   108)    1.077/   2.610 above Zip2
  (    54)    2.020/   5.346 above Zip3
RVK: called 4 times, time in 0./0.
  (     4)        0/       0 under ProfileRoot
```

In[•]:=
```
Timing@Block[{$k = 3}, Z[Knot[3, 1]]]
```

Out[•]=

$$\left\{3841.72,\ \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-4\,t\,\hbar-\text{Log}\left[\left(\frac{1}{T^3}-\frac{2}{T^2}+\frac{2}{T}\right)^2\right]-\text{Log}\left[\left(1+\frac{T}{1-2T+2T^2}-\frac{T^2}{1-2T+2T^2}\right)^2\right]\right),\right.\right.$$

$$\frac{a\left(-2\hbar+2T^2\hbar\right)}{1-T+T^2}+\frac{-2\hbar+3T\hbar-2T^2\hbar+T^3\hbar}{1-2T+3T^2-2T^3+T^4}+\frac{xy\left(-2\hbar^2-2T\hbar^2\right)}{1-T+T^2},\ \frac{a^2\left(2T\hbar^2-8T^2\hbar^2+2T^3\hbar^2\right)}{1-2T+3T^2-2T^3+T^4}+$$

$$\frac{a\left(2T\hbar^2-14T^2\hbar^2+12T^3\hbar^2-6T^4\hbar^2+2T^5\hbar^2\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{T\hbar^2-11T^2\hbar^2+16T^3\hbar^2-12T^4\hbar^2+8T^5\hbar^2-3T^6\hbar^2+T^7\hbar^2}{2-8T+20T^2-32T^3+38T^4-32T^5+20T^6-8T^7+2T^8}+$$

$$\frac{axy\left(8T\hbar^3-8T^2\hbar^3-4T^3\hbar^3\right)}{1-2T+3T^2-2T^3+T^4}+\frac{xy\left(-2\hbar^3-2T^2\hbar^3-6T^3\hbar^3+2T^5\hbar^3\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{x^2y^2\left(\hbar^4+5T\hbar^4+T^2\hbar^4\right)}{1-2T+3T^2-2T^3+T^4},$$

$$\frac{a^3\left(-4T\hbar^3+28T^2\hbar^3-28T^4\hbar^3+4T^5\hbar^3\right)}{3-9T+18T^2-21T^3+18T^4-9T^5+3T^6}+\frac{a^2\left(-2T\hbar^3+24T^2\hbar^3-12T^3\hbar^3-32T^4\hbar^3+20T^5\hbar^3-8T^6\hbar^3+2T^7\hbar^3\right)}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+$$

$$\left(a\left(-T\hbar^3+19T^2\hbar^3-19T^3\hbar^3-34T^4\hbar^3+40T^5\hbar^3-22T^6\hbar^3+11T^7\hbar^3-3T^8\hbar^3+T^9\hbar^3\right)\right)/$$

$$\left(1-5T+15T^2-30T^3+45T^4-51T^5+45T^6-30T^7+15T^8-5T^9+T^{10}\right)+$$

$$\left(-T\hbar^3+29T^2\hbar^3-43T^3\hbar^3-71T^4\hbar^3+131T^5\hbar^3-84T^6\hbar^3+53T^7\hbar^3-23T^8\hbar^3+11T^9\hbar^3-3T^{10}\hbar^3+T^{11}\hbar^3\right)/$$

$$\left(6-36T+126T^2-300T^3+540T^4-756T^5+846T^6-756T^7+540T^8-300T^9+126T^{10}-36T^{11}+6T^{12}\right)+$$

$$\frac{a^2xy\left(-8T\hbar^4+8T^2\hbar^4+36T^3\hbar^4-20T^4\hbar^4-4T^5\hbar^4\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+$$

$$\frac{axy\left(12T\hbar^4-16T^2\hbar^4+40T^3\hbar^4-16T^4\hbar^4-56T^5\hbar^4+8T^6\hbar^4+4T^7\hbar^4\right)}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+$$

$$\left(xy\left(-4\hbar^4+3T\hbar^4-6T^2\hbar^4-9T^3\hbar^4-15T^4\hbar^4-63T^5\hbar^4-9T^6\hbar^4+42T^7\hbar^4+3T^8\hbar^4-4T^9\hbar^4\right)\right)/$$

$$\left(3-15T+45T^2-90T^3+135T^4-153T^5+135T^6-90T^7+45T^8-15T^9+3T^{10}\right)+$$

$$\frac{ax^2y^2\left(-14T\hbar^5-6T^2\hbar^5+30T^3\hbar^5+4T^4\hbar^5\right)}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+$$

$$\frac{x^2y^2\left(2\hbar^5+23T\hbar^5-10T^2\hbar^5+11T^3\hbar^5+42T^4\hbar^5-29T^5\hbar^5-8T^6\hbar^5\right)}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+$$

$$\left.\left.\frac{x^3y^3\left(-2\hbar^6-24T\hbar^6-24T^2\hbar^6-2T^3\hbar^6\right)}{3-9T+18T^2-21T^3+18T^4-9T^5+3T^6}\right]\right\}$$

In[•]:= **PrintProfile**[]

Out[•]=
```
ProfileRoot is root. Profiled time: 5816.8
  (     5)     2.375/ 5816.798 above Z
  (     5)        0/       0 above RVK
CCF: called 169628 times, time in 2757.93/2757.93
  ( 169628)  2757.925/ 2757.925 under CF
CF: called 71886 times, time in 2708.74/5466.67
  (   913)     8.985/   20.634 under Z
  (   372)     0.485/    1.144 under Boot
  (   967)  1234.497/ 2279.516 under EZip3
  (   514)     0.661/    1.821 under Zip1
  (  1934)   470.278/ 1266.607 under Zip2
  ( 67186)   993.839/ 1896.948 under Zip3
  ( 169628)  2757.925/ 2757.925 above CCF
EZip3: called 257 times, time in 297.784/2604.69
  (   180)   296.975/ 2598.603 under Z
  (    77)     0.809/    6.089 under Boot
  (   967)  1234.497/ 2279.516 above CF
  (   257)     9.115/   27.392 above Zip3
Zip3: called 514 times, time in 38.768/1935.72
  (   180)    26.381/ 1899.794 under Z
  (    77)     3.272/    8.530 under Boot
  (   257)     9.115/   27.392 under EZip3
  ( 67186)   993.839/ 1896.948 above CF
Zip2: called 514 times, time in 6.096/1272.7
  (   360)     4.563/ 1267.181 under Z
  (   154)     1.533/    5.522 under Boot
  (  1934)   470.278/ 1266.607 above CF
Zip1: called 257 times, time in 4.697/6.518
  (   180)     2.752/    4.148 under Z
  (    77)     1.945/    2.370 under Boot
  (   514)     0.661/    1.821 above CF
Z: called 5 times, time in 2.375/5816.8
  (     5)     2.375/ 5816.798 under ProfileRoot
  (    19)     0.031/   24.063 above Boot
  (   913)     8.985/   20.634 above CF
  (   180)   296.975/ 2598.603 above EZip3
  (   180)     2.752/    4.148 above Zip1
  (   360)     4.563/ 1267.181 above Zip2
  (   180)    26.381/ 1899.794 above Zip3
Boot: called 71 times, time in 0.408/60.783
  (    19)     0.031/   24.063 under Z
  (    52)     0.377/   36.720 under Boot
  (    52)     0.377/   36.720 above Boot
  (   372)     0.485/    1.144 above CF
  (    77)     0.809/    6.089 above EZip3
  (    77)     1.945/    2.370 above Zip1
  (   154)     1.533/    5.522 above Zip2
  (    77)     3.272/    8.530 above Zip3
RVK: called 5 times, time in 0./0.
  (     5)        0/       0 under ProfileRoot
```

In[•]:=
```
Timing@Block[{$k = 3}, Z[Knot[8, 17]]]
```

Out[•]=

$$\left\{79\,170.2,\ \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-2\,t\,\hbar-\text{Log}\left[\left(-1-\frac{1}{T^4}+\frac{4}{T^3}-\frac{6}{T^2}+\frac{5}{T}\right)^2\right]-\right.\right.\right.$$

$$\text{Log}\left[\left(1+\frac{T}{1-4\,T+6\,T^2-5\,T^3+T^4}-\frac{2\,T^2}{1-4\,T+6\,T^2-5\,T^3+T^4}+\frac{T^3}{1-4\,T+6\,T^2-5\,T^3+T^4}\right)^2\right]-$$

$$\text{Log}\left[\left(1-\frac{T}{1-3\,T+4\,T^2-4\,T^3+T^4}+\frac{4\,T^2}{1-3\,T+4\,T^2-4\,T^3+T^4}-\frac{7\,T^3}{1-3\,T+4\,T^2-4\,T^3+T^4}+\right.\right.$$

$$\left.\left.\left.\left.\frac{7\,T^4}{1-3\,T+4\,T^2-4\,T^3+T^4}-\frac{4\,T^5}{1-3\,T+4\,T^2-4\,T^3+T^4}+\frac{T^6}{1-3\,T+4\,T^2-4\,T^3+T^4}\right)^2\right]\right)\right],$$

$$\frac{-3\,\hbar+8\,T\,\hbar-8\,T^2\,\hbar+8\,T^4\,\hbar-8\,T^5\,\hbar+3\,T^6\,\hbar}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}+\frac{a\left(-6\,\hbar+16\,T\,\hbar-16\,T^2\,\hbar+16\,T^4\,\hbar-16\,T^5\,\hbar+6\,T^6\,\hbar\right)}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}+$$

$$\frac{x\,y\left(-6\,\hbar^2+10\,T\,\hbar^2-6\,T^2\,\hbar^2-6\,T^3\,\hbar^2+10\,T^4\,\hbar^2-6\,T^5\,\hbar^2\right)}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6},$$

$$\left(a\left(8\,T\,\hbar^2-64\,T^2\,\hbar^2+262\,T^3\,\hbar^2-608\,T^4\,\hbar^2+952\,T^5\,\hbar^2-1096\,T^6\,\hbar^2+952\,T^7\,\hbar^2-608\,T^8\,\hbar^2+262\,T^9\,\hbar^2-64\,T^{10}\,\hbar^2+8\,T^{11}\,\hbar^2\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(a^2\left(8\,T\,\hbar^2-64\,T^2\,\hbar^2+262\,T^3\,\hbar^2-608\,T^4\,\hbar^2+952\,T^5\,\hbar^2-1096\,T^6\,\hbar^2+952\,T^7\,\hbar^2-608\,T^8\,\hbar^2+262\,T^9\,\hbar^2-64\,T^{10}\,\hbar^2+8\,T^{11}\,\hbar^2\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(4\,T\,\hbar^2-50\,T^2\,\hbar^2+307\,T^3\,\hbar^2-1160\,T^4\,\hbar^2+3062\,T^5\,\hbar^2-6127\,T^6\,\hbar^2+9760\,T^7\,\hbar^2-12\,754\,T^8\,\hbar^2+13\,916\,T^9\,\hbar^2-12\,754\,T^{10}\,\hbar^2+9760\,T^{11}\,\hbar^2-6127\,T^{12}\,\hbar^2+3062\,T^{13}\,\hbar^2-1160\,T^{14}\,\hbar^2+307\,T^{15}\,\hbar^2-50\,T^{16}\,\hbar^2+4\,T^{17}\,\hbar^2\right)\Big/\left(2-24\,T+144\,T^2-578\,T^3+1728\,T^4-4056\,T^5+7708\,T^6-12\,072\,T^7+15\,744\,T^8-17\,194\,T^9+15\,744\,T^{10}-12\,072\,T^{11}+7708\,T^{12}-4056\,T^{13}+1728\,T^{14}-578\,T^{15}+144\,T^{16}-24\,T^{17}+2\,T^{18}\right)+$$

$$\left(a\,x\,y\left(28\,T\,\hbar^3-168\,T^2\,\hbar^3+544\,T^3\,\hbar^3-1000\,T^4\,\hbar^3+1248\,T^5\,\hbar^3-1096\,T^6\,\hbar^3+656\,T^7\,\hbar^3-216\,T^8\,\hbar^3-20\,T^9\,\hbar^3+40\,T^{10}\,\hbar^3-12\,T^{11}\,\hbar^3\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(x\,y\left(-18\,\hbar^3+78\,T\,\hbar^3-146\,T^2\,\hbar^3+110\,T^3\,\hbar^3+78\,T^4\,\hbar^3-274\,T^5\,\hbar^3+274\,T^6\,\hbar^3-78\,T^7\,\hbar^3-110\,T^8\,\hbar^3+146\,T^9\,\hbar^3-78\,T^{10}\,\hbar^3+18\,T^{11}\,\hbar^3\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right)+$$

$$\left(x^2\,y^2\left(3\,\hbar^4-37\,T^2\,\hbar^4+153\,T^3\,\hbar^4-261\,T^4\,\hbar^4+325\,T^5\,\hbar^4-261\,T^6\,\hbar^4+153\,T^7\,\hbar^4-37\,T^8\,\hbar^4+3\,T^{10}\,\hbar^4\right)\right)\Big/\left(1-8\,T+32\,T^2-86\,T^3+168\,T^4-248\,T^5+283\,T^6-248\,T^7+168\,T^8-86\,T^9+32\,T^{10}-8\,T^{11}+T^{12}\right),$$

$$\left(a^2\left(-8\,T\,\hbar^3+96\,T^2\,\hbar^3-594\,T^3\,\hbar^3+2016\,T^4\,\hbar^3-4264\,T^5\,\hbar^3+5994\,T^6\,\hbar^3-5824\,T^7\,\hbar^3+3536\,T^8\,\hbar^3-3536\,T^{10}\,\hbar^3+5824\,T^{11}\,\hbar^3-5994\,T^{12}\,\hbar^3+4264\,T^{13}\,\hbar^3-2016\,T^{14}\,\hbar^3+594\,T^{15}\,\hbar^3-96\,T^{16}\,\hbar^3+8\,T^{17}\,\hbar^3\right)\right)\Big/\left(1-12\,T+72\,T^2-289\,T^3+864\,T^4-2028\,T^5+3854\,T^6-6036\,T^7+7872\,T^8-8597\,T^9+7872\,T^{10}-6036\,T^{11}+3854\,T^{12}-2028\,T^{13}+864\,T^{14}-289\,T^{15}+72\,T^{16}-12\,T^{17}+T^{18}\right)+$$

$$\left(a^3\left(-16\,T\,\hbar^3+192\,T^2\,\hbar^3-1188\,T^3\,\hbar^3+4032\,T^4\,\hbar^3-8528\,T^5\,\hbar^3+11\,988\,T^6\,\hbar^3-11\,648\,T^7\,\hbar^3+7072\,T^8\,\hbar^3-7072\,T^{10}\,\hbar^3+11\,648\,T^{11}\,\hbar^3-11\,988\,T^{12}\,\hbar^3+8528\,T^{13}\,\hbar^3-4032\,T^{14}\,\hbar^3+1188\,T^{15}\,\hbar^3-192\,T^{16}\,\hbar^3+16\,T^{17}\,\hbar^3\right)\right)\Big/\left(3-36\,T+216\,T^2-867\,T^3+2592\,T^4-6084\,T^5+11\,562\,T^6-18\,108\,T^7+23\,616\,T^8-25\,791\,T^9+23\,616\,T^{10}-18\,108\,T^{11}+11\,562\,T^{12}-6084\,T^{13}+2592\,T^{14}-867\,T^{15}+216\,T^{16}-36\,T^{17}+3\,T^{18}\right)+$$

$$\left(-4\,T\,\hbar^3+76\,T^2\,\hbar^3-641\,T^3\,\hbar^3+2816\,T^4\,\hbar^3-6940\,T^5\,\hbar^3+8124\,T^6\,\hbar^3+4904\,T^7\,\hbar^3-39\,224\,T^8\,\hbar^3+82\,152\,T^9\,\hbar^3-101\,684\,T^{10}\,\hbar^3+71\,608\,T^{11}\,\hbar^3-71\,608\,T^{13}\,\hbar^3+101\,684\,T^{14}\,\hbar^3-82\,152\,T^{15}\,\hbar^3+39\,224\,T^{16}\,\hbar^3-4904\,T^{17}\,\hbar^3-8124\,T^{18}\,\hbar^3+6940\,T^{19}\,\hbar^3-2816\,T^{20}\,\hbar^3+641\,T^{21}\,\hbar^3-76\,T^{22}\,\hbar^3+4\,T^{23}\,\hbar^3\right)\Big/\left(6-96\,T+768\,T^2-4104\,T^3+16\,416\,T^4-52\,128\,T^5+136\,092\,T^6-298\,752\,T^7+559\,776\,T^8-904\,416\,T^9+1\,268\,640\,T^{10}-1\,551\,744\,T^{11}+1\,659\,090\,T^{12}-1\,551\,744\,T^{13}+1\,268\,640\,T^{14}-904\,416\,T^{15}+\right.$$

$$559\,776\,T^{16} - 298\,752\,T^{17} + 136\,092\,T^{18} - 52\,128\,T^{19} + 16\,416\,T^{20} - 4104\,T^{21} + 768\,T^{22} - 96\,T^{23} + 6\,T^{24}\big) +$$
$$\big(a\,\big(-4\,T\,\hbar^3 + 68\,T^2\,\hbar^3 - 561\,T^3\,\hbar^3 + 2688\,T^4\,\hbar^3 - 8380\,T^5\,\hbar^3 + 18\,212\,T^6\,\hbar^3 - 28\,776\,T^7\,\hbar^3 + 33\,688\,T^8\,\hbar^3 -$$
$$29\,096\,T^9\,\hbar^3 + 18\,052\,T^{10}\,\hbar^3 - 7384\,T^{11}\,\hbar^3 + 7384\,T^{13}\,\hbar^3 - 18\,052\,T^{14}\,\hbar^3 + 29\,096\,T^{15}\,\hbar^3 - 33\,688\,T^{16}\,\hbar^3 +$$
$$28\,776\,T^{17}\,\hbar^3 - 18\,212\,T^{18}\,\hbar^3 + 8380\,T^{19}\,\hbar^3 - 2688\,T^{20}\,\hbar^3 + 561\,T^{21}\,\hbar^3 - 68\,T^{22}\,\hbar^3 + 4\,T^{23}\,\hbar^3\big)\big) \Big/$$
$$\big(1 - 16\,T + 128\,T^2 - 684\,T^3 + 2736\,T^4 - 8688\,T^5 + 22\,682\,T^6 - 49\,792\,T^7 + 93\,296\,T^8 - 150\,736\,T^9 +$$
$$211\,440\,T^{10} - 258\,624\,T^{11} + 276\,515\,T^{12} - 258\,624\,T^{13} + 211\,440\,T^{14} - 150\,736\,T^{15} + 93\,296\,T^{16} -$$
$$49\,792\,T^{17} + 22\,682\,T^{18} - 8688\,T^{19} + 2736\,T^{20} - 684\,T^{21} + 128\,T^{22} - 16\,T^{23} + T^{24}\big) + \big(a^2\,x\,y$$
$$\big(-28\,T\,\hbar^4 + 224\,T^2\,\hbar^4 - 960\,T^3\,\hbar^4 + 1948\,T^4\,\hbar^4 - 928\,T^5\,\hbar^4 - 5472\,T^6\,\hbar^4 + 17\,332\,T^7\,\hbar^4 - 30\,256\,T^8\,\hbar^4 + 38\,100\,T^9\,\hbar^4 -$$
$$37\,328\,T^{10}\,\hbar^4 + 28\,980\,T^{11}\,\hbar^4 - 17\,460\,T^{12}\,\hbar^4 + 7600\,T^{13}\,\hbar^4 - 2084\,T^{14}\,\hbar^4 + 228\,T^{15}\,\hbar^4 + 32\,T^{16}\,\hbar^4 - 12\,T^{17}\,\hbar^4\big)\big) \Big/$$
$$\big(1 - 12\,T + 72\,T^2 - 289\,T^3 + 864\,T^4 - 2028\,T^5 + 3854\,T^6 - 6036\,T^7 + 7872\,T^8 - 8597\,T^9 + 7872\,T^{10} -$$
$$6036\,T^{11} + 3854\,T^{12} - 2028\,T^{13} + 864\,T^{14} - 289\,T^{15} + 72\,T^{16} - 12\,T^{17} + T^{18}\big) +$$
$$\big(a\,x\,y\,\big(132\,T\,\hbar^4 - 1192\,T^2\,\hbar^4 + 5460\,T^3\,\hbar^4 - 15\,300\,T^4\,\hbar^4 + 28\,772\,T^5\,\hbar^4 - 37\,188\,T^6\,\hbar^4 +$$
$$30\,672\,T^7\,\hbar^4 - 8188\,T^8\,\hbar^4 - 19\,080\,T^9\,\hbar^4 + 36\,036\,T^{10}\,\hbar^4 - 35\,760\,T^{11}\,\hbar^4 +$$
$$23\,580\,T^{12}\,\hbar^4 - 10\,236\,T^{13}\,\hbar^4 + 2428\,T^{14}\,\hbar^4 + 12\,T^{15}\,\hbar^4 - 168\,T^{16}\,\hbar^4 + 36\,T^{17}\,\hbar^4\big)\big) \Big/$$
$$\big(1 - 12\,T + 72\,T^2 - 289\,T^3 + 864\,T^4 - 2028\,T^5 + 3854\,T^6 - 6036\,T^7 + 7872\,T^8 - 8597\,T^9 + 7872\,T^{10} -$$
$$6036\,T^{11} + 3854\,T^{12} - 2028\,T^{13} + 864\,T^{14} - 289\,T^{15} + 72\,T^{16} - 12\,T^{17} + T^{18}\big) +$$
$$\big(x\,y\,\big(-108\,\hbar^4 + 1184\,T\,\hbar^4 - 6228\,T^2\,\hbar^4 + 20\,559\,T^3\,\hbar^4 - 46\,545\,T^4\,\hbar^4 + 72\,963\,T^5\,\hbar^4 - 70\,761\,T^6\,\hbar^4 +$$
$$10\,415\,T^7\,\hbar^4 + 101\,655\,T^8\,\hbar^4 - 202\,605\,T^9\,\hbar^4 + 208\,095\,T^{10}\,\hbar^4 - 88\,665\,T^{11}\,\hbar^4 - 88\,665\,T^{12}\,\hbar^4 +$$
$$208\,095\,T^{13}\,\hbar^4 - 202\,605\,T^{14}\,\hbar^4 + 101\,655\,T^{15}\,\hbar^4 + 10\,415\,T^{16}\,\hbar^4 - 70\,761\,T^{17}\,\hbar^4 +$$
$$72\,963\,T^{18}\,\hbar^4 - 46\,545\,T^{19}\,\hbar^4 + 20\,559\,T^{20}\,\hbar^4 - 6228\,T^{21}\,\hbar^4 + 1184\,T^{22}\,\hbar^4 - 108\,T^{23}\,\hbar^4\big)\big) \Big/$$
$$\big(3 - 48\,T + 384\,T^2 - 2052\,T^3 + 8208\,T^4 - 26\,064\,T^5 + 68\,046\,T^6 - 149\,376\,T^7 + 279\,888\,T^8 - 452\,208\,T^9 +$$
$$634\,320\,T^{10} - 775\,872\,T^{11} + 829\,545\,T^{12} - 775\,872\,T^{13} + 634\,320\,T^{14} - 452\,208\,T^{15} + 279\,888\,T^{16} -$$
$$149\,376\,T^{17} + 68\,046\,T^{18} - 26\,064\,T^{19} + 8208\,T^{20} - 2052\,T^{21} + 384\,T^{22} - 48\,T^{23} + 3\,T^{24}\big) +$$
$$\big(x^2\,y^2\,\big(18\,\hbar^5 + 60\,T\,\hbar^5 - 988\,T^2\,\hbar^5 + 4723\,T^3\,\hbar^5 - 12\,050\,T^4\,\hbar^5 + 19\,335\,T^5\,\hbar^5 - 19\,017\,T^6\,\hbar^5 + 7023\,T^7\,\hbar^5 + 12\,209\,T^8\,\hbar^5 -$$
$$27\,675\,T^9\,\hbar^5 + 31\,059\,T^{10}\,\hbar^5 - 23\,091\,T^{11}\,\hbar^5 + 11\,368\,T^{12}\,\hbar^5 - 3167\,T^{13}\,\hbar^5 + 74\,T^{14}\,\hbar^5 + 228\,T^{15}\,\hbar^5 - 60\,T^{16}\,\hbar^5\big)\big) \Big/$$
$$\big(1 - 12\,T + 72\,T^2 - 289\,T^3 + 864\,T^4 - 2028\,T^5 + 3854\,T^6 - 6036\,T^7 + 7872\,T^8 - 8597\,T^9 + 7872\,T^{10} -$$
$$6036\,T^{11} + 3854\,T^{12} - 2028\,T^{13} + 864\,T^{14} - 289\,T^{15} + 72\,T^{16} - 12\,T^{17} + T^{18}\big) +$$
$$\big(a\,x^2\,y^2\,\big(-48\,T\,\hbar^5 + 340\,T^2\,\hbar^5 - 1314\,T^3\,\hbar^5 + 2512\,T^4\,\hbar^5 - 1962\,T^5\,\hbar^5 - 2646\,T^6\,\hbar^5 + 10\,398\,T^7\,\hbar^5 - 18\,098\,T^8\,\hbar^5 +$$
$$21\,762\,T^9\,\hbar^5 - 19\,854\,T^{10}\,\hbar^5 + 13\,914\,T^{11}\,\hbar^5 - 7092\,T^{12}\,\hbar^5 + 2386\,T^{13}\,\hbar^5 - 392\,T^{14}\,\hbar^5 + 12\,T^{16}\,\hbar^5\big)\big) \Big/$$
$$\big(1 - 12\,T + 72\,T^2 - 289\,T^3 + 864\,T^4 - 2028\,T^5 + 3854\,T^6 - 6036\,T^7 + 7872\,T^8 - 8597\,T^9 + 7872\,T^{10} -$$
$$6036\,T^{11} + 3854\,T^{12} - 2028\,T^{13} + 864\,T^{14} - 289\,T^{15} + 72\,T^{16} - 12\,T^{17} + T^{18}\big) +$$
$$\big(x^3\,y^3\,\big(-6\,\hbar^6 - 30\,T\,\hbar^6 + 336\,T^2\,\hbar^6 - 1514\,T^3\,\hbar^6 + 3288\,T^4\,\hbar^6 - 4650\,T^5\,\hbar^6 + 3954\,T^6\,\hbar^6 - 1728\,T^7\,\hbar^6 -$$
$$1728\,T^8\,\hbar^6 + 3954\,T^9\,\hbar^6 - 4650\,T^{10}\,\hbar^6 + 3288\,T^{11}\,\hbar^6 - 1514\,T^{12}\,\hbar^6 + 336\,T^{13}\,\hbar^6 - 30\,T^{14}\,\hbar^6 - 6\,T^{15}\,\hbar^6\big)\big) \Big/$$
$$\big(3 - 36\,T + 216\,T^2 - 867\,T^3 + 2592\,T^4 - 6084\,T^5 + 11\,562\,T^6 - 18\,108\,T^7 + 23\,616\,T^8 - 25\,791\,T^9 +$$
$$23\,616\,T^{10} - 18\,108\,T^{11} + 11\,562\,T^{12} - 6084\,T^{13} + 2592\,T^{14} - 867\,T^{15} + 216\,T^{16} - 36\,T^{17} + 3\,T^{18}\big)\big]\big\}$$

In[•]:=
```
PrintProfile[]
```

Out[•]= ProfileRoot is root. Profiled time: 84987.

```
  (     6)      9.612/ 84986.954 above Z
  (     6)         0/        0 above RVK
CF: called 89125 times, time in 48618.5/81272.3
  (  1341)     44.693/    95.975 under Z
  (   396)      0.563/     1.393 under Boot
  (  1272)  34578.313/ 53311.779 under EZip3
  (   636)      1.037/     2.495 under Zip1
  (  2544)   4229.157/ 10143.332 under Zip2
  ( 82936)   9764.711/ 17717.324 under Zip3
  ( 386706)  32653.824/ 32653.824 above CCF
CCF: called 386706 times, time in 32653.8/32653.8
  ( 386706)  32653.824/ 32653.824 under CF
EZip3: called 318 times, time in 3558.87/56927.8
  (   237)   3557.966/ 56921.426 under Z
  (    81)      0.902/     6.400 under Boot
  (  1272)  34578.313/ 53311.779 above CF
  (   318)     17.799/    57.179 above Zip3
Zip3: called 636 times, time in 126.055/17843.4
  (   237)    104.814/ 17777.296 under Z
  (    81)      3.442/     8.904 under Boot
  (   318)     17.799/    57.179 under EZip3
  ( 82936)   9764.711/ 17717.324 above CF
Zip2: called 636 times, time in 13.689/10157.
  (   474)     12.060/ 10150.950 under Z
  (   162)      1.629/     6.071 under Boot
  (  2544)   4229.157/ 10143.332 above CF
Z: called 6 times, time in 9.612/84987.
  (     6)      9.612/ 84986.954 under ProfileRoot
  (    21)      0.046/    25.766 above Boot
  (  1341)     44.693/    95.975 above CF
  (   237)   3557.966/ 56921.426 above EZip3
  (   237)      3.907/     5.929 above Zip1
  (   474)     12.060/ 10150.950 above Zip2
  (   237)    104.814/ 17777.296 above Zip3
Zip1: called 318 times, time in 5.913/8.408
  (   237)      3.907/     5.929 under Z
  (    81)      2.006/     2.479 under Boot
  (   636)      1.037/     2.495 above CF
Boot: called 75 times, time in 0.519/63.596
  (    21)      0.046/    25.766 under Z
  (    54)      0.473/    37.830 under Boot
  (    54)      0.473/    37.830 above Boot
  (   396)      0.563/     1.393 above CF
  (    81)      0.902/     6.400 above EZip3
  (    81)      2.006/     2.479 above Zip1
  (   162)      1.629/     6.071 above Zip2
  (    81)      3.442/     8.904 above Zip3
RVK: called 6 times, time in 0./0.
  (     6)         0/        0 under ProfileRoot
```