Pensieve header: Full testing of the sl_2 portfolio. Continues pensieve://Projects/SL2Portfolio2/PortfolioTesting.nb. Time : 274.386.

Startup

```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine-210101.m
<< Objects.m
<< KT.m
```

(Alt) Out[∘]= {2021, 1, 4, 8, 22, 40.8035811}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

(Alt) In[∘]:=

```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

(Alt) In[∘]:=

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Testing

(Alt) In[∘]:=
```
Block[{$k = 1}, {
    am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j},
    aS → aS_i, aS̄ → aS̄_i, bS → bS_i, bS̄ → bS̄_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k},
    dΔ → dΔ_{i→j,k}, C → C_i, C̄ → C̄_i, Kink → Kink_i, K̄ink → K̄ink_i, b2t → b2t_i, t2b → t2b_i
  }] //
 Column
```

$\mathrm{am} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\left(\alpha_i+\alpha_j\right)+x_k\left(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\right),\ 0\right]$

$\mathrm{bm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\left(\beta_i+\beta_j\right)+y_k\left(\eta_i+\eta_j\right),\ -y_k\,\beta_i\,\eta_j\right]$

$\mathrm{dm} \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\left(\alpha_i+\alpha_j\right)+b_k\,\beta_i+b_k\,\beta_j+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,\right.$
$\left.-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(1-3\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(1-3\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{\left(1-4\,B_k+3\,B_k^2\right)\eta_j^2\,\xi_i^2}{4\,\hbar}\right]$

$\mathrm{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i+\hbar\,x_j\,y_i,\ -\frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right]$

$\overline{\mathrm{R}} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i-\frac{\hbar\,x_j\,y_i}{B_i},\ -\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right]$

$\mathrm{P} \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar}+\frac{\eta_i\,\xi_j}{\hbar},\ \frac{\eta_i^2\,\xi_j^2}{4\,\hbar}\right]$

$\mathrm{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ -\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$

$\overline{\mathrm{aS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-x_i\,\mathcal{A}_i\,\xi_i,\ \hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right]$

$\mathrm{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ -\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$

$\overline{\mathrm{bS}} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i-\frac{y_i\,\eta_i}{B_i},\ \frac{\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right]$

(Alt) Out[∘]= $\mathrm{dS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-b_i\,\beta_i-\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},\right.$
$\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-\frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}+\frac{(-\mathcal{A}_i+B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i}+$
$\left.\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}+\frac{y_i\left(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2\,B_i^2}-\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\left(3\,\mathcal{A}_i^2-B_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2\,B_i}+\frac{\left(-3\,\mathcal{A}_i^2+4\,B_i\,\mathcal{A}_i^2-B_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\right]$

$\mathrm{a\Delta} \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+x_j\,\xi_i+x_k\,\xi_i,\ -\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$

$\mathrm{b\Delta} \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[\left(b_j+b_k\right)\beta_i+B_k\,y_j\,\eta_i+y_k\,\eta_i,\ \frac{1}{2}\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\right]$

$\mathrm{d\Delta} \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+\left(b_j+b_k\right)\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\right.$
$\left.\frac{1}{2}\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right]$

$\mathrm{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2},\ -\frac{\hbar\,a_i}{2}\right]$

$\overline{\mathrm{C}} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2},\ \frac{\hbar\,a_i}{2}\right]$

$\mathrm{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[\frac{\hbar\,b_i}{2}+\hbar\,a_i\,b_i+\hbar\,x_i\,y_i,\ \frac{\hbar\,a_i}{2}-\frac{1}{4}\,\hbar^3\,x_i^2\,y_i^2\right]$

$\overline{\mathrm{Kink}} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\frac{\hbar\,b_i}{2}-\hbar\,a_i\,b_i-\frac{\hbar\,x_i\,y_i}{B_i},\ -\frac{\hbar\,a_i}{2}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right]$

$\mathrm{b2t} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i-t_i\,\beta_i+y_i\,\eta_i+x_i\,\xi_i,\ a_i\,\beta_i\right]$

$\mathrm{t2b} \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+y_i\,\eta_i+x_i\,\xi_i-b_i\,\tau_i,\ a_i\,\tau_i\right]$

Check that on the generators this agrees with our conventions in the handout:

```
(Alt) In[]:= 𝔼2Λ[ℰ_] := Module[{k}, Sum[ℰ[k] ϵ^(k-1), {k, 0, ℰ[$]}]];
Timing@Block[{$k = 2}, {
   {
    "[a,x]" → 𝔼2Λ[𝔼{}→{1,2}[0, a2 x1] // am1,2→1] - 𝔼2Λ[𝔼{}→{1,2}[0, a1 x2] // am1,2→1],
    "[b,y]" → 𝔼2Λ[𝔼{}→{1,2}[0, y2 b1, 0] // bm1,2→1] - 𝔼2Λ[𝔼{}→{1,2}[0, y1 b2, 0] // bm1,2→1]
   } /. z_1 → z,
   {
    "Δ[y]" → Last[𝔼{}→{1}[0, y1] // bΔ1→1,2],
    "Δ[b]" → Last[𝔼{}→{1}[0, b1] // bΔ1→1,2],
    "Δ[a]" → Last[𝔼{}→{1}[0, a1] // aΔ1→1,2],
    "Δ[x]" → Last[𝔼{}→{1}[0, x1] // aΔ1→1,2]},
   {
    "S(a)" → ((𝔼{}→{1}[0, a1] // aS1)[1]),
    "S(x)" → ((𝔼{}→{1}[0, x1] // aS1)[1]),
    "S(b)" → ((𝔼{}→{1}[0, b1] // bS1)[1]),
    "S(y)" → ((𝔼{}→{1}[0, y1] // bS1)[1])
   } /. z_1 → z
  }]
```

(Alt) Out[]= $\{4.57813, \{\{[a,x] \to -x,\ [b,y] \to -y\,\epsilon\},\ \{\Delta[y] \to B_2 y_1 + y_2,\ \Delta[b] \to b_1 + b_2,\ \Delta[a] \to a_1 + a_2,\ \Delta[x] \to x_1 + x_2\},\ \{S(a) \to -a,\ S(x) \to -x,\ S(b) \to -b,\ S(y) \to -\frac{y}{B}\}\}\}$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
(Alt) In[]:= Timing@Block[{$k = 3},
   HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
  ]
```

(Alt) Out[]= {0.40625, {True, True}}

R and *P* are inverses:

```
(Alt) In[]:= Timing@Block[{$k = 3}, {Ri,j, Pi,k, HL[(Ri,j // Pi,k) ≡ aσk→j]}]
```

(Alt) Out[]= $\{0.515625, \{\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j b_i + \hbar\, x_j y_i,\ -\frac{1}{4}\hbar^3 x_j^2 y_i^2,\ \frac{1}{9}\hbar^5 x_j^3 y_i^3,\ \frac{1}{48}\left(\hbar^5 x_j^2 y_i^2 - 3\hbar^7 x_j^4 y_i^4\right)\right],$
$\mathbb{E}_{\{i,k\}\to\{\}}\left[\frac{\alpha_k \beta_i}{\hbar} + \frac{\eta_i \xi_k}{\hbar},\ \frac{\eta_i^2 \xi_k^2}{4\hbar},\ \frac{1}{8}\eta_i^2 \xi_k^2 + \frac{5\eta_i^3 \xi_k^3}{36\hbar},\ \frac{1}{24}\hbar\,\eta_i^2 \xi_k^2 + \frac{1}{6}\eta_i^3 \xi_k^3 + \frac{5\eta_i^4 \xi_k^4}{48\hbar}\right], \text{True}\}\}$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

```
(Alt) In[]:= Timing[HL /@ {(aS̅1 // aS1) ≡ aσ1→1, (bS̅1 // bS1) ≡ bσ1→1}]
```

(Alt) Out[]= {1.04688, {True, True}}

(co)-associativity on both sides

(Alt) In[∘]:=
```
Timing[
 HL /@ {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
   (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
```

(Alt) Out[∘]= {1.0625, {True, True, True, True}}

Δ is an algebra morphism

(Alt) In[∘]:=
```
Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
   (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
```

(Alt) Out[∘]= {1.90625, {True, True}}

An explicit formula for aS_i

(Alt) In[ ]:=
```
Timing@Block[{$k = 4}, HL[aS_i ≡ (𝔼_{i}→{i,j}[-α_i a_j, -ξ_i x_i,
    Sum[Expand[(e^(ξ_i x_i) (-ℏ γ ϵ)^k)/(2^k k!) Nest[Expand[x_i^2 ∂_{x_i,2} #] &, e^(-ξ_i e^(ℏ ϵ a_i) x_i), k]], {k, 0, $k}]]_$k //
    am_{i,j→i})]]
```

(Alt) Out[ ]=

$$\Big\{6.07813,\ \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\alpha_i - x_i\mathcal{A}_i\xi_i,\ -\hbar a_i x_i\mathcal{A}_i\xi_i - \frac{1}{2}\hbar x_i^2\mathcal{A}_i^2\xi_i^2,\ -\frac{1}{2}\hbar^2 a_i^2 x_i\mathcal{A}_i\xi_i + \frac{1}{4}\hbar^2 x_i^2\mathcal{A}_i^2\xi_i^2 - \hbar^2 a_i x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{1}{2}\hbar^2 x_i^3\mathcal{A}_i^3\xi_i^3,\ -\frac{1}{6}\hbar^3 a_i^3 x_i\mathcal{A}_i\xi_i - \frac{1}{12}\hbar^3 x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{1}{2}\hbar^3 a_i x_i^2\mathcal{A}_i^2\xi_i^2 - \hbar^3 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{2}{3}\hbar^3 x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{3}{2}\hbar^3 a_i x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{2}{3}\hbar^3 x_i^4\mathcal{A}_i^4\xi_i^4,\ -\frac{1}{24}\hbar^4 a_i^4 x_i\mathcal{A}_i\xi_i + \frac{1}{48}\hbar^4 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{1}{6}\hbar^4 a_i x_i^2\mathcal{A}_i^2\xi_i^2 + \frac{1}{2}\hbar^4 a_i^2 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{2}{3}\hbar^4 a_i^3 x_i^2\mathcal{A}_i^2\xi_i^2 - \frac{13}{24}\hbar^4 x_i^3\mathcal{A}_i^3\xi_i^3 + 2\hbar^4 a_i x_i^3\mathcal{A}_i^3\xi_i^3 - \frac{9}{4}\hbar^4 a_i^2 x_i^3\mathcal{A}_i^3\xi_i^3 + \frac{13}{8}\hbar^4 x_i^4\mathcal{A}_i^4\xi_i^4 - \frac{8}{3}\hbar^4 a_i x_i^4\mathcal{A}_i^4\xi_i^4 - \frac{25}{24}\hbar^4 x_i^5\mathcal{A}_i^5\xi_i^5\Big] \equiv$$

$$\mathbb{E}_{\{i,j\}\to\{i\}}\Big[a_i(\alpha_i+\alpha_j) + x_i\Big(\frac{\xi_i}{\mathcal{A}_j}+\xi_j\Big),\ 0,\ 0,\ 0,\ 0\Big]\Big[$$

$$\mathbb{E}_{\{i\}\to\{i,j\}}\Big[-a_j\alpha_i,\ -x_i\xi_i,\ e^{x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i} - \frac{1}{2}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma\epsilon\hbar x_i^2\xi_i^2 + \frac{1}{4}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^2\xi_i^2 - \frac{1}{12}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^2\xi_i^2 + \frac{1}{48}e^{2\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^2\xi_i^2 - \frac{1}{2}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^3\xi_i^3 + \frac{2}{3}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^3\xi_i^3 - \frac{13}{24}e^{3\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^3\xi_i^3 + \frac{1}{8}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^2\epsilon^2\hbar^2 x_i^4\xi_i^4 - \frac{19}{24}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^4\xi_i^4 + \frac{163}{96}e^{4\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^4\xi_i^4 + \frac{1}{4}e^{5\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^5\xi_i^5 - \frac{3}{2}e^{5\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^5\xi_i^5 - \frac{1}{48}e^{6\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^3\epsilon^3\hbar^3 x_i^6\xi_i^6 + \frac{47}{96}e^{6\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^6\xi_i^6 - \frac{1}{16}e^{7\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^7\xi_i^7 + \frac{1}{384}e^{8\epsilon\hbar a_i + x_i\xi_i - e^{\epsilon\hbar a_i}x_i\xi_i}\gamma^4\epsilon^4\hbar^4 x_i^8\xi_i^8\Big]_4\Big]\Big\}$$

S is convolution inverse of id

(Alt) In[ ]:=
```
Timing[HL[# ≡ sϵ_1 sη_1] & /@ {
    (aΔ_{1→1,2} // aS_1) // am_{1,2→1}, (aΔ_{1→1,2} // aS_2) // am_{1,2→1},
    (bΔ_{1→1,2} // bS_1) // bm_{1,2→1}, (bΔ_{1→1,2} // bS_2) // bm_{1,2→1}}]
```

(Alt) Out[ ]= {1.90625, {True, True, True, True}}

But not with the opposite product:

```
(Alt) In[∘]:= Timing[Short[# ≡ sϵ₁ sη₁] & /@ {
      (aΔ₁→₁,₂ ~ B₁ ~ aS₁) ~ B₁,₂ ~ am₂,₁→₁, (aΔ₁→₁,₂ ~ B₂ ~ aS₂) ~ B₁,₂ ~ am₂,₁→₁,
      (bΔ₁→₁,₂ ~ B₁ ~ bS₁) ~ B₁,₂ ~ bm₂,₁→₁, (bΔ₁→₁,₂ ~ B₂ ~ bS₂) ~ B₁,₂ ~ bm₂,₁→₁}]

(Alt) Out[∘]= {0.015625,
   {B₁,₂[B₁[𝔼{1}→{1,2}[a₁ α₁ + a₂ α₁ + x₁ ξ₁ + x₂ ξ₁, ≪1≫ + ≪1≫, ≪1≫], ≪1≫], ≪1≫] ≡ ≪1≫,
    B₁,₂[B₂[𝔼{1}→{1,2}[a₁ α₁ + a₂ α₁ + x₁ ξ₁ + x₂ ξ₁, ≪1≫ + ≪1≫, ≪1≫], ≪1≫], ≪1≫] ≡ ≪1≫,
    B₁,₂[B₁[𝔼{1}→{1,2}[≪1≫], 𝔼{1}→{1}[-b₁ β₁ - ≪1≫/≪1≫, -≪1≫ - ≪1≫, ≪1≫]], ≪1≫] ≡ ≪1≫,
    B₁,₂[B₂[𝔼{1}→{1,2}[≪1≫], 𝔼{2}→{2}[-b₂ β₂ - ≪1≫/≪1≫, -≪1≫ - ≪1≫, ≪1≫]], ≪1≫] ≡
     ≪1≫}}
```

S is an algebra anti-(co)morphism

```
(Alt) In[∘]:= Timing[HL /@ {(am₁,₂→₁ // aS₁) ≡ ((aS₁ aS₂) // am₂,₁→₁), (bm₁,₂→₁ // bS₁) ≡ ((bS₁ bS₂) // bm₂,₁→₁),
      (aS₁ // aΔ₁→₁,₂) ≡ (aΔ₁→₂,₁ // (aS₁ aS₂)), (bS₁ // bΔ₁→₁,₂) ≡ (bΔ₁→₂,₁ // (bS₁ bS₂))}]

(Alt) Out[∘]= {2.07813, {True, True, True, True}}
```

Pairing axioms

```
(Alt) In[∘]:= Timing[HL /@ {((bm₁,₂→₁ sϒ₃→₀,₀,₃,₃ // sϵ₀) // P₁,₃) ≡
       (((sϒ₁→₁,₁,₀,₀ // sϵ₀) (sϒ₂→₂,₂,₀,₀ // sϵ₀) aΔ₃→₄,₅) // P₁,₄ // P₂,₅),
      ((bΔ₁→₁,₂ (sϒ₃→₀,₀,₃,₃ // sϵ₀) (sϒ₄→₀,₀,₄,₄ // sϵ₀)) // P₁,₃ // P₂,₄) ≡
       (((sϒ₁→₁,₁,₀,₀ // sϵ₀) am₃,₄→₃) // P₁,₃)}]

(Alt) Out[∘]= {2.07813, {True, True}}

(Alt) In[∘]:= Timing[HL /@ {((bS₁ aσ₂→₂) // P₁,₂) ≡ ((bσ₁→₁ aS₂) // P₁,₂),
      ((bS̄₁ aσ₂→₂) // P₁,₂) ≡ ((bσ₁→₁ aS̄₂) // P₁,₂)}]

(Alt) Out[∘]= {1.21875, {True, True}}
```

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

(Alt) In[∘]:=
```
(*Timing@{{
      "[a,y]"→ ((𝔼{}→{1,2}[0,0,y2a1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,y1a2]~B1,2~dm1,2→1)[3]),
      "[b,x]"  →
       ((𝔼{}→{1,2}[0,0,x2b1]~B1,2~dm1,2→1)[3]-(𝔼{}→{1,2}[0,0,x1b2]~B1,2~dm1,2→1)[3]),"xy-qyx"→
       ((𝔼{}→{1,2}[0,0,x1y2]~B1,2~dm1,2→1)[3]-(1+∈)(𝔼{}→{1,2}[0,0,y1x2]~B1,2~dm1,2→1)[3])
     }/.{z_1→z}//Expand//Factor,
   {
    "Δ(a)"→((𝔼{}→{1}[0,0,a1]~B1~dΔ1→1,2)[3]),
    "Δ(x)"→((𝔼{}→{1}[0,0,x1]~B1~dΔ1→1,2)[3]),
    "Δ(b)"→((𝔼{}→{1}[0,0,b1]~B1~dΔ1→1,2)[3]),
    "Δ(y)"→((𝔼{}→{1}[0,0,y1]~B1~dΔ1→1,2)[3])
   }//Simplify,
   {
     "S(a)"→((𝔼{}→{1}[0,0,a1]~B1~dS1)[3]),
     "S(x)"→((𝔼{}→{1}[0,0,x1]~B1~dS1)[3]),
     "S(b)"→((𝔼{}→{1}[0,0,b1]~B1~dS1)[3]),
     "S(y)"→((𝔼{}→{1}[0,0,y1]~B1~dS1)[3])
    }/.{z_1→z}//Simplify
  }*)
```

(Alt) In[∘]:=
```
{HL[((sϒ1→0,0,1,1 // sϵ0) (sϒ2→0,0,2,2 // sϵ0) // dm1,2→1) ≡ am1,2→1],
 HL[((sϒ1→1,1,0,0 // sϵ0) (sϒ2→2,2,0,0 // sϵ0) // dm1,2→1) ≡ bm1,2→1]}
```

(Alt) Out[∘]= {True, True}

(co)-associativity

(Alt) In[∘]:=
```
Timing[Block[{$k = 1},
  HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2→1)}]
]
```

(Alt) Out[∘]= {0.859375, {True, True}}

Δ is an algebra morphism

(Alt) In[∘]:=
```
Timing@HL[(dm1,2→1 // dΔ1→1,2) ≡ ((dΔ1→1,3 dΔ2→2,4) // (dm3,4→2 dm1,2→1))]
```

(Alt) Out[∘]= {3.85938, True}

dS and $\overline{\text{dS}}$ are inverses:

(Alt) In[∘]:=
```
Timing@HL[(d̅S̅1 // dS1) ≡ dσ1→1]
```

(Alt) Out[∘]= {4.67188, True}

S_2 inverts R, but not S_1:

```
(Alt) In[∘]:= Timing@{(R1,2 // dS1) ≡ R̄1,2, HL[(R1,2 // dS2) ≡ R̄1,2]}
```

(Alt) Out[∘]= $\left\{0.78125, \left\{\frac{\hbar^2 x_2 y_1}{B_1} - \frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} == -\frac{\hbar^2 a_2 x_2 y_1}{B_1} - \frac{3\hbar^3 x_2^2 y_1^2}{4 B_1^2} \,\&\&\, -\frac{\hbar^3 x_2 y_1}{2 B_1} + \frac{\hbar^3 a_2 x_2 y_1}{B_1} - \frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{2\hbar^4 x_2^2 y_1^2}{B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3} == -\frac{\hbar^3 a_2^2 x_2 y_1}{2 B_1} + \frac{\hbar^4 x_2^2 y_1^2}{2 B_1^2} - \frac{3\hbar^4 a_2 x_2^2 y_1^2}{2 B_1^2} - \frac{10\hbar^5 x_2^3 y_1^3}{9 B_1^3}, \text{True}\right\}\right\}$

dS is convolution inverse of id

```
(Alt) In[∘]:= Timing[HL[# ≡ dϵ1 dη1] & /@ {(dΔ1→1,2 // dS1) // dm1,2→1, (dΔ1→1,2 // dS2) // dm1,2→1}]
(Alt) Out[∘]= {5.53125, {True, True}}
```

dS is a (co)-algebra anti-morphism

```
(Alt) In[∘]:= Timing[HL /@
   Expand /@ {(dm1,2→1 // dS1) ≡ ((dS1 dS2) // dm2,1→1), (dS1 // dΔ1→1,2) ≡ (dΔ1→2,1 // (dS1 dS2))}]
(Alt) Out[∘]= {12.1094, {True, True}}
```

Quasi-triangular axiom 1:

```
(Alt) In[∘]:= Timing[
  HL /@ {(R1,3 // dΔ1→1,2) ≡ ((R1,4 R2,3) // dm3,4→3), (R1,2 // dΔ2→2,3) ≡ ((R1,2 R4,3) // dm1,4→1)}]
(Alt) Out[∘]= {0.9375, {True, True}}
```

Quasi-triangular axiom 2:

```
(Alt) In[∘]:= Timing@HL[((dΔ1→1,2 R3,4) // (dm1,3→1 dm2,4→2)) ≡ ((R1,2 dΔ1→3,4) // (dm1,4→1 dm2,3→2))]
(Alt) Out[∘]= {3.60938, True}
```

The Drinfel'd element inverse property, $(u_1\,\bar{u}_2)$ // $\text{dm}_{1,2\to 1} \equiv \text{d}\epsilon_i$:

```
(Alt) In[∘]:= Timing@HL[(((R1,2 // dS1 // dm2,1→i) (R1,2 // dS2 // dS2 // dm2,1→j)) // dmi,j→i) ≡ dηi]
```

(Alt) Out[∘]= $\left\{9.28125, \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - \text{Log}\left[B_i^2\right]\right) == 0\right\}$

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

```
(Alt) In[∘]:= Timing@
  Block[{$k = 2}, (((R1,2 // dS1 // dm2,1→i) // dSi) (R1,2 // dS2 // dS2 // dm2,1→j)) // dmi,j→i]
```

(Alt) Out[∘]= $\left\{16.6094, \mathbb{E}_{\{\}\to\{i\}}\left[\frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right), \hbar\, a_i, 0\right]\right\}$

```
(Alt) In[∘]:= Timing@Block[{$k = 2}, HL /@ {((C_i C̄_j) // dm_{i,j→i}) ≡ dη_i, ((C̄_i C̄_j) // dm_{i,j→i}) ≡
        (((R_{1,2} // dS_1 // dm_{2,1→i}) // dS_i) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j}) // dm_{i,j→i})}]
```

(Alt) Out[∘]= $\left\{17.3594, \left\{\text{True}, \hbar\, b_i == \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right)\right\}\right\}$

```
(Alt) In[∘]:= Timing@Block[{$k = 2}, HL /@ {((C_i C̄_j) // dm_{i,j→i}) ≡ dη_i, ((C̄_i C̄_j) // dm_{i,j→i}) ≡
        ((((R_{1,2} // dS_1 // dm_{2,1→i}) // dS_i) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j})) // dm_{i,j→i})}]
```

(Alt) Out[∘]= $\left\{17.0156, \left\{\text{True}, \hbar\, b_i == \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_i^2}\right] - 2\,\text{Log}\left[B_i^2\right]\right)\right\}\right\}$

Reidemeister 2:

```
(Alt) In[∘]:= Timing[HL[# ≡ dη_1 dη_2] & /@ {(R̄_{1,2} R_{3,4}) // (dm_{1,3→1} dm_{2,4→2}), (R_{1,2} R̄_{3,4}) // (dm_{1,3→1} dm_{2,4→2})}]
```

(Alt) Out[∘]= {2.48438, {True, True}}

Cyclic Reidemeister 2:

```
(Alt) In[∘]:= Timing@HL[((R_{1,4} R̄_{5,2} C̄_3) // dm_{2,4→2} // dm_{1,3→1} // dm_{1,5→1}) ≡ C̄_1 dη_2]
```

(Alt) Out[∘]= {1.28125, True}

Reidemeister 3:

```
(Alt) In[∘]:= Timing@HL[(R_{1,2} R_{6,3} R_{4,5} // dm_{1,6→1} dm_{2,4→2} dm_{3,5→3} ) ≡ (R_{2,3} R_{1,4} R_{5,6} // dm_{1,5→1} dm_{2,6→2} dm_{3,4→3})]
```

(Alt) Out[∘]= {5.21875, True}

Relations between the four kinks:

```
(Alt) In[∘]:= Timing[HL /@ {Kink_i ≡ ((R_{3,1} C_2) // dm_{1,2→1} // dm_{1,3→i}),
      K̄ink_j ≡ ((R̄_{3,1} C̄_2) // dm_{1,2→1} // dm_{1,3→j}), ((Kink_i K̄ink_j) // dm_{i,j→1}) ≡ dη_1}]
```

(Alt) Out[∘]= $\left\{9.73438, \left\{\frac{\hbar\, b_i}{2} + \hbar\, a_i\, b_i + \hbar\, x_i\, y_i == \hbar\, a_i\, b_i + \frac{1}{2}\left(-\text{Log}\left[B_i^2\right] - \hbar\, b_i\right) + \hbar\, x_i\, y_i,\right.\right.$

$\left.\left. -\frac{\hbar\, b_j}{2} - \hbar\, a_j\, b_j - \frac{\hbar\, x_j\, y_j}{B_j} == -\hbar\, a_j\, b_j + \frac{1}{2}\left(-\text{Log}\left[\frac{1}{B_j^2}\right] + \hbar\, b_j\right) - \frac{\hbar\, x_j\, y_j}{B_j}, \text{True}\right\}\right\}$

The Trefoil

(Alt) In[∘]:=
```
Timing@Block[{$k = 1},
  Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4 K̄ink_8 K̄ink_9 K̄ink_10;
  Do[Z31 = Z31 // dm_{1,r→1}, {r, 2, 10}];
  {Simplify /@ Z31, Simplify /@ (Z31 // b2t_1 /. T_1 → T)}]
```

(Alt) Out[∘]= $\left\{6.0625, \left\{\mathbb{E}_{\{\}\to\{1\}}\left[-\frac{1}{2}\text{Log}\left[\left(1-B_1+B_1^2\right)^2\right]-\hbar\, b_1, -\frac{\hbar\left(B_1-2B_1^2-2B_1^4-a_1\left(-1+B_1-B_1^3+B_1^4\right)+2\hbar\, x_1 y_1+B_1^3\left(3+2\hbar\, x_1 y_1\right)\right)}{\left(1-B_1+B_1^2\right)^2}\right], \mathbb{E}_{\{\}\to\{1\}}\left[-\frac{1}{2}\text{Log}\left[\left(1-T_1+T_1^2\right)^2\right]+\hbar\, t_1, -\frac{\hbar\left(T_1-2T_1^2-2T_1^4-2a_1\left(-1+T_1-T_1^3+T_1^4\right)+2\hbar\, x_1 y_1+T_1^3\left(3+2\hbar\, x_1 y_1\right)\right)}{\left(1-T_1+T_1^2\right)^2}\right]\right\}\right\}$

b2t, t2b, knot tensors.

(Alt) In[∘]:= `HL[(b2t_i // t2b_i) ≡ dσ_{i→i}]`

(Alt) Out[∘]= True

(Alt) In[∘]:= `t2b_i // b2t_i`

(Alt) Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}[a_i\,\alpha_i + y_i\,\eta_i + x_i\,\xi_i + t_i\,\tau_i,\ 0,\ 0]$

Reidemeister 2:

(Alt) In[∘]:= `Timing[HL[# ≡ dη_1 dη_2] & /@ {(k̄R_{1,2} kR_{3,4}) // (km_{1,3→1} km_{2,4→2}), (kR_{1,2} k̄R_{3,4}) // (km_{1,3→1} km_{2,4→2})}]`

(Alt) Out[∘]= {3.96875, {True, True}}

Cyclic Reidemeister 2:

(Alt) In[∘]:= `Timing@HL[((kR_{1,4} k̄R_{5,2} k̄C_3) // km_{2,4→2} // km_{1,3→1} // km_{1,5→1}) ≡ k̄C_1 dη_2]`

(Alt) Out[∘]= {1.25, True}

Reidemeister 3:

(Alt) In[∘]:=
```
Timing@HL[(kR_{1,2} kR_{4,3} kR_{5,6} // km_{1,4→1} // km_{2,5→2} // km_{3,6→3}) ≡
  (kR_{1,6} kR_{2,3} kR_{4,5} // km_{1,4→1} // km_{2,5→2} // km_{3,6→3})]
```

(Alt) Out[∘]= {2.57813, True}

Relations between the four kinks:

(Alt) In[∘]:= $\text{Timing}\left[\text{HL} /@ \left\{\text{kKink}_i \equiv ((\text{kR}_{3,1}\,\text{kC}_2) \;//\; \text{km}_{1,2\to 1} \;//\; \text{km}_{1,3\to i}),\; \overline{\text{kKink}}_j \equiv \left(\left(\overline{\text{kR}}_{3,1}\,\overline{\text{kC}}_2\right) \;//\; \text{km}_{1,2\to 1} \;//\; \text{km}_{1,3\to j}\right),\; \left(\left(\text{kKink}_i\,\overline{\text{kKink}}_j\right) \;//\; \text{km}_{i,j\to 1}\right) \equiv \text{d}\eta_1\right\}\right]$

(Alt) Out[∘]= $\left\{4.25, \left\{-\frac{t\,\hbar}{2} - t\,\hbar\,a_i + \hbar\,x_i\,y_i == \frac{1}{2}\left(t\,\hbar - \text{Log}\left[T^2\right]\right) - t\,\hbar\,a_i + \hbar\,x_i\,y_i,\; \frac{t\,\hbar}{2} + t\,\hbar\,a_j - \frac{\hbar\,x_j\,y_j}{T} == \frac{1}{2}\left(-t\,\hbar - \text{Log}\left[\frac{1}{T^2}\right]\right) + t\,\hbar\,a_j - \frac{\hbar\,x_j\,y_j}{T},\; \text{True}\right\}\right\}$

The Trefoil

(Alt) In[∘]:= $\text{Timing@Block}\Big[\{\$k = 1\},\; \text{Z31} = \text{kR}_{1,5}\,\text{kR}_{6,2}\,\text{kR}_{3,7}\,\overline{\text{kC}}_4\,\overline{\text{kKink}}_8\,\overline{\text{kKink}}_9\,\overline{\text{kKink}}_{10};\; \text{Do}[\text{Z31} = \text{Z31} \;//\; \text{km}_{1,r\to 1}, \{r, 2, 10\}];\; \text{Simplify} /@ \text{Z31}\Big]$

(Alt) Out[∘]= $\left\{5.14063, \mathbb{E}_{\{\}\to\{1\}}\left[t\,\hbar - \frac{1}{2}\,\text{Log}\left[\left(1 - T + T^2\right)^2\right],\; -\frac{\hbar\left(T - 2\,T^2 + 3\,T^3 - 2\,T^4 - 2\times\left(-1 + T - T^3 + T^4\right)a_1 + 2\times\left(1 + T^3\right)\hbar\,x_1\,y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$

(Alt) In[∘]:= $\text{Timing@Block}\Big[\{\$k = 1\},\; \text{Z31} = \text{kR}_{1,5}\,\text{kR}_{6,2}\,\text{kR}_{3,7}\,\overline{\text{kC}}_4\,\overline{\text{kKink}}_8\,\overline{\text{kKink}}_9\,\overline{\text{kKink}}_{10};\; \text{Do}[\text{Z31} = \text{Z31} \;//\; \text{km}_{1,r\to 1}, \{r, 2, 10\}];\; \text{Simplify} /@ \text{Z31}\Big]$

(Alt) Out[∘]= $\left\{4.1875, \mathbb{E}_{\{\}\to\{1\}}\left[t\,\hbar - \frac{1}{2}\,\text{Log}\left[\left(1 - T + T^2\right)^2\right],\; -\frac{\hbar\left(T - 2\,T^2 + 3\,T^3 - 2\,T^4 - 2\times\left(-1 + T - T^3 + T^4\right)a_1 + 2\times\left(1 + T^3\right)\hbar\,x_1\,y_1\right)}{\left(1 - T + T^2\right)^2}\right]\right\}$

(Alt) In[•]:=
```
Timing@Block[{$k = 1}, Z[Knot[8, 17]]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[•]=

$$\left\{84.6094, \mathbb{E}_{\{\}\to\{0\}}\left[\frac{1}{2}\times\left(-2\,t\,\hbar - \mathrm{Log}\left[\left(-1-\frac{1}{T^4}+\frac{4}{T^3}-\frac{6}{T^2}+\frac{5}{T}\right)^2\right] - \mathrm{Log}\left[\left(1+\frac{T}{1-4T+6T^2-5T^3+T^4}-\frac{2T^2}{1-4T+6T^2-5T^3+T^4}+\frac{T^3}{1-4T+6T^2-5T^3+T^4}\right)^2\right] - \mathrm{Log}\left[\left(1-\frac{T}{1-3T+4T^2-4T^3+T^4}+\frac{4T^2}{1-3T+4T^2-4T^3+T^4}-\frac{7T^3}{1-3T+4T^2-4T^3+T^4}+\frac{7T^4}{1-3T+4T^2-4T^3+T^4}-\frac{4T^5}{1-3T+4T^2-4T^3+T^4}+\frac{T^6}{1-3T+4T^2-4T^3+T^4}\right)^2\right]\right), \frac{-3\hbar+8T\hbar-8T^2\hbar+8T^4\hbar-8T^5\hbar+3T^6\hbar}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}+\frac{a\left(-6\hbar+16T\hbar-16T^2\hbar+16T^4\hbar-16T^5\hbar+6T^6\hbar\right)}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}+\frac{x\,y\left(-6\hbar^2+10T\hbar^2-6T^2\hbar^2-6T^3\hbar^2+10T^4\hbar^2-6T^5\hbar^2\right)}{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}\right]\right\}$$

CU

Associativity of CU:

(Alt) In[•]:=

$$\mathrm{Timing@Block}[\{\$k = 3\}, \mathrm{HL}[(cm_{1,2\to1} \,//\, cm_{1,3\to1}) \equiv (cm_{2,3\to2} \,//\, cm_{1,2\to1})]]$$

(Alt) Out[•]= {2.01563, True}

Associativity, co-associativity, and Δ is an algebra morphism:

(Alt) In[•]:=

$$\mathrm{Timing@Block}[\{\$k = 3\}, \mathrm{HL} \,/@\, \{(cm_{1,2\to1} \,//\, cm_{1,3\to1}) \equiv (cm_{2,3\to2} \,//\, cm_{1,2\to1}), (c\Delta_{1\to1,2} \,//\, c\Delta_{2\to2,3}) \equiv (c\Delta_{1\to1,3} \,//\, c\Delta_{1\to1,2}), (cm_{1,2\to1} \,//\, c\Delta_{1\to1,2}) \equiv ((c\Delta_{1\to1,3}\, c\Delta_{2\to2,4}) \,//\, (cm_{3,4\to2}\, cm_{1,2\to1}))\}]$$

(Alt) Out[•]= {3.59375, {True, True, True}}

S is convolution inverse of id:

(Alt) In[•]:=

$$\mathrm{Timing@Block}[\{\$k = 3\}, \mathrm{HL}[\# \equiv c\epsilon_1\, c\eta_1]\ \&\ /@\ \{(c\Delta_{1\to1,2} \,//\, cS_1) \,//\, cm_{1,2\to1}, (c\Delta_{1\to1,2} \,//\, cS_2) \,//\, cm_{1,2\to1}\}]$$

(Alt) Out[•]= {3.35938, {True, True}}

S is an algebra anti-(co)morphism

(Alt) In[•]:=

$$\mathrm{Timing@Block}[\{\$k = 3\}, \mathrm{HL} \,/@\, \{(cm_{1,2\to1} \,//\, cS_1) \equiv ((cS_1\, cS_2) \,//\, cm_{2,1\to1}), (cS_1 \,//\, c\Delta_{1\to1,2}) \equiv (c\Delta_{1\to2,1} \,//\, (cS_1\, cS_2))\}]$$

(Alt) Out[•]= {6.29688, {True, True}}

Classical is the $\hbar \to 0$ limit of quantum:

(Alt) In[∘]:=
```
ClassicalLimit[ƒ_] := Normal@Series[Normal[ƒ] // U2l, {ℏ, 0, 0}] // l2U;
Timing[HL /@ Simplify /@
  {cm_{1,2→3} ≡ ClassicalLimit /@ dm_{1,2→3},
   (cΔ_{1→2,3} /. τ_1 → 0) ≡ ClassicalLimit /@ dΔ_{1→2,3}, cS_1 ≡ ClassicalLimit /@ dS_1}]
```
(Alt) Out[∘]= {1.90625, {True, True, True}}

(Alt) In[∘]:=
```
PrintProfile[]
```
(Alt) Out[∘]=
```
ProfileRoot is root. Profiled time: 274.386
  (      1)    0.311/  84.594 above Z
  (     59)    1.001/  33.249 above Boot
  (   1314)    3.274/   9.170 above CF
  (      1)        0/       0 above RVK
  (    197)    4.083/   5.830 above Zip1
  (    394)    4.564/  40.872 above Zip2
  (    394)   21.909/ 100.670 above Zip3
CCF: called 124215 times, time in 122.133/122.133
  ( 124215)  122.130/ 122.130 under CF
CF: called 87109 times, time in 99.341/221.474
  (    214)    1.248/   2.876 under Z
  (    407)    0.672/   1.419 under Boot
  (   1314)    3.274/   9.170 under ProfileRoot
  (    680)    1.327/   3.198 under Zip1
  (   2526)   32.735/  91.592 under Zip2
  (  81968)   60.085/ 113.220 under Zip3
  ( 124215)  122.130/ 122.130 above CCF
Zip3: called 680 times, time in 34.936/148.155
  (    114)    5.120/  27.528 under Z
  (    172)    7.907/  19.956 under Boot
  (    394)   21.909/ 100.670 under ProfileRoot
  (  81968)   60.085/ 113.220 above CF
Zip2: called 680 times, time in 8.316/99.908
  (    114)    1.394/  51.629 under Z
  (    172)    2.358/   7.407 under Boot
  (    394)    4.564/  40.872 under ProfileRoot
  (   2526)   32.735/  91.592 above CF
Zip1: called 340 times, time in 8.255/11.453
  (     57)    1.174/   1.922 under Z
  (     86)    2.998/   3.701 under Boot
  (    197)    4.083/   5.830 under ProfileRoot
  (    680)    1.327/   3.198 above CF
Boot: called 86 times, time in 1.094/48.296
  (      3)    0.016/   0.328 under Z
  (     24)    0.077/  14.719 under Boot
  (     59)    1.001/  33.249 under ProfileRoot
  (     24)    0.077/  14.719 above Boot
  (    407)    0.672/   1.419 above CF
```

```
  (    86)   2.998/  3.701 above Zip1
  (   172)   2.358/  7.407 above Zip2
  (   172)   7.907/ 19.956 above Zip3
Z: called 1 times, time in 0.311/84.594
  (     1)   0.311/ 84.594 under ProfileRoot
  (     3)   0.016/  0.328 above Boot
  (   214)   1.248/  2.876 above CF
  (    57)   1.174/  1.922 above Zip1
  (   114)   1.394/ 51.629 above Zip2
  (   114)   5.120/ 27.528 above Zip3
RVK: called 1 times, time in 0./0.
  (     1)      0/     0 under ProfileRoot
```