Pensieve header: Exponentiation in ybax algebras (old code and attempt on $k=0$ exponentiation using a matrix representation).

Startup

In[∘]:=
```
Date[]
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\FullDoPeGDO"];
Once[<< KnotTheory`];
Once[Get@"../Profile/Profile.m"];
BeginProfile[];
$k = 1;
<< Engine.m
<< Objects.m
<< KT.m
```

Out[∘]= {2021, 8, 5, 10, 48, 36.7946331}

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
This version: April 2020. Original version: July 1994.
```

In[∘]:=
```
$k = 2; (*ℏ=γ=1;*)
```

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

```
(* Bug: The first line is valid only if 𝕆(e^P0) == e^𝕆(P0). *)
Exp_{m_,i_,0}[P_] := Module[{LQ = Normal@P /. ϵ → 0},
  𝔼[LQ /. (x | y)_i → 0, LQ /. (b | a | t)_i → 0, 1]];
```

```
Expm_,i_,k_[P_] := Block[{$k = k},
  Module[{P0, λ, φ, φs, F, j, rhs, eqn, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Expm,i,k-1[λ P];
   While[
    rhs = mi,j→i[𝔼{}→{i}[λ P0 /. (x | y)i → 0, λ P0 /. (b | a | t)i → 0, F]k
        sσi→j@𝔼{}→{i}[0, 0, P]k] // Last // Normal;
    eqn = CF[(∂λ F) + P0 F - rhs];
    eqn =!= 0, (*do*)
    pows = First /@ CoefficientRules[eqn, {yi, bi, ai, xi}];
    F += Sum[ϵ^k φjs[λ] Times @@ {yi, bi, ai, xi}^js, {js, pows}];
    rhs = mi,j→i[𝔼{}→{i}[λ P0 /. (x | y)i → 0, λ P0 /. (b | a | t)i → 0, F]k
        sσi→j@𝔼{}→{i}[0, 0, P]k] // Last // Normal;
    eqn = CF[(∂λ F) + P0 F - rhs];
    φs = Table[φjs[λ], {js, pows}];
    at0 = Table[φjs[0] == 0, {js, pows}];
    atλ = (# == 0) & /@ (pows /. CoefficientRules[eqn, {yi, bi, ai, xi}]);
    F = F /. DSolve[And @@ (at0 ⋃ atλ), φs, λ]⟦1⟧
   ];
   𝔼{}→{i}[P0 /. (x | y)i → 0, P0 /. (b | a | t)i → 0, F + O[ϵ]^(k+1) /. λ → 1]]]
```

Exponentials

Task. Define $\mathrm{Exp}_{m,k}[\mathbb{U}_{\{\_\_\_\}\to\{i\_\}}[U\_\_]]$ to compute $e^{\mathbb{O}(U)}$ to order ϵ^k using the $m_{i,i\to i}$ multiplication, where U is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form.
Methodology. If $U_0 := U_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(U)} = \mathbb{O}(e^{\lambda U_0} F(\lambda))$, then $F(\lambda=0)=1$ and we have:
$\mathbb{O}(e^{\lambda U_0}(U_0 F(\lambda) + \partial_\lambda F)) = \mathbb{O}(\partial_\lambda e^{\lambda U_0} F(\lambda)) = \partial_\lambda \mathbb{O}(e^{\lambda U_0} F(\lambda)) = \partial_\lambda e^{\lambda\,\mathbb{O}(U)} = e^{\lambda\,\mathbb{O}(U)}\,\mathbb{O}(U) = \mathbb{O}(e^{\lambda U_0} F(\lambda))\,\mathbb{O}(U)$.
This is a linear ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

In[ ]:=
```
ME = MatrixExp; b[A_, B_] := A.B - B.A;
{ρ0 = (0 0 0; 0 0 0; 0 0 0);
  ρy = (0 1 0; 0 0 0; 0 0 0), ρb = (0 0 -1; 0 0 0; 0 0 0), ρa = (0 0 0; 0 1 0; 0 0 0), ρx = (0 0 0; 0 0 1; 0 0 0), ρxy = (b 0 0; 0 0 0; 0 0 b)};
HL /@ {b[ρa, ρx] == ρx, b[ρa, ρy] == -ρy, b[ρb, ρy] == ρ0, b[ρb, ρx] == ρ0, b[ρx, ρy] == ρb,
  b[ρxy, ρx] == -b ρx, b[ρxy, ρy] == b ρy, b[ρxy, ρa] == ρ0, b[ρxy, ρb] == ρ0}
```

Out[ ]= {True, True, True, True, True, True, True, True, True}

In[∘]:=
```
MatrixForm[lhs = ME[c1 ρy + c2 ρb + c3 ρa + c4 ρx]]
MatrixForm[
 rhs = ME[d1 ρy].ME[d2 ρb].ME[d3 ρa].ME[d4 ρx] /.
   {d1 → (1 - e^-c3) c1 / c3, d2 → c2 + c1 c4 (e^-c3 - 1 + c3) / c3^2, d3 → c3, d4 → (1 - e^-c3) c4 / c3}
]
HL@Simplify[lhs == rhs]
```

$$\mathrm{lhs}=\begin{pmatrix} 1 & \frac{\left(-1+e^{c_3}\right) c_1}{c_3} & \frac{-c_2 c_3^2-c_1 c_4+e^{c_3} c_1 c_4-c_1 c_3 c_4}{c_3^2} \\ 0 & e^{c_3} & \frac{\left(-1+e^{c_3}\right) c_4}{c_3} \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{e^{c_3}\left(1-e^{-c_3}\right) c_1}{c_3} & -c_2+\frac{e^{c_3}\left(1-e^{-c_3}\right)^2 c_1 c_4}{c_3^2}-\frac{c_1\left(-1+e^{-c_3}+c_3\right) c_4}{c_3^2} \\ 0 & e^{c_3} & \frac{e^{c_3}\left(1-e^{-c_3}\right) c_4}{c_3} \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]= True

In[∘]:=
```
MatrixForm[lhs = ME[c1 ρy + c2 ρb + c3 ρa + c4 ρx + c5 ρxy]]
MatrixForm[
 rhs = ME[d1 ρy].ME[d2 ρb].ME[d3 ρa].ME[d4 ρx].ME[d5 ρxy] /. {d1 → (1 - e^(-c3+b c5)) c1 / (c3 - b c5),
   d2 → c2 + e^-c3 c1 c4 (e^(b c5) + e^c3 (-1 + c3 - b c5)) / (c3 - b c5)^2, d3 → c3, d4 → -(e^-c3 - e^(-b c5)) c4 / (c3 - b c5), d5 → c5}
]
HL@Simplify[lhs == rhs]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} e^{b c_5} & \frac{\left(e^{c_3}-e^{b c_5}\right) c_1}{c_3-b c_5} & -\frac{e^{b c_5} c_2 c_3^2-e^{c_3} c_1 c_4+e^{b c_5} c_1 c_4+e^{b c_5} c_1 c_3 c_4-2 b e^{b c_5} c_2 c_3 c_5-b e^{b c_5} c_1 c_4 c_5+b^2 e^{b c_5} c_2 c_5^2}{\left(c_3-b c_5\right)^2} \\ 0 & e^{c_3} & \frac{\left(e^{c_3}-e^{b c_5}\right) c_4}{c_3-b c_5} \\ 0 & 0 & e^{b c_5} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} e^{b c_5} & \frac{e^{c_3}\left(1-e^{-c_3+b c_5}\right) c_1}{c_3-b c_5} & e^{b c_5}\left(-c_2-\frac{e^{c_3}\left(e^{-c_3}-e^{-b c_5}\right)\left(1-e^{-c_3+b c_5}\right) c_1 c_4}{\left(c_3-b c_5\right)^2}-\frac{e^{-c_3} c_1 c_4\left(e^{b c_5}+e^{c_3}\left(-1+c_3-b c_5\right)\right)}{\left(c_3-b c_5\right)^2}\right) \\ 0 & e^{c_3} & -\frac{e^{c_3+b c_5}\left(e^{-c_3}-e^{-b c_5}\right) c_4}{c_3-b c_5} \\ 0 & 0 & e^{b c_5} \end{pmatrix}$$

Out[∘]= True